Circuit Analysis for Analog Designers
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Lecture - 20
The Butterworth Approximation (contd)

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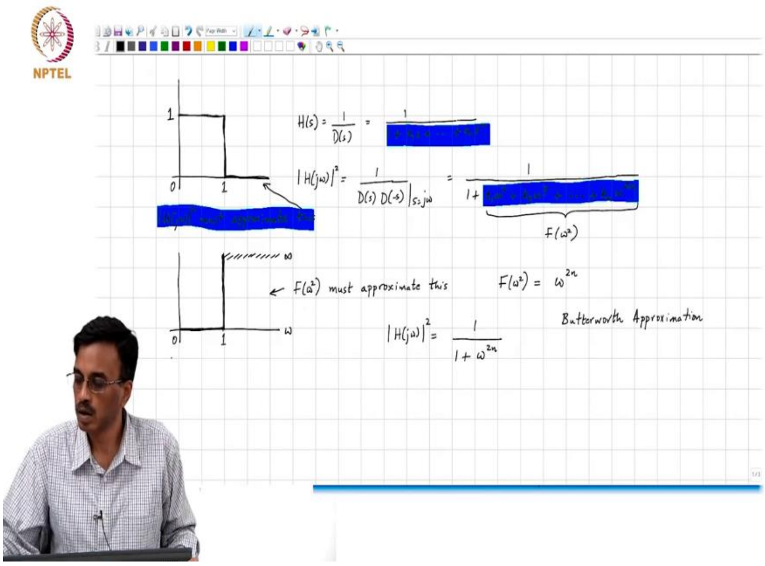


In the last class we started our study of filters and we started off trying to figure out how to achieve or how to obtain what is called an all-pole filter right. Remember that any transfer function is the ratio of polynomials in s and to make our life simple we said OK.

Let us assume that the numerator polynomial is just simply 1 and we would like to find what the denominator polynomial must be so that the magnitude response of this transfer function which is simply 1 over D of s and D of minus s evaluated at s equals to D omega must approximate this.

$$|H(j\omega)|^2 = \frac{1}{D(s)D(-s)|_{s=j\omega}}$$

So, in other words the magnitude square magnitude response must approximate the ideal brick wall and since the magnitude squared of any transfer function is simply an even function of omega it must follow that the denominator polynomial must be at even polynomial.

So, it must be a polynomial in omega square and it stands to reason that there must be you know n coefficients that appear in the denominator, and we denoted this part of that denominator of squared polynomial by F of omega square right. Signifying the fact that the function is, it is a function of omega square.

$$k_1\omega^2 + k_2\omega^4 + \cdots + k_n\omega^{2n} = F(\omega^2)$$

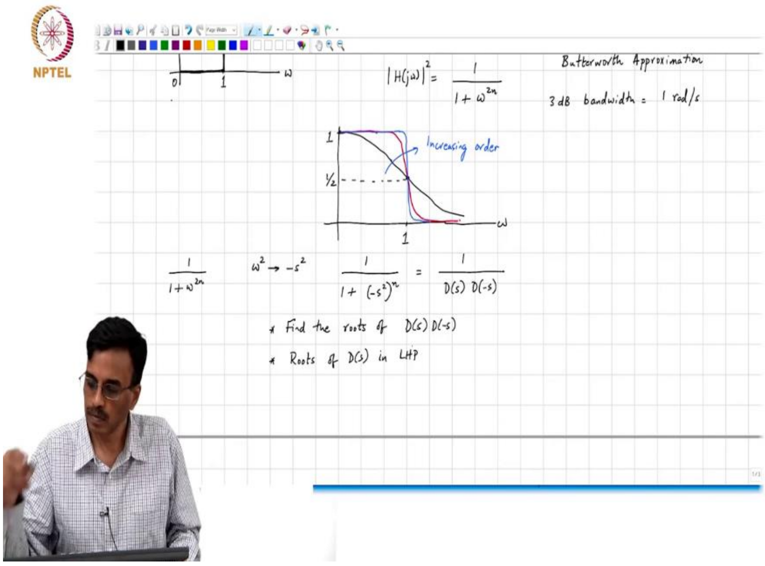
Now, if 1 over 1 plus F of omega square approximates or is attempting to approximate the ideal brick wall, it follows that F of omega square itself must approximate this function here, where it is 0 in the range omega equal to 0 to 1 and becomes infinity thereafter right.

And we said well one way of doing this is to basically use what is called the maximally flat approximation, where you basically say $F(\omega^2) = \omega^{2n}$ right, and as we saw yesterday as we you know at DC of course, F of omega square is 0 and as omega keeps increasing F of omega square is as flat as possible about which point?

It is as flat as possible about DC. I mean there is nothing holy about DC in principle. We could have move the maximally flat point to someplace else in the pass back which corresponds to 0 to 1.

But I mean traditionally this is how the stuff came about they said $F(\omega^2) = \omega^{2n}$ and this will give us magnitude response where mod of H of j omega whole square is 1 over 1 plus omega to the 2 n and this approximation is what is called the maximally flat or the Butterworth approximation and such filters are called Butterworth filters.

$$|H(j\omega)|^2 = \frac{1}{1 + \omega^{2n}}$$



And if you plot the squared magnitude response, it will turn out as you can imagine that this is 1 and this is 1, what is the magnitude response? Regardless of n the magnitude response squared at omega equal to 1 is what? Is half. So, magnitude squared is always associated with power. So, that is the half power frequency. If the squared magnitude is half what comment can we make about the magnitude itself. 1 by row 2 which is the 3dB frequency right.

So, in other words the bandwidth of the filter a 3dB bandwidth is simply 1 radian per second. Does it make sense? And so, for different values of n, so for n equal to 2 for instance let say this is the 1 for n equal to 1, n equal to 2 will be flatter in the pass band and sharper in the stop band and you know as you keep increasing n you will eventually achieve the ideal right.

So, this is increasing order. Does it make sense? Alright. So, that is as far as the approximation is concerned. Now, but that is not really, I mean that is only part of the puzzle what we are actually interested in finding is?

We would like to find the H of s which corresponds to this magnitude response, $\frac{1}{1+\omega^{2n}}$ ok and how do we get to the magnitude response here? Given H of j omega the whole square what do we, how do we get back to the magnitude, how do we get back to the transfer function? Well, we saw yesterday that replace omega square with minus s square.

And then so what we end up with? So, we basically get 1 plus omega square is minus s square whole power n ok and this must be equal to 1 over what is that? That is nothing, but D of s times? D of minus s right and therefore, what is our job now?

$$\frac{1}{1+(-s^2)^n} = \frac{1}{D(s)D(-s)}$$

We find, how do we find D of s? Well, we find the roots of. So, find the roots of D of s times D of minus s and the poles of D of s or the roots of D of s, I mean roots of D of s in the left half plane are what? We are interested because we are interested in realizing a stable filter transfer function, well. Now, that we know this now it is just a matter of doing the stuff.

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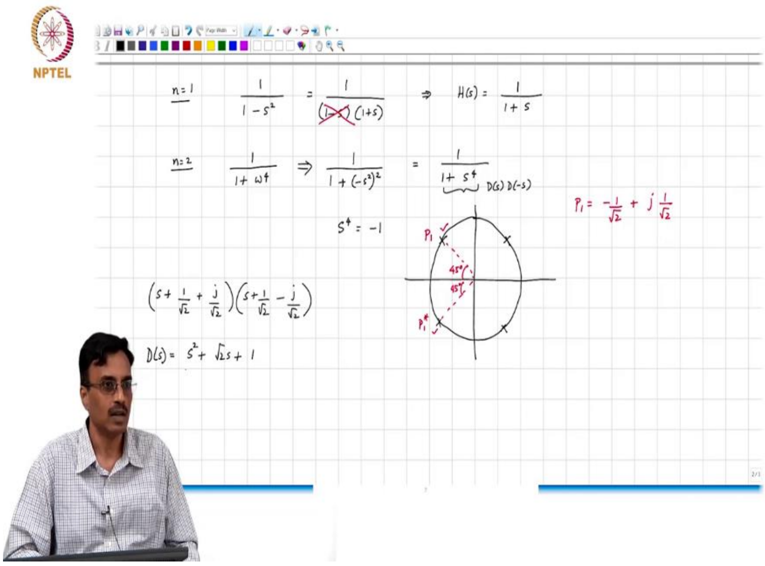

So, n is 1 is trivial. So, we have 1 by 1 minus s square, which is 1 over 1 minus s into 1 plus s.

$$\frac{1}{1-s^2} = \frac{1}{(1-s)(1+s)}$$

So, which is the stuff that we discard? 1 minus s. We discard that. So, H of s is 1 over 1 plus s alright.

$$H(s) = \frac{1}{1+s}$$

So, let us come to something which is a little more interesting, n equal to 2 mod H of j omega square is? 1 by Mod H of j omega the whole square equals what? One plus omega raise to the power 4 and therefore, when we replace from here we get to 1 by? 1 plus minus s square whole square right, which is basically 1 by 1 plus s power 4. We need to find the roots of 1 plus s power 4 and what are the roots of s power 4 equal to minus 1? Yes.

$$n = 2$$

$$\Rightarrow \frac{1}{1+\omega^4} \Rightarrow \frac{1}{1+(-s^2)^2} = \frac{1}{1+s^4}$$

$$s^4 = -1$$

Oh I mean it is minus s power 4 equals minus 1 yeah. So, what are the roots? Where are the roots located at? 5 by 4. They are all lie on a the unit circle right, and let assume this a circle. So, we have root there. If you have root there you will have root there. You already know that if h of you know P is root minus P is also root and minus P star will also be a root. We have 4 roots alright and which 2 roots are we interested in. Left half. So, we are interested in these 2 roots and remember what is this angle? 45 degrees. So, what are the poles that we were interested in? P 1 is what? In Cartesian form. Minus 1 by root 2 Plus.j times 1 by root 2 and P 1 star is of course, conjugate of this.

$$P_1 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

So, what is the polynomial? This is the root right; we know the roots of the polynomial. How do we what is the polynomial itself? So, this is remembered this is D of s times D of minus s. So, what is the polynomial? It is s plus 1 by root 2 plus j by root 2 times S plus 1 by root 2 minus j by root 2 and that is simply nothing but let us please do the math. This nothing but s square plus root 2 s plus.1, alright.

$$\left(s + \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right)\left(s + \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)$$

$$D(s) = s^2 + \sqrt{2}s + 1$$

So, this is D of s. Remember we got a little bit lucky because when s is equal to 0, this automatically becomes 0, otherwise you will have to have a coefficient here which basically makes sure that the denominator polynomial is 0 when is I am sorry when s is 0 here this becomes automatically 1 right. Otherwise, you will have a constant of multiplication where which you may and then that constant is derived. So, that when s is equal to 0, you get 1 ok.

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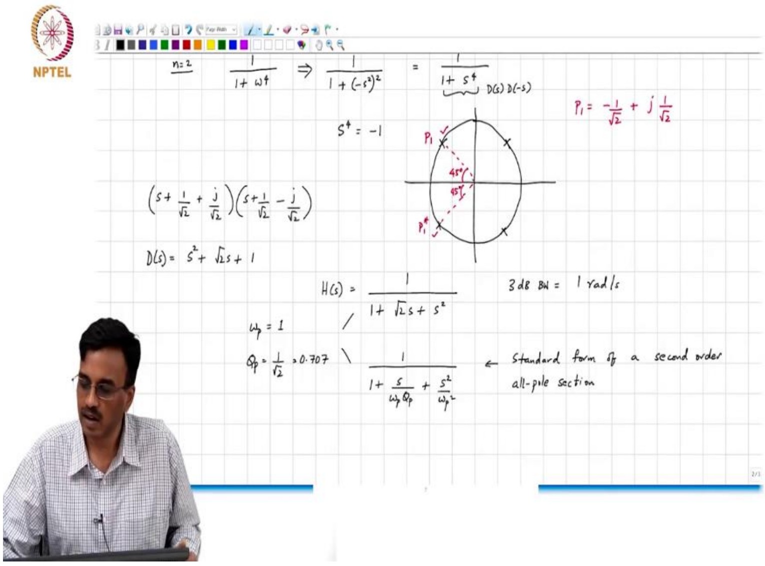

So, this is therefore, the second order Butterworth filter as a transfer function which is of the form 1 over 1 plus root 2 s plus s square ok.

$$H(s) = \frac{1}{1 + \sqrt{2}s + s^2}$$

And what is the 3 dB bandwidth? This is 1 radian per second, ok. So, in general if you want omega c radians per second what would we do? Simply replace s with s by omega s alright.

Now, this is I am sure you are aware of you know the standard form of writing a second order transfer function and in the standard form is 1 plus s by omega p Q p plus s square by the standard form of a second order all pole sections ok.

$$\frac{1}{1+\frac{s}{\omega_P Q_P}+\frac{s^2}{\omega_P^2}} \leftarrow Standard\ form\ of\ a\ secon\ order\ all - pole\ section$$

So, simply looking at these 2 what do we what do we see as comparing these two? What comment can you make about omega p? 1, and what about Q p? 1 by root 2 or 0.707 right.

$$\omega_P = 1$$

$$Q_P = \frac{1}{\sqrt{2}} = 0.707$$

So, will there be overshoot or will there be no overshoot in the step response? Well, we overshoot why? The Q is greater than half and therefore, there will be overshoot in the step response right. It turns out that the overshoot is about 4 percent. Anyway, so that is again as you can see the second order Butterworth filter is also very straight forward.

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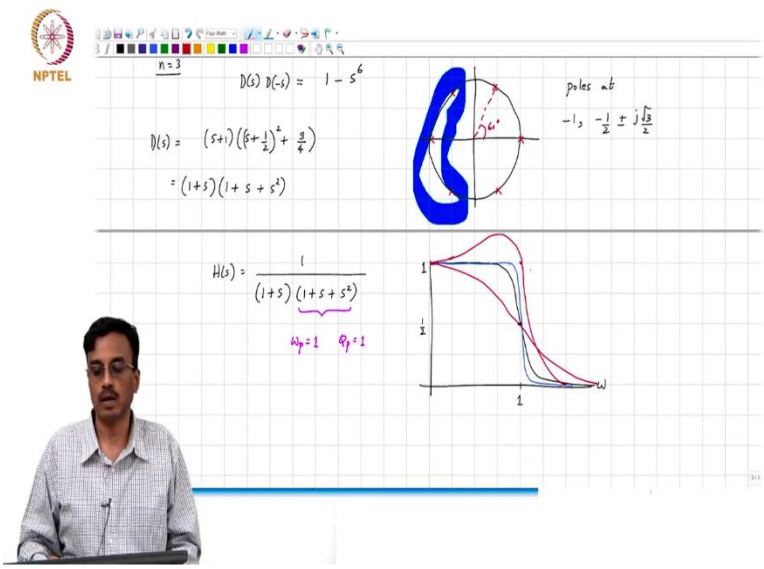

And therefore, I mean and similarly higher order filters I am just going to do the third order one and stop. So, now, there is no point in going through the same rig model all over again D of s into D of minus s is what now hm? 1 minus S power 6.

$$D(s)D(-s) = 1 - s^6$$

So, where the roots are basically, again we have our circle alright, and where are the where are the roots? They must be rooted; obviously, at 1 and plus 1 and then you know you divide the whole pizza pie into 6 slices right. So, basically, I have something there, something there and therefore, something right, and this is what is this angle? 60 degrees.

So, which are the poles that we are interested in? Those in the left half plane. So, there is basically a pole at. So, what is the polynomial? So, there the poles that we are interested in are at what locations? There is 1 at. There is 1 at minus 1 right. Then? Minus 1 by 2 plus minus j root 3 by 2 alright.

$$poles\ at\ \left(-1, -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}\right)$$

So, therefore, the polynomial is simply s plus 1 times s plus half the whole square plus 3 by 4 right, which is 1 plus s times 1 plus s plus s square.

$$D(s) = (s+1)\left(\left(s+\frac{1}{2}\right)^2 + \frac{s}{4}\right)$$

$$= (1+s)(1+s+s^2)$$

So, therefore, the third order Butterworth filter H of s is simply given by 1 over 1 plus s times 1 plus s plus s square and what comment can be. So, you can think of this; obviously, as a cascade of a first order section and a second order section, and what comment can you make about the location of I mean if you express the second order section, this second order section in standard form what comment can we make about omega p? Omega p still remains 1 and what is the Q p now? Q p is 1 ok.

$$\omega_P = 1;\ \ Q_P = 1$$

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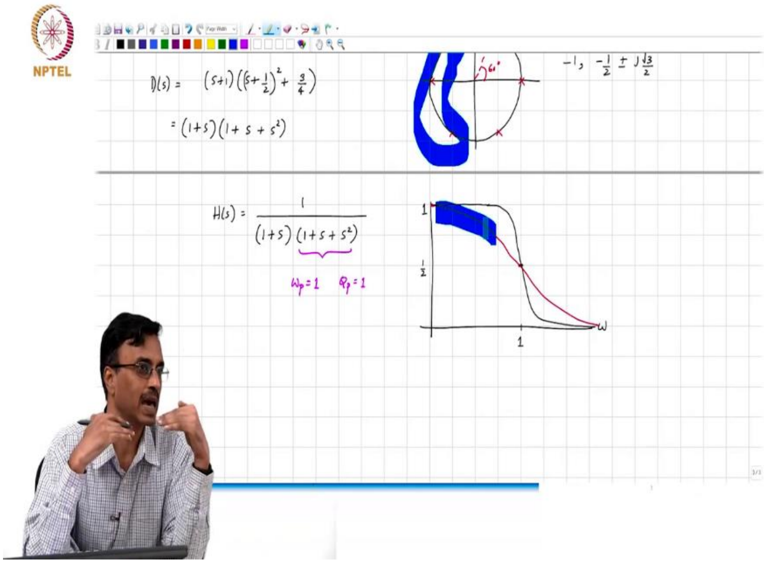

So, it would be interesting to kind of compare the response of the second order Butterworth third order Butterworth. This is 1 this is 1 and this is half and the second order Butterworth squared magnitude transfer function probably does something like this right. Now, the third order transfer function does something I mean I would like to express this as simply the product of the magnitudes of the first order response and the second order response. So, what comment can you make about the first order response? How does that look like? The squared magnitude response of the first order part of that third order transfer function, how would it look like? So, how would that like you draw that what would, how would it look like compared to this black curve how would that look like? It would be more gradual. It will start at the same point and then it will also go through do something like that. Now, what comment can we make about the second order section? Pardon.

So, ok let us start it the origin, what will it be at the origin? 1 ok. Slightly beyond omega equal to 0 how will it, how will it look like?

The one answer I got was oh it will be flat ok. So, remember the total response is the product of the first order response, if the second order response right. The second order response is flat and the first order is falling what if you multiply the two what will you get?

Something which is falling right, but the fourth the third order response is actually? How does the third order Butterworth response behave at the origin? It is flat. So, if the first order section has a response which is falling, it must follow that the second order section must have a response which is? Which is going? Up, right. So, the second order response will basically should be expected to do something like this right and the overall third order response will look like what? Should look like? How will it be compared to the second order Butterworth response?

It will be even flatter than the second order one. So, it you probably look like this right, and the overall response is the product of the first order response and the second order response and therefore, the second order response must look like, how will it look like? Pardon.

It will have a magnitude 1 at omega equal to 1 right, but more importantly ok. So, here it will have a magnitude of 1 right. So, it starts off like that and eventually what does it do? Eventually the second order response falls of as? As 1 over Omega square, correct?

So, eventually the second order response must do this, must do something like this ok, alright. And so therefore, it starts off going this way and eventually is falling off in the middle what must it do? Well, there must be a peak in that picture, but I am sure you get the idea.