

Circuit Analysis for Analog Designers
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Lecture - 43
Solving the wave equation in an ideal transmission line

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And so, differentiate 1 with respect to say for example, let us say we want to eliminate i , you differentiate 1 with respect to with respect to space. And therefore, you have double square v of x of t by double x square equals minus L double square i of t double x .

$$1) \rightarrow \frac{\partial^2 v(x, t)}{\partial x^2} = -L \frac{\partial^2 i(x, t)}{\partial t \partial x}$$

And similarly, you differentiate the 2nd one with respect to with respect to time, right. And so, therefore, double square i of x comma t double x double t equals minus C double square v of x comma t divided by double t square, alright.

$$2) \rightarrow \frac{\partial^2 i(x, t)}{\partial x \partial t} = -C \frac{\partial^2 v(x, t)}{\partial t^2}$$

And so, what does this mean? We get double square v by double x square must be L times C times square i double square t square, straight forward substitution.

$$\frac{\partial^2 v(x, t)}{\partial x^2} = LC \frac{\partial^2 v(x, t)}{\partial t^2}$$

I mean let us say quickly make sure that the stuff is dimensionally consistent, right. Sanity check. What are the units on the left-hand side? Whole square per meter square. On the right-hand side, what are the units of this guy here? Remember L in this notation is inductance per unit length. I mean you would bear that in mind, I mean it is a little bit of notational abuse, right.

In a book you would have some you know nicely drawn calligraphic symbol for L, right. I cannot do that unfortunately. So, I will just call it L. So, LC is basically has got dimensions of time square per meter square, and the time square in the numerator and the time square in the denominator cancel and therefore, we are all a happy family. Does makes sense folks? Very good.

So, alright. So, this by the way as you all know is called the, this is called the wave equation, alright.

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The slide content is as follows:

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$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}$$

Wave equation

d'Alembert: $f(x-ct)$ is a solution if

$$\frac{\partial^2 v}{\partial x^2} = f''(x-ct) \quad LC \frac{\partial^2 v}{\partial t^2} = c^2 f''(x-ct) LC$$

$$c^2 = \frac{1}{LC} \Rightarrow c = \pm \frac{1}{\sqrt{LC}} \quad \text{Dimensions of velocity}$$

$$\therefore v(x,t) = f_1(x-ct) + f_2(x+ct)$$

$$c = \frac{1}{\sqrt{LC}} \quad \text{Velocity of Light}$$

And the question is you know how on do, we solve this thing now, right. And fortunately, it turns out that there is a smart guy who figured this out right, by this by his name d'Alembert solution to the wave equation. And we just discovered that oh any function of the form f of x minus this is supposed to be c, alright. This c is not to be confused with the C of the LC, right.

You know as you can see, we are running even we are running out of letters here. And so, this is f of x minus ct , $(f_1(x - ct))$, he claims that any function of this form, right, oh f_1 of x minus ct satisfies the wave equation. And well you know who are we to argue with the great man, we just basically substitute, this in the equation and check if it works or not, right.

Oh, well, what is the partial derivative with respect to x , the second partial second derivative with respect to x ? It simply, its f_1 double prime of x minus ct , $(f_1''(x - ct))$, where this prime refers to the derivative with respect to the argument, correct by just using chain rule, alright.

So, $\frac{\partial^2 v}{\partial x^2}$ by $\frac{\partial^2 v}{\partial t^2}$ is going to be, oh is going to be c^2 times f_1 double prime of x minus ct , alright.

$$\frac{\partial^2 v}{\partial x^2} = f_1''(x - ct)$$

$$\frac{\partial^2 v}{\partial t^2} = c^2 f_1''(x - ct)$$

And so, if you want to make these two equals, what do we do? We want to make c^2 times the derivative with respect to, the second derivative with respect to time which is this guy here, to be equal to the second derivative with respect to with respect to with respect to space, alright.

$$c^2 = \frac{1}{LC}$$

So, stare at this and tell me what do we see? Well, that is true if the c^2 is equal to 1 over L times C . And c is basically or rather this calligraphic c is the has got dimensions of in c equals plus minus 1 over square root LC , alright.

$$c = \pm \frac{1}{\sqrt{LC}}$$

And this basically has dimensions of dimensions of velocity, right, that makes sense. I mean xc times t must have dimensions of distance you know; it is consistent with the fact that x minus ct must have dimensions of distance. Is that clear.

So, therefore, the solution to the wave equation is you can write this as v of x comma t , can be written as some function f_1 times x minus c t , right, where c is plus minus 1 over square root LC .

$$\therefore v(x, t) = f_1(x - ct)$$

$$c = \pm \frac{1}{\sqrt{LC}}$$

So, if we define, if we deliberately say c is 1 by square root LC , then you can think of the solution as f_1 of x minus c t plus some other function f_2 of x plus c t .

$$c = \frac{1}{\sqrt{LC}}$$

$$\text{then } v(x, t) = f_1(x - ct) + f_2(x + ct)$$

And why we justified in adding another function? It is a linear differential equation if x by f_1 of x minus c t is a solution and f_2 of x plus c t is a solution, then any linear combination of the two solutions will also be a solution, right. If you are not convinced you can go and do the, I mean you can plop this into the wave equation and see that it satisfies it, right. So, c is basically 1 over square root LC which is basically the velocity of light, alright. Now, well that is the voltage.

What comment can we make about the current? We go back to our, once we know the voltage, we go back to this thing here.

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The whiteboard contains the following derivations:

$$f_1'(x-ct) = -L \frac{\partial i(x,t)}{\partial t} \Rightarrow i(x,t) = \frac{1}{Lc} f_1(x-ct)$$

$$= \frac{f_1(x-ct)}{\sqrt{LC}}$$

Annotations on the right side of the whiteboard:

$$\frac{1}{Lc} = \frac{1}{\sqrt{LC}}$$

Characteristic Impedance

$$\frac{1}{\sqrt{LC}} \rightarrow \text{Impedance} = Z_0$$

$$v(x,t) = f_1(x-ct) + f_2(x+ct)$$

$$i(x,t) = \frac{f_1(x-ct)}{Z_0} - \frac{f_2(x+ct)}{Z_0}$$

So, let us say the voltage f_1 of x minus ct . What is the derivative with respect to with respect to x ? This is simply f_1 prime of x minus ct , correct. And this must be minus L times the partial derivative of i with respect to time.

$$\frac{\partial v(x,t)}{\partial x} = -L \frac{\partial i(x,t)}{\partial t} \rightarrow 1$$

$$f_1'(x - ct) = -L \frac{\partial i(x,t)}{\partial t}$$

So, what comment can we make about i of x comma t ? 1 over L times c times f_1 of x minus ct .

$$\Rightarrow i(x,t) = \frac{1}{Lc} f_1(x - ct)$$

Does it make sense? Now, this 1 by L , again I would like to warn you that this c is not the capacitance, it is the velocity of light. And before we get confused any further let us replace c with the formula that we already derived.

What is c in terms of L and C ? It is 1 by square root of which is therefore, 1 by square root of L by C , ok.

$$\frac{1}{L \frac{1}{\sqrt{LC}}} = \frac{1}{\sqrt{L/C}}$$

So, this is f_1 of x minus divided by square root of L by C .

$$i(x, t) = \frac{f_1(x - Ct)}{\sqrt{L/C}}$$

What are the dimensions of square root L by C ? Impedance and since it basically depends on L and C , which are characteristics of the transmission line, right. This is called the characteristic impedance and often abbreviated by, is often called z naught, (Z_0).

So, therefore, if we have the voltage which is f_1 of x minus $c t$ plus f_2 of x plus $c t$, then the current i of x comma t is basically f_1 of x minus $c t$ by z naught.

$$v(x, t) = f_1(x - Ct) + f_2(x + Ct)$$

$$i(x, t) = \frac{f_1(x - Ct)}{Z_0} -$$

What comment can we make about the correspond current corresponding to f_2 of x plus $c t$? Minus f_2 of x plus $c t$ over z , alright.

$$i(x, t) = \frac{f_1(x - Ct)}{Z_0} - \frac{f_2(x + Ct)}{Z_0}$$

Fortunately, it turns out that there is a you know nice way of interpreting these results.

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The slide displays the following content:

- NPTEL logo and a toolbar at the top.
- Equations for voltage and current:

$$v(x,t) = f_1(x-ct) + f_2(x+ct)$$

$$i(x,t) = \frac{f_1(x-ct)}{Z_0} - \frac{f_2(x+ct)}{Z_0}$$
 The first term in both equations is labeled "Forward wave" and the second is labeled "Backward wave".
- A diagram of a transmission line with a dashed vertical line representing a point x .
- Three graphs showing the pulse $f_1(x)$ at different times:
 - Graph 1: $f_1(x)$ at $t=0$.
 - Graph 2: $f_1(x-ct)$ at $t=t_1$, showing the pulse shifted to the right by a distance ct_1 .
 - Graph 3: $f_1(x-ct)$ at $t=t_2 > t_1$, showing the pulse shifted further to the right.

What this math is telling us is that, now, basically this is the symbol for a transmission line. At any point x along the transmission line, the voltage can be represented as the sum of two quantities f_1 of x minus c t and f_2 of x plus c t . So, let us try and take a look at what this f_1 of x minus c t actually means.

So, let us say f_1 as a function of its argument is some function like this, alright. So, in other words, let us say if I took a snapshot, if I took a photograph of the voltage on the line if that was possible with at t equal to 0, what would you get? You would get f_1 of x , right. And what have I done to get f_1 of x ? At t equal to 0, right I had an imaginary camera which would basically capture the voltage at all positions along the transmission line and then you know give me a picture, right.

Now, at t equal to t_1 , what do you think will how will the picture look like? I do the same thing again. Well, all that it says is this is going to be f_1 of x minus c times t_1 . So, it is going to be, this is c times t , alright. So, at t_2 greater than t_1 , how will this look like? The shape will look the same except that how much would it have traveled.

This will go, this will be this is f_1 of x minus c t_2 , right. So, the interpretation of f_1 of x minus c t is simply a wave travelling in the forward direction, right. And likewise, this is a voltage wave travelling in the backward direction, right. So, this is the forward wave, this is the backward wave, correct.

And corresponding to the forward voltage wave we also have, what does that represent?

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The slide contains the following content:

- Equations:**

$$v(x,t) = \underbrace{f_1(x-ct)}_{\text{Forward wave}} + \underbrace{f_2(x+ct)}_{\text{Backward wave}}$$

$$i(x,t) = \underbrace{f_1(x-ct)}_{\text{Forward wave}} - \underbrace{f_2(x+ct)}_{\text{Backward wave}}$$
- Diagram:** A horizontal transmission line with a dashed vertical line in the center. Arrows above the line indicate the direction of wave propagation: a red arrow pointing right for the forward wave and a black arrow pointing left for the backward wave.
- Graphs:** Three graphs showing the voltage waveform $f_1(x)$ at different times:
 - Graph 1: $f_1(x)$ at $t=0$. The waveform is a pulse moving to the right.
 - Graph 2: $f_1(x-ct)$ at $t=t_1$. The pulse has moved to the right by a distance ct_1 .
 - Graph 3: $f_1(x-ct)$ at $t=t_2 > t_1$. The pulse has moved further to the right.

We have a forward current wave which is simply the voltage wave divided by an impedance, right.

And the forward wave is going this way, the reverse wave or the backward wave is going that way. So, it stands to reason that the current that corresponds to the backward travelling wave, is going in which direction? It is going in the?

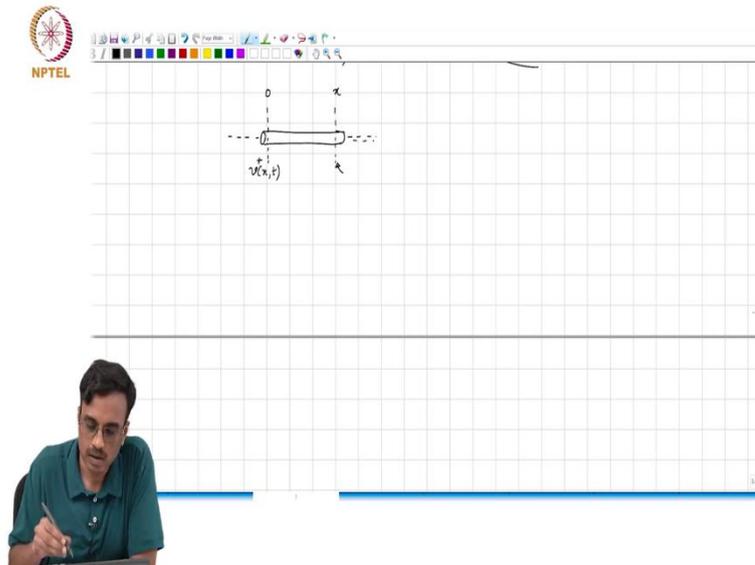
Yes, it is going in the opposite direction, alright, ok. So, basically this is the backward current wave.

So, at any point on the transmission line you can write the voltage waveform as a sum of the voltage and I mean the forward and the backward travelling voltage waves. And the current at that point is basically the difference between the forward and backward traveling waves, ok. And the amplitude of the current wave is simply 1 over Z_0 times, is simply the amplitude of the forward going voltage wave divided by the characteristic impedance and likewise for the backward, right, ok.

This is all you need to understand, distributed circuits at least you know for a whole you know for a good first level understanding of distributed circuits, ok. The rest of it is simply a derivative of I mean just simply applying these laws to. This is another circuit element

that is all, right. And then now you just do circuit analysis you know with this new element in place. So, that is kind of build on what we have learnt so far.

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So, let us say we have a transmission line, right which extends to infinity. So, at some point x , let us say at point z at x equal to 0 we have a forward going wave and a backward going wave, ok. v plus with this superscript plus is often used to denote the forward going wave and v minus is used to define is to denote the backward going wave, right.

And so, what comment can we make; remember this line goes to infinity, we do not know where it goes. So, at a distance say x to the right, what comment can we make about the forward going wave at position x or rather I mean if you knew that the voltage here was v of x comma t , what comment can we make about the voltage there?

About let us call it sorry v plus of x comma t , right, what comment can we make about the voltage waveform at place x , the forward going voltage wave at x ?

Student: (Refer Time: 20:46).

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Sorry, this is $v(0, t)$, yeah. This is $v(x, t)$, right. How is $v(x, t)$ related to $v(0, t)$?

Student: Because the x is the (Refer Time: 21:08) of 0 minus (Refer Time: 21:10) equal 0 .

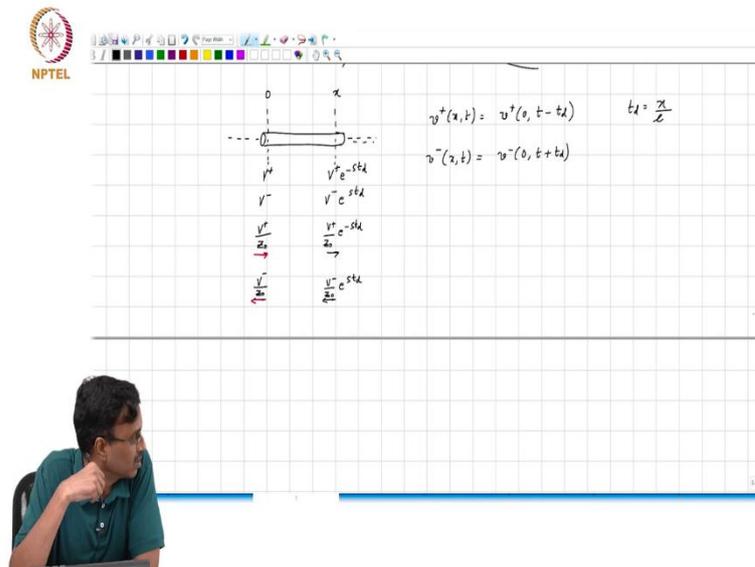
Exactly, right. So, remember this forward going wave is travelling in the positive x direction with the speed c , correct. So, whatever happens at position $x = 0$ will happen at x at a time x/c later, right because it's going this way, right. So, basically $v(x, t)$ is simply $v(0, t - t_d)$, where t_d is nothing, but x over the velocity of light.

$$v^+(x, t) = v^+(0, t - t_d) \quad t_d = \frac{x}{c}$$

Likewise, if this is $v(0, t)$ and this is $v(x, t)$, what comment can we make about the relationship between these two guys? $v(x, t)$ is if, all that this is saying is that the backward going wave at 0 is delayed with respect to the backward wave at x . Does it make sense people? Right.

$$v^-(x, t) = v^-(0, t + t_d)$$

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In the Laplace domain, V plus if this if we call this V plus and we call the voltage here V minus what would be the Laplace transform of the forward going wave at x ? This would be V plus delayed by $t d$. So, it would be? e to the power minus s times $t d$, $(V^+ e^{-s t d})$. And likewise, this would be v minus e to the plus s times $t d$, $(V^- e^{s t d})$. Its clear people. Very nice.

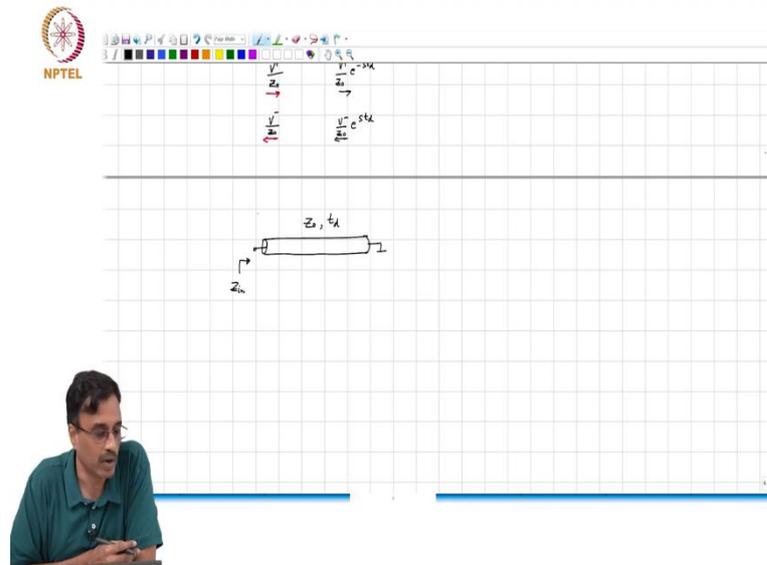
Now, what comment can we make about I plus, the forward current wave at this point at x equal to 0 ? We know the forward voltage wave. So, what is the forward current wave?

Oh, well, the current here is V plus, the forward current is V plus over z naught, $(\frac{V^+}{Z_0})$. The backward current is; the forward current I mean is this, the backward current going in that direction is; since the direction is already there is simply V minus by z naught, $(\frac{V^-}{Z_0})$, alright.

And what can we say about the forward current here? It is simply V plus over z naught e to the minus s times $t d$, $(\frac{V^+}{Z_0} e^{-s t d})$ and likewise this is V minus over z naught e to the s times $t d$, $(\frac{V^-}{Z_0} e^{s t d})$, correct. Does it make sense people? I mean have you guys seen all this, you must have seen all this before, right. Yes.

Now, all of you might remember these bounce diagrams where you cannot draw pictures of waves bouncing around, so on, right. It is all pretty straightforward stuffs, is meant as a refresher, ok alright.

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So, now let us see you know what happens to a short circuit. Well, let us say I mean remember we are trying to measure our y parameters and we connect you know this is you know we can only short the ends of some conductor of some finite length, right. So, let us call the length. We though, we know that it is going to be physically long, we are going to call the delay. So, we are going to denote the transmission line by 2 parameters, one is the characteristic impedance, right.

And the next thing we need to know is how long it is, right; how long basically does not really matter as much it is more, how long the wave takes to go from point a to point b, correct. So, it is basically the length of the line divided by the velocity along the line is basically the delay of the line. So, we will simply characterize the line by two parameters, one is the impedance, the other one is; the one is the characteristic impedance, other is the delay. Does it make sense? Ok, alright.

So, now you know let us try and see you know we have a transmission line with this characteristic impedance and delay. I shot the other end of the line, right, what would be the impedance looking in here, correct. So, what do we do? I mean if you want to find

impedance, what do we know I mean what do we do normally? You apply voltage measure the current or vice versa, right.

We need to figure out you know how the voltage here is related to the voltage at this end, on the left end is related to the voltage at the current, I mean sorry how the voltage is at the left end is related to the current at the left end. So, what do we do well? We know for sure that.