Circuit Analysis for Analog Designers
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Lecture - 45
Transmission line circuit analysis, the reflection coefficient, open- and short-circuited lines

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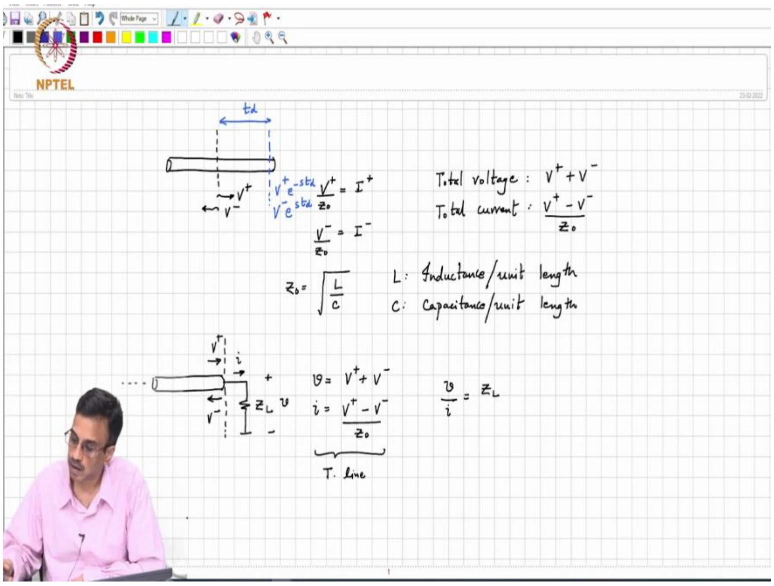

In the last class, we were talking about transmission lines and the telegraphers equations and we concluded that at any point on ideal transmission line, the voltage can be expressed as the sum of a forward going wave V plus and a backward going wave V minus right. The forward going wave is associated with a current which is I plus and the backward going wave is by the same token I minus.

$$\frac{V^{+}}{Z_0} = I^{+}$$

$$\frac{V^{-}}{Z_0} = I^{-}$$

The total voltage at any point therefore, is V plus plus V minus.

$$Total\ voltage: V^{+} + V^{-}$$

While the total current at any point a line is V plus minus V minus by Z.

$$Total\ current\text{:}\frac{V^+ - V^-}{Z_0}$$

Recall that Z naught is the characteristic impedance of the transmission line and is given by a square root of L by C where, L is the inductance per unit length and C is the capacitance of a unit length.

$$Z_0 = \sqrt{\frac{L}{C}}$$

So, as the wave propagates, the positive going wave in other words the wave going towards the right keeps getting delayed. So, for instance if the delay corresponding to this section of the line is t d, then the positive going wave at the right hand is going to be given by V plus e to the minus s t d and the frequency domain because the wave which is travelling towards you are right is going to get delayed as it travels.

So, and e to the minus s t d is simply the transfer function of an ideal delay with a value t d. Similarly V minus on the other hand must be advanced when you look at it at the blue boundary. So, that is getting delayed as it moves towards the left and therefore, at the black boundary, it appears V minus alright.

Now, let us see some applications of it turns out that these are very useful relations to know while analyzing circuits with transmission lines. So, in this course we are only going to look at idle transmission lines.

And so for example, consider what happens when you are at the boundary. So, here we do not know what is happening towards the left side the trans the ideal transmission line is terminated by an impedance Z sub L and just at the interface we have let us call the positive going wave as V plus and the negative going wave as V minus and the total voltage is simply V plus plus V minus, the total current is V plus minus V minus by Z naught.

$$v = V^+ + V^-$$

$$i = \frac{V^+ - V^-}{Z_0}$$

So, that is the total current and that is voltage and therefore, this is the constraint imposed by the transmission line and the terminating impedance; however, imposes a constraint due to Ohms law or v by i, its nothing, but Z L, alright.

$$\frac{v}{i} = Z_L$$

So, this means that we should now be able to figure out what the reverse going wave is in terms of the forward going wave and that is what we will do going forward.

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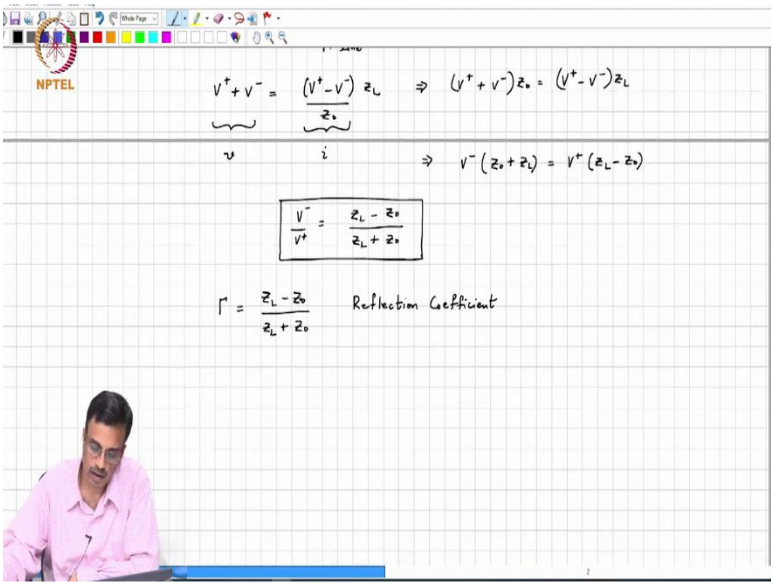

So, V plus plus V minus must be V plus minus V minus by Z naught times Z L.

$$V^{+} + V^{-} = \frac{(V^{+} - V^{-})}{Z_0} Z_L$$

So, this is the voltage, this on the other hand is the current and therefore, V plus plus V minus in to Z naught is V plus minus V minus into Z L which means let V minus into Z naught plus Z L is V plus in to Z L minus Z naught. So, we therefore, have V minus over V plus is nothing, but that Z L minus Z naught by Z L plus Z naught.

$$\Rightarrow (V^{+} + V^{-}) Z_0 = (V^{+} - V^{-}) Z_L$$

$$\Rightarrow V^{-}(Z_0 + Z_L) = V^{+}(Z_L - Z_0)$$

$$\frac{V^-}{V^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

In other words, if you have a transmission line which is terminated at one end by an impedance Z sub L, then Ohm's law has to be satisfied as far as the impedance is concerned as far as the terminating impedance Z L is concerned. However, the forward and backward going waves on the transmission line must. So, what this basically means is that the impedance Z L imposes a constraint due to Ohm's law; however, the transmission line also imposes constraints in the sense that the quantities on the transmission line must obey the telegrapher's equation.

So, at the interface both these relationships must be satisfied and that is why we have a constraint that the reflected wave is dependent on the terminating impedance and the characteristic impedances of the transmission line of course, its proportional to the amplitude of the incident way.

So, this quantity Z L minus Z naught by Z L plus Z naught is called the reflection coefficient and is often denoted by the Greek symbol capital gamma. So, this is the reflection coefficient, alright.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

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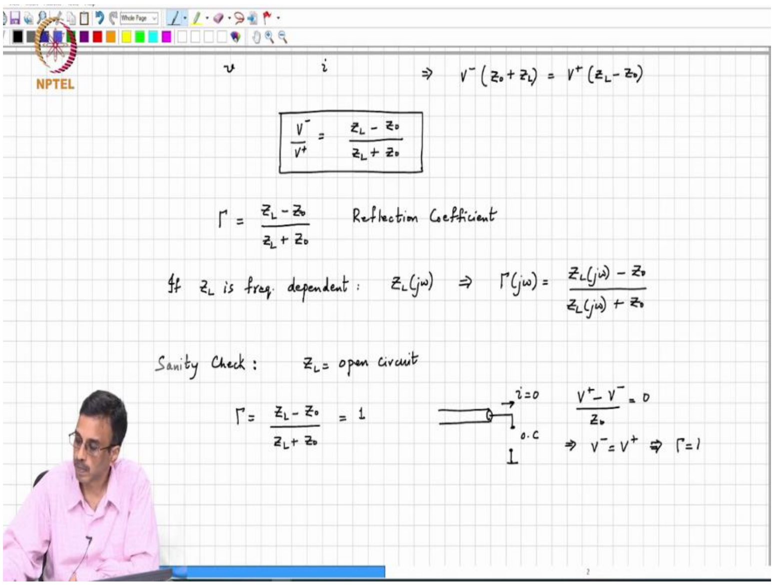

And if Z L is frequency dependent, in other words Z L of j omega, then the reflection coefficient is also a function of frequency and is given by Z L of j omega minus Z naught by Z L of j omega plus Z naught, alright.

$$Z_L(j\omega) \ \Rightarrow \Gamma(j\omega) = \frac{Z_L(j\omega) - Z_0}{Z_L(j\omega) + Z_0}$$

And let us do some sanity checks. The first case we consider is when Z L is an open circuit. So, in this case gamma, our formula is telling us that gamma which is Z L minus Z naught by Z L plus Z naught is simply plus 1 because Z L tends to infinite right.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = 1$$

And why does that make intuitive sense? At the end of the transmission line where we had Z L being infinity. So, we have an open circuit and the property of an open circuit as we all know is that this current i is 0, if the current i is 0, the current is nothing, but V plus minus V minus by Z naught which must be 0 which can only happen when V minus is equal to V plus which is equivalent to saying gamma is 1, ok.

$$\frac{V^+ - V^-}{Z_0} = 0$$

$$\Rightarrow V^- = V^+ \ \Rightarrow \Gamma = 1$$

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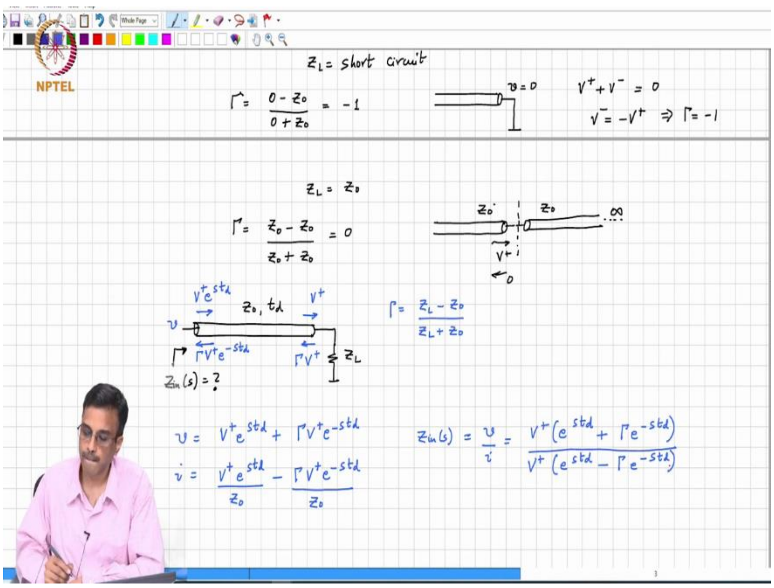

Let us now look at another example, Z L is a short circuit. So, gamma then it is nothing, but 0 minus Z naught by 0 plus Z naught which is equal to minus 1

$$\Gamma = \frac{0 - Z_0}{0 + Z_0} = -1$$

And how do we figure this out? This is when the voltage here is 0 and the voltage at the end of the transmission line is V plus plus V minus which is equal to 0 which means that V minus is equal to plus, alright. So, so this means that is equivalent to saying that gamma is minus 1.

$$V^+ + V^- = 0$$

$$V^- = -V^+$$

$$\Gamma = -1$$

Now, let us take third case where the impedance is exactly the same as the characteristic impedance of the line

$$Z_L = Z_0$$

And the formula is telling us that gamma is Z L which is equal to Z naught minus Z naught by Z naught plus Z naught which is equal to 0 right.

$$\Gamma = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

And why does this make intuitive sense? This is saying that if you terminate a transmission line from ideal transmission line with its characteristic impedance, the reflected wave is 0 amplitude to the reflected wave is 0.

And why does that make sense? Well, you can think of this as follows, I assume that you had an infinitely long line with the same characteristic impedance. So, this is our line which Z naught and this goes to infinity. So, this is also Z naught right. As the wave is concerned this looks like a continuous transmission line which goes to infinity. So, nothing can get reflected back from infinity because it takes an infinitely long time to come back.

So, as far as the transmission line is the backward going wave is concerned that is got to be 0 which of course, is also borne out by the by the reform right. So, intuitively of course, you know you can think of it as the transmission line which is continuing on up to 2 infinity.

Now, if you have a transmission line with the time delay t d and the characteristic impedance Z naught and it is terminated by an arbitrary impedance Z L. The question is what happens, what is the looking in impedance here? Alright. And to do that you know this we just use the techniques that we have developed just now. So, let us assume that this is V plus, this as we just discussed is gamma times V plus where gamma is Z L minus Z naught by Z L plus Z naught, ok write that down here ok.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Now, if the wave at the right end of the transmission line was V plus, then it must follow that the forward going wave at the left end of the line must be, must have been there a time td earlier. So, this must be V plus e to the s plus s times t d because after delaying it by t d it is V plus at the right end right. In a similar fashion the left going wave at the left end is basically or the backward going way at the left end its nothing, but gamma V plus e to then minus s t d because the wave going towards the left has gotten delayed by a time d.

So, what comment can we make about the voltage at the total voltage at the left end of the line? The voltage is nothing, but the sum of the forward going wave and the backward going wave. So, that is nothing, but V plus e to the s t d plus gamma V plus e to the minus s t d, the current is nothing, but V plus e to the s t d by Z naught minus gamma V plus e to the minus s t d by Z naught.

$$v = V^+ e^{st_d} + \Gamma V^+ e^{-st_d}$$

$$i = \frac{V^+ e^{st_d}}{Z_0} - \frac{\Gamma V^+ e^{-st_d}}{Z_0}$$

So, the impedance Z n of s is simply the ratio of the voltage to the current at the left end of the line and that is nothing, but V plus into e to the power s t d plus gamma e to the minus s t d divided by V plus e to the s t d minus gamma e to the minus s t d.

$$Z_{in}(s) = \frac{v}{i} = \frac{V^{+}(e^{st_d} + \Gamma e^{-st_d})}{V^{+}(e^{st_d} - \Gamma e^{-st_d})} Z_0$$

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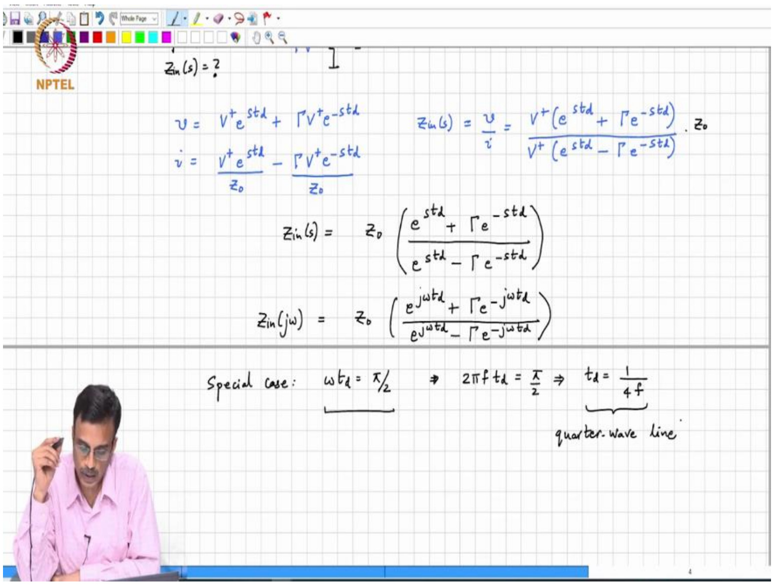

So, the impedance is simply given by sorry e this times Z, impedance is simply given by there in a s is Z naught times e to the s t d plus gamma e to the minus s t d by e to the s t d minus gamma minus s t d ok.

$$Z_{in}(s) = Z_0 \left(\frac{e^{st_d} + \Gamma e^{-st_d}}{e^{st_d} - \Gamma e^{-st_d}} \right)$$

And all that we have the, all that we have used to get this relationship is simply observe two things. One at the boundary condition Ohm's law must be satisfied along with the telegraphers equations and at the along the line the forward wave propagates towards the right, the backward wave propagates towards the left and they have the same delay.

At the left end of the line the total voltage is the sum of the forward and the backward going waves and similarly the current is the sum of the current due to the forward going wave which is in the right and the current that is due to the backward going wave which is in the opposite direction. So, the current has to be the two currents have to be subtracted from each other.

So, and the impedance is simply the ratio of the voltage to be cut and with that we get this expression. Now, let us take a look at this expression you know not only to get familiar with it, but also a useful practical case. A special case occurs remember that when you are doing this the sinusoidal steady state then this now all these become s equal to j omega.

So, this is nothing, but Z naught e to the j omega td plus gamma e to the minus j omega t d divided by e to the j omega t d minus gamma e to the minus j omega t d.

$$Z_{in}(j\omega) = Z_0 \left(\frac{e^{j\omega t_d} + \Gamma e^{-j\omega t_d}}{e^{j\omega t_d} - \Gamma e^{-j\omega t_d}} \right)$$

A special case occurs I when omega t d is pi by 2, alright.

$$\omega t_d = \frac{\pi}{2}$$

So, what does this, what does it actually mean? Remember right omega is 2 pi f times t d is pi over 2 which basically means that t d is one-fourth f ok

$$\Rightarrow 2\pi f t_d = \frac{\pi}{2} \Rightarrow t_d = \frac{1}{4f}$$

And let us stare at this and try and see if we can interpret this physically 1 over f is nothing but the time period of the of the sinusoidal right the input sinusoidal is at a frequency f, 1 over f therefore, is a time period and in one time period of the wave a wave will travel one wavelength. So, this t d if it is chosen such that omega t d is 90 degrees pi by 2 radiance intuitively or physically that corresponds to the delay of the transmission line being equal to quarter wave line. So, this is what is called a quarter wave line, ok.

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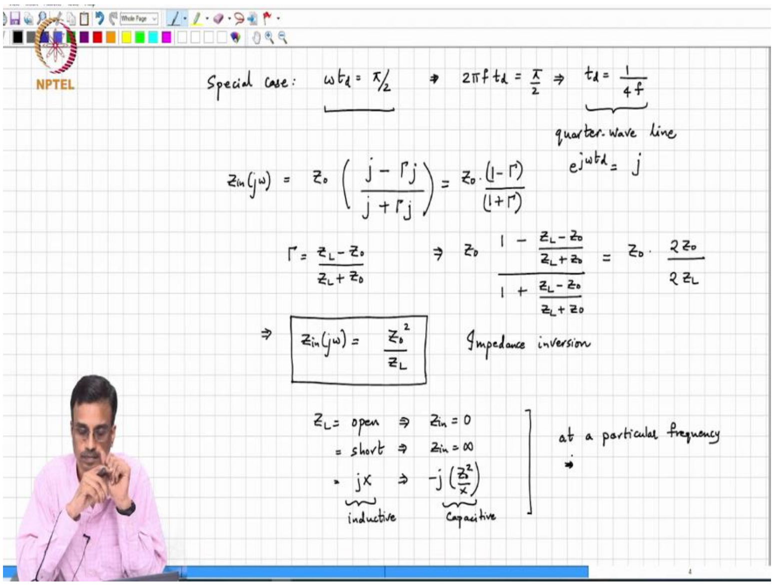

Now, so Z n or j omega if this is the case then e to the j by e to the j omega t d is simply given by cos 5 by 2 plus j sin 5 by 2 which is simply nothing but j, alright. So, Z n of j omega is nothing but j. So, I mean either j omega t d is nothing, but j. So, Z n is nothing but Z naught times j plus gamma times minus j. So, that becomes and this becomes j plus gamma which is nothing, but Z naught times 1 minus gamma by 1 plus gamma and recall that gamma is nothing but Z L minus Z naught by Z L plus Z naught.

So, which basically means that looking in impedance is nothing but Z naught times that Z L plus Z naught 1 minus Z L minus Z naught by Z L plus Z naught divided by 1 plus Z L minus Z naught by Z L plus Z L naught ok. And this is basically Z naught times 2 Z naught divided by 2 Z L therefore, Z in of j omega is Z naught square by Z naught L, alright.

$$e^{j\omega t_d} = j$$

$$Z_{in}(j\omega) = Z_0\left(\frac{j - \Gamma j}{j + \Gamma j}\right) = Z_0\frac{(1-\Gamma)}{(1+\Gamma)}$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \Rightarrow Z_0\frac{1 - \dfrac{Z_L - Z_0}{Z_L + Z_0}}{1 + \dfrac{Z_L - Z_0}{Z_L + Z_0}} = Z_0\frac{2Z_0}{2Z_L}$$

$$Z_{in}(j\omega) = \frac{Z_0^2}{Z_L}$$

So, as you can see the impedance looking on the left is an inverted version of the terminating impedance. So, and, so if you put a capacitor it will look like an inductor, an open circuit. So, if Z L is open, Z in will look like a short, if that Z L is a short Z in will look like an open circuit, if Z L is inductive, then it will look like, so, this is inductive it will look like minus j times Z naught square by X, $\left(-j\ \left(\frac{Z_0^2}{x}\right)\right)$ and therefore, this those capacitive.

With a key point to note is that all this only happens at that frequency ok which makes the length of the line equal to quarter wave.