Circuit Analysis for Analog Designers
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Lecture - 37
Bode's Noise Theorem- Frequency domain

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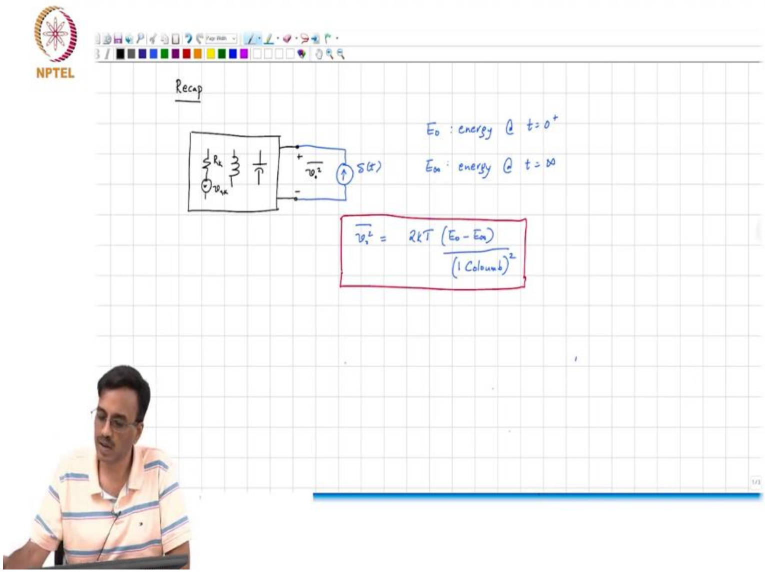

A quick recap of what we did in the last class. So, what we said was if you have an RLC network and you are measuring, you are interested in finding the mean square noise at the output of the network and this is the calculation that often comes about in practice.

And if you want to do this, the straightforward approach would be to as we discussed several times every resistor R sub k is associated with the noise voltage source with a voltage spectral density 4kT R sub k. We find the transfer functions from every noise source to the output from which we can find the spectral density and you can then integrate the spectral density all the way from 0 to infinity to obtain v 0s the mean square noise of the output.

Unfortunately, right that is a very you know it is obviously, a very laborious way of doing things and it does not make sense to do simply because we are working so hard to basically get all these transfer functions and all these spectral densities. And finally, we are throwing away I would say 99 percent of that information because you simply integrating that whole thing from 0 to infinite hm.

So, yesterday we saw how one can use exploit reciprocity and the idea was the following, what you do is to inject an impulse current into the network and let E 0 be the network the energy stored in the network at t equal to 0 plus right and E infinity be the energy at t equal to infinity alright.

And we said that the mean square noise is nothing but 2kT times E 0 minus e infinity by 1 coulomb square I hope I got this spelling of coulomb right alright,

$$\overline{v_o^2} = 2kT \, \frac{(E_o - E_\infty)}{(1\, Coloumb)^2}$$

And we saw yesterday cases where E infinity was 0, we also saw cases where E infinity was what do you call non-zero and this formula basically covers all the cases does make sense? Ok.

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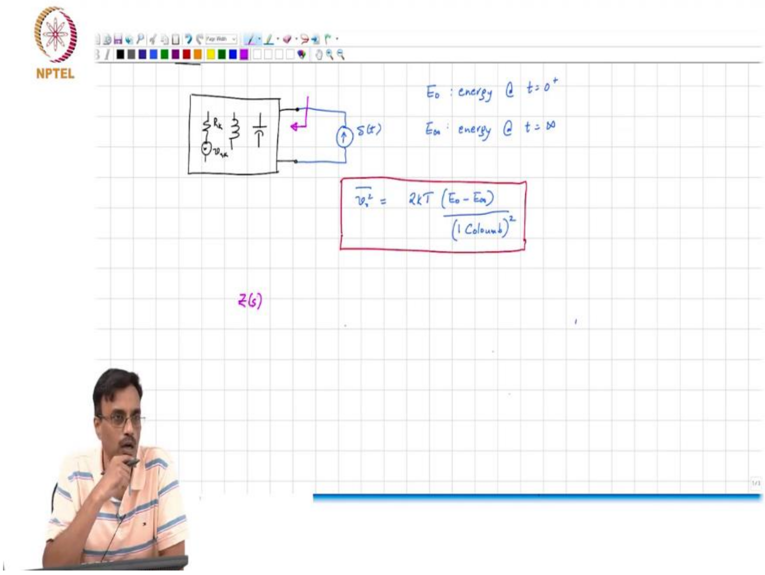

Now, let us kind of see this from another perspective, the same result from another perspective. So, when you inject a current into the network right, what common can you make about the Laplace transform of the voltage developed across these two terminals?

Pardon. It is simply the impedance correct. So, if you inject a current impulse, the voltage developed across the two terminals of the network has a Laplace transform which is simply given by the driving point impedance of the network correct.

So, and what comment can we make about the initial energy stored in the network? How can we relate it to the initial voltage developed across the network or rather ok let me put it in another way, what comment can we make about the voltage developed across the network at t equal to 0 plus? Let me call that v o pardon.

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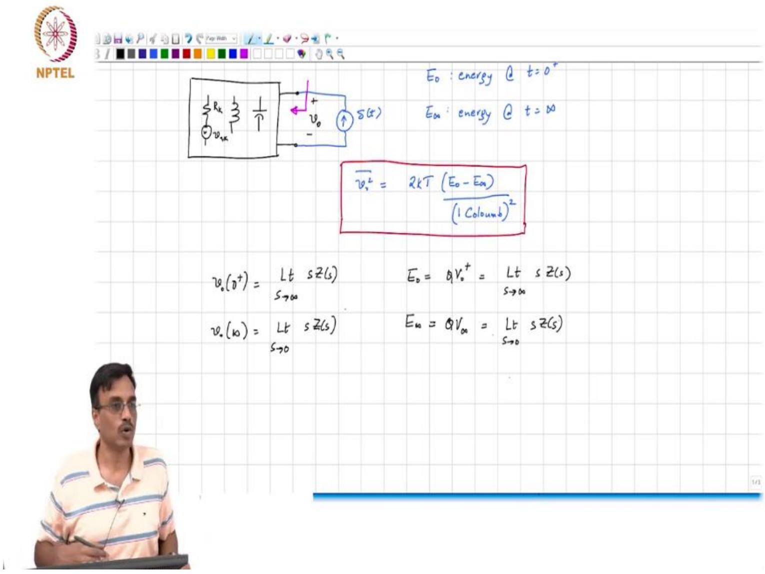

No. Pardon. Yeah, use and tell me what it is. v 0 plus is simply nothing but? I mean what are all those theorems for? The voltage waveform is z of s correct has a Laplace transform given by z of s. So, what is the voltage at 0 plus? A limit s tends to infinity s times z of s alright

$$v_o(0^+) = \lim_{s\to\infty} s\mathcal{Z}(s)$$

And so, what comment can we make about the initial energy delivered to the network? You pumped in a current impulse correct, what comment can we make about the initial energy that is been delivered into the network into the network?

Pardon. Yeah, it is v 0 plus times the charge so, the so, E 0 therefore, is nothing but? The charge is unity so, what is this? Or E 0 by 1 coulomb is nothing but is v 0 plus which is limit s tends to infinity of s times z of s correct alright

$$E_o = \frac{1}{2}QV_o^+ = \frac{1}{2}\lim_{s\to\infty} s\mathcal{Z}(s)$$

And what is E infinity by the same token? Well, E infinity is simply Q times V infinity and what is V infinity? Limit s tends to 0 s times z of s alright ok.

$$E_\infty = \frac{1}{2} Q V_\infty = \frac{1}{2} \lim_{s\to 0} s\mathcal{Z}(s)$$

$$v_o(\infty) = \lim_{s\to 0} s\mathcal{Z}(s)$$

So, what is E 0 minus E infinity? This is nothing but z of s alright.

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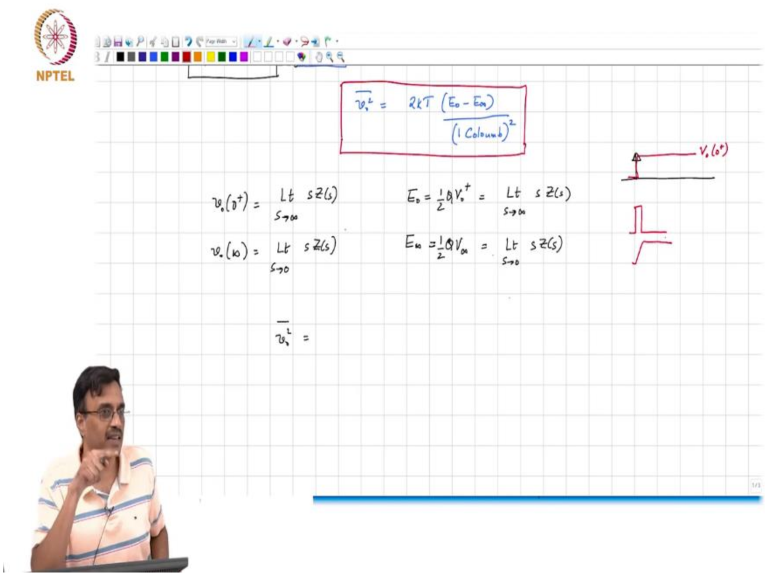

So, what is this equivalent to? So, mean square noise is nothing but, I am sorry actually we missed a factor of 2 right, why?

Remember that we have injected an impulse right and how much of and what is the energy delivered by the empire, what is the energy that is going into the network? The voltage across the network has gone from 0 to? 0 to v 0 plus right v 0 or 0 plus so, what comment can we make about the energy in the network? What is the energy inside the network?

It is simply the integral of voltage times; the voltage waveform times the current waveform that has gone into the network and you can see that I mean there is of course a discontinuity, but if you interpret the impulses being say for instance a thin pulse and the voltage waveform doing this right is very clear that the energy that is going in that is there inside the network is half Q times V of 0 plus.

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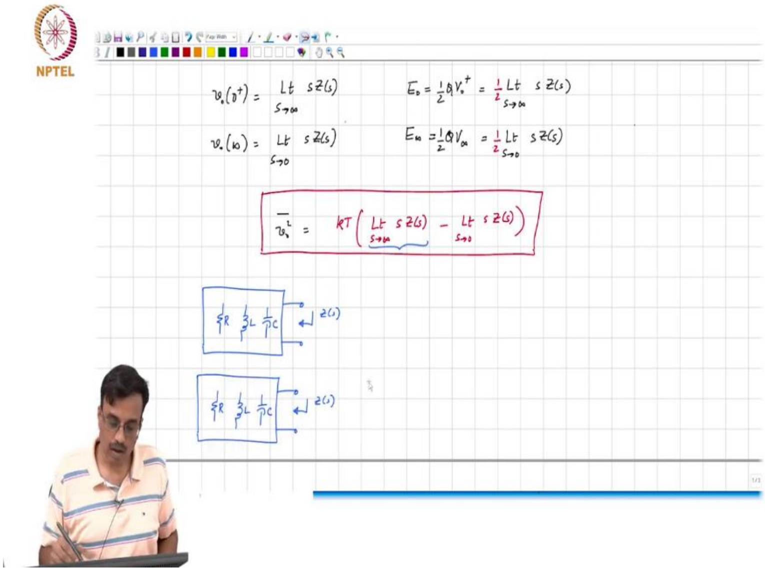

So, this is one-half times this and likewise this is one-half times s z of s s tends to infinity ok alright. So, the mean square noise is therefore, what is the mean square noise people?

Yes. It is simply k T times limit s tending to infinity of s times z of s minus limit s tending to 0 s times z of s ok. Now, it turns out that you know there is a simple way of interpreting this s times z of s as s tends to infinity.

$$\overline{v_O^2} = kT\left(\lim_{s\to\infty} s\mathcal{Z}(s) - \lim_{s\to 0} s\mathcal{Z}(s)\right)$$

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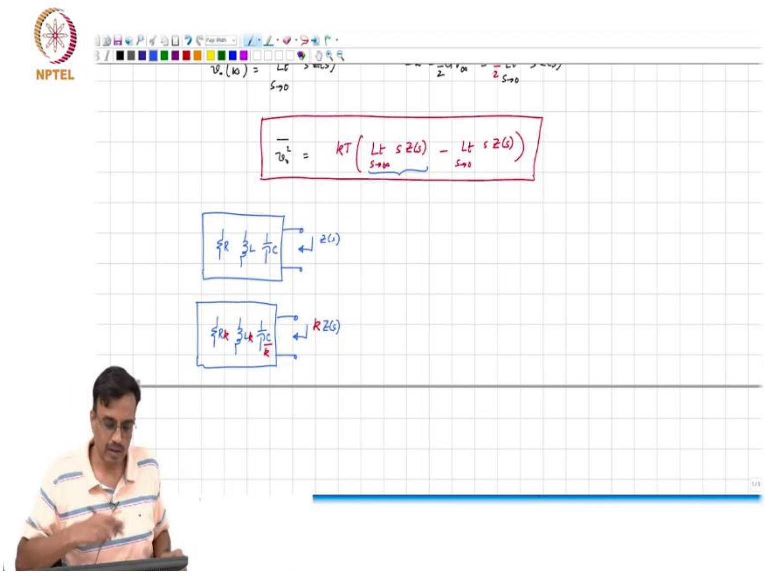

Remember, let us say you have a network ok and let us say you had the input impedance was some z of s. Now, if let us say this is R, this is L and this is C. Now, if I multiply this by k, by k and by k, sorry and what will the impedance be?

It will be k times z of s, does make sense correct. All impedances have been scaled by the same factor and therefore, the total impedance will also scale by the same factor k. Now, there is no necessity for that k to be real, it can be; it can be complex.

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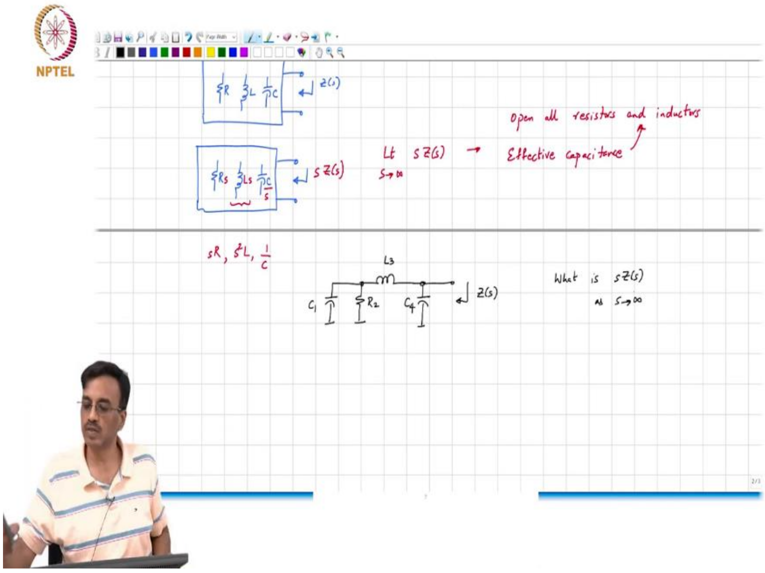

So, for instance, if I make k equal to s right so, in other words, I divide I multiply every impedance by the complex number s, what comment can we make about the looking in impedance?

Students: s times z of s.

It will simply be s times z of s, does make sense people, right? Ok. Now, remember that this s is the multiplying factor for every inductor so, what is in reality the impedance of this inductor now at a certain frequency s?

Students: (Refer Time: 12:27).

Remember that the impedance of the inductor is was s L that is now been multiplied by an extra factor s so, every resistor has now become s times R, the impedance of the inductor has become s square times L and the impedance of the capacitor has become?

What is the impedance of the capacitor earlier? It is 1 over s C, it is multiplied by this complex number s so, this is become? 1 over C correct. Now, if we let s tend to infinity right, how can you interpret this, as s tends to infinity s times r becomes? s times R becomes infinite similarly. Oh, s square times L also becomes infinite and therefore, how do you interpret this limit of limit as s tends to infinity of s times z of s you open all resistors and inductors right. Once you do that, you will get a network with only with capacitors correct ok so, yes, therefore, how can you interpret s times z of s; how can you interpret s times z of s?

Yeah, so, basically it is the effective capacitance of what remains. Effective capacitance is just simply the capacitance, that number correct that s goes away because so, let us say C 1, R 2, L 3 ok so, this is z of s alright and so, the question is what is s times z of s as s tends to infinity and one way of doing this would be to find the actual z of s multiplied by s and let s tend to infinity, the easier thing to do is do it on a branch by branch basis as s tends to infinity and then, look at the resulting network.

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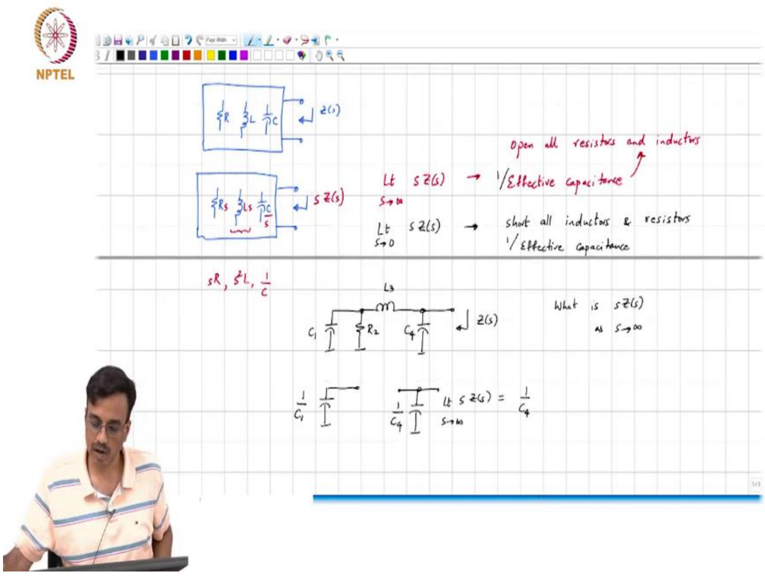

So, what will this become? The capacitance will become 1 over C 1 right. The resistance will become an open circuit, the inductor will become an open circuit and what does this become? A resistance of value 1 over C 4 equivalently right. So, what is so, therefore, what is z of s times z of s as s tends to infinity? Staring at this what do you see? 1 by C 4 and I mean in English it basically means that you open up all resistors and inductors ok, you will get a network which only consists of capacitors, the equivalent capacitance that you see right is s times z of s as s tends to infinity alright.

$$\lim_{s \to \infty} s\mathcal{Z}(s) = \frac{1}{C_4}$$

Pardon. Because the s times z of s does not have dimensions of capacitance, I mean s times z of s has dimensions of capacitance not it is not resistance right. It is 1 by yeah, if the effective capacitance that you look it is 1 by it is 1 by the effective capacitance that you look looks like alright ok. So, by the same token, therefore, how do you interpret limit s tends to 0 of s time z of s what should you do?

Well, that is straightforward as s tends to 0 s R becomes a short circuit, s square L also becomes a short circuit. So, you basically say short all inductors and resistors and you look at the you know the 1 over actually it should be 1 over effective capacitance looking alright.

Now, let us do this for the for our example we have chosen. What should you do for our network? What happens to C 1?

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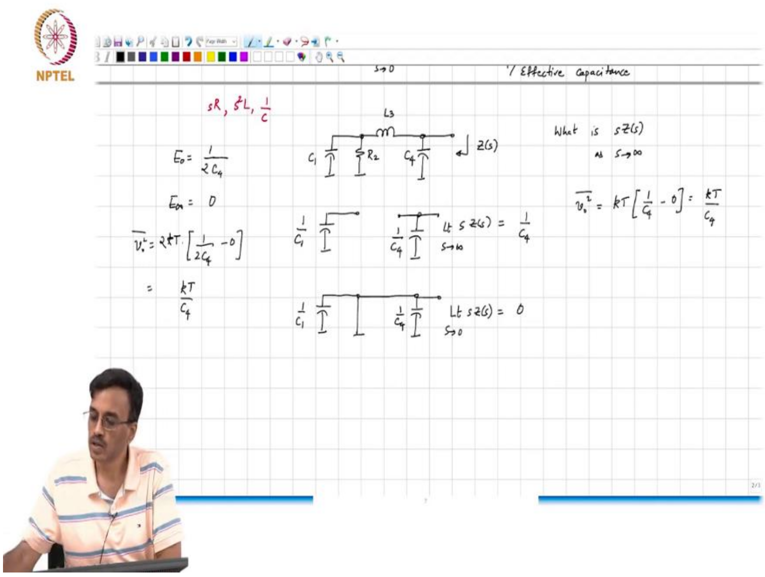

Yeah, it becomes you know a resistor value 1 over C 1 right and then, what happens to R 2? R 2 becomes a short circuit L 3 becomes a short circuit and C 4 again as 1 over C 1 ok. So, what comment can you make therefore, as limit s tends to 0 of s times z of s?

What is the effective capacitance looking in? Effective capacitance looking in? It is infinite right so, basically because a dead shot so, therefore, the 1 over the capacitance looking in is 0 correct.

$$\lim_{s\to 0} s\mathcal{Z}(s) = 0$$

So, what comment can we make about the mean square noise looking so, what is the mean square noise is simply k T times 1 over C 4 minus 0 which is k T over C 4.

$$\overline{v_o^2} = kT\left[\frac{1}{C_4} - 0\right] = \frac{kT}{C_4}$$

Does it make sense and is this consistent with our energy of you know formulation? Well, when you what is E 0? You inject an impulse, where does all that current go?

It is on C 4 because in the beginning all the inductors are open and you know what you call the capacitance of the shorts so, E 0 is nothing, but 1 over 2 C 4,

$$E_0 = \frac{1}{2C_4}$$

E infinity what happens? There is some initial voltage on the capacitor C 4 as T tends to infinity well, the inductors become short circuits and this must be no current flowing through R 2 that is what will happen at T equal to infinity and therefore, at T equal to infinity is very apparent that both C 1, you know L 3 and C 4 are all discharged right or disfluxed ok and therefore, E infinity is what is E infinity? 0

So, mean square noise, therefore, is 1 half a k T 2kT times 1 over 2C 4 minus 0 and this is basically k T over C 4 alright.

$$\overline{v_o^2} = 2kT\ \left[\frac{1}{2C_4} - 0\right]$$

$$= \frac{kT}{C_4}$$

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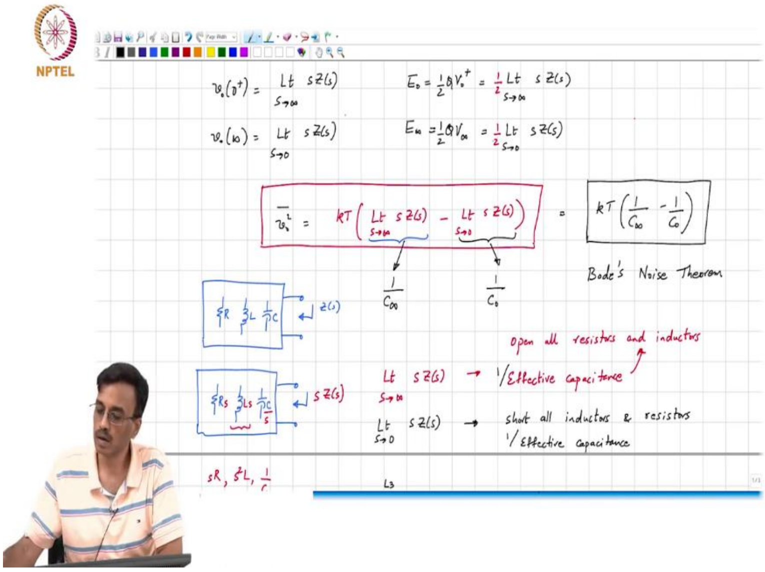

So, ok so, this term basically has you know is often written as 1 over C infinity where that 1 over C infinity is basically denoting, I mean this is not just a symbol that is all 1 over C infinity is simply, what is C infinity? It is the capacitance that you see when you open up all the inductors and resistors and likewise this is nothing but if you call this C, 1 over C infinity, what will you call this? 1 over C 0 right. So, this is written in ok, it is also written in this form in the books, and this is what is called Bode's noise theorem.

$$\overline{v_o^2} = kT\left(\lim_{s\to\infty} s\mathcal{Z}(s) - \lim_{s\to 0} s\mathcal{Z}(s)\right) = kT\left(\frac{1}{C_\infty} - \frac{1}{C_0}\right)$$

Students: (Refer Time: 21:58) we have (Refer Time: 22:00) should be (Refer Time: 22:04) resistance all effective capacitors.

Come again.

What you need to do is the following, you to find C infinity, what you do is you open up all resistors and inductors, you will get a network with only capacitors, you find the effective capacitance looking in right using all your you know high school physics right, if you have series parallel or some combination, you can do whatever you need to do and the that effective capacitance is C infinity so, limit s tends to infinity s times z of s is 1 over that that effective capacitance that you see at infinite frequency right.

And likewise, limit s tends to 0 of s times z of s is the effective capacitance you see at when all the resistors and inductors are shorted, and the total noise is simply nothing but k T times 1 over C infinity minus 1 over C 0 yes.

Students: Sir, if the and a network contains only with L and R (Refer Time: 23:18) the.

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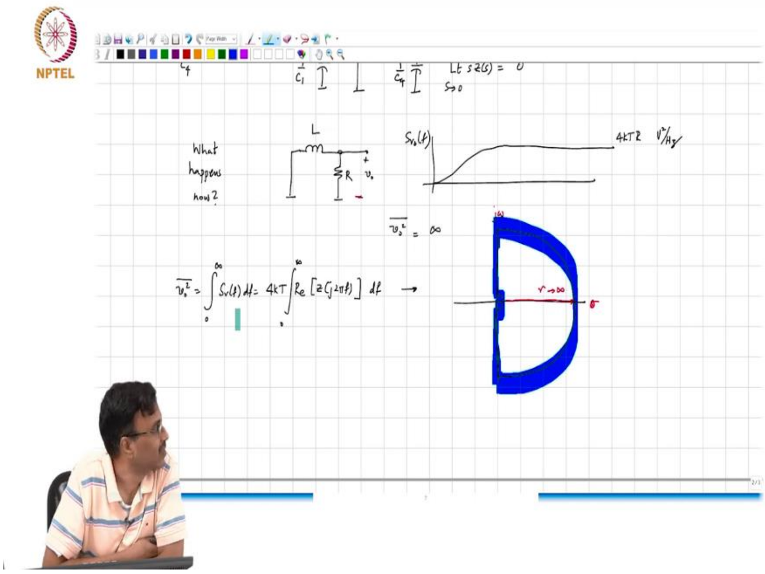

I mean the question I mean I was hoping somebody ask this question. So, a very legitimate one he says well, what happens if yeah, you have a network like this correct, what comment can we make about the spectral density at the output at this node? How is this look like? At low frequency this will be.

What will it be at low frequency? 0 because it is a high pass filter right, at high frequency what will happen?

It will eventually tend to 4kTR volt square per hertz. So, what is the total integrated noise of mean square value of so, what common can we make about is simply the integral of this noise spectral density and what is this telling us?

Well, this is evidently infinity right because that spectral density is constant and you keep integrating infinite frequency and therefore, ok. So, this basically for this network, therefore, for networks of this kind right, you basically theoretically have this integral does not converge correct ok.

And again, this is telling us that well, you know if I simply put an inductor or resistor like that, I will get infinite mean square voltage which basically means how can I you know evidently something again which sounds too good to be true.

So, in reality, you will never have a network like this because well, every node is associated with the parasitic capacitance correct and eventually, the spectrum must fall off ok and so, in other words, this formula is only valid when the when integral converges right which basically means in all practical networks is this is this one ok.

Likewise, you can show it is straight forward to see that if you have networks with R, L and C and you are interested in finding the mean square, if you short circuit the network and find the mean square noise current, you will be able to show I mean you will be able to write a similar relationship right involving L infinity and L 0 for the mean square current noise integral across all frequencies alright.

So, I as I said this is called Bode's noise theorem and is a nifty little trick to know to basically solve or to look at a complicated network and estimate what the or determine exactly what the total integrated noise will be without going through any algebra right or complicated integrals ok.

This is not the original proof of the theorem, the original proof of the theorem you know starts from Nyquist result which is 4k which you know to be s v of f is 4kT times real part of z of s a real part j 2 pi f correct and we need to find the integral of so, what we need is basically integral s v of f df 0 to infinity which is infinity and this actually it turns out that you can think of it as a contour integral in the z plane, this is the s plane this is the sigma and this is the j omega plane.

$$\overline{v_o^2} = \int_0^\infty S_v(f)df = 4kT\int_0^\infty Re[\mathcal{Z}(j2\pi f)]df$$

So, you take a contour like this and you integrate you go along a contour like that in the z plane, what common can we make about the poles and zeros of z? Let say you have a passive impedance like this, what comment can we make about the poles of z?

Poles are in the left half plane, what comment can you make about the zeros of the impedance? Where is the; where is the poles and zeros of a passive impedance located?

They must be in the left half s plane so, if you go around this contour, therefore, does not enclose any poles and zeros so, the contour integral around this entire contour must be.

If you take z of s which is a complex function, and the contour does not enclose any poles or zeros right, the integral around the contour must be? Must be 0 right and the and it turns out that the radius, you choose the radius of this contour as you know r as r tends to infinity correct.

So, its I leave this as an exercise for you to work out, but this basically it can be broken up into 2 into 3 parts, one on that small circle whose radius tends to 0 and that is necessary to take care of potential pole at the origin. And this guy here and it is a I do not know if you are able to see this, but that contour integral is basically it will turn out to be one of those circles the large circle will turn out to be limit of s times z of s as s tends to infinity right.

And the small contour will turn out, the small semicircular contour will turn out to be limit s tends s times z of s as s tends to 0 and along the axis, it will be, it will turn out to be integral minus infinity to infinity real part of z of j 2 pi f ok and you can relate the two and get the answer right.

I mean contour integration is of course, you know a very valid way of finding the proof right, but it is a at least to me it seems a lot more intuitive to basically look at the energy going into the network and ok it also turns out that the same energy-based formulation I mean you know approach also works.

When you as we will see later in this course, when we have periodically operated switches inside the network where the network no longer becomes says is no longer time invariant, but becomes periodically time varying, then you will find that this notion of using energy is more general because energy does not depend on the concept of energy is independent of time invariant or time variant or linear or non-linear right. Whereas, when you say impedance, you already mean that the network is time invariant.