Circuit Analysis for Analog Designers
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Lecture - 19
The maximally flat (Butterworth) approximation

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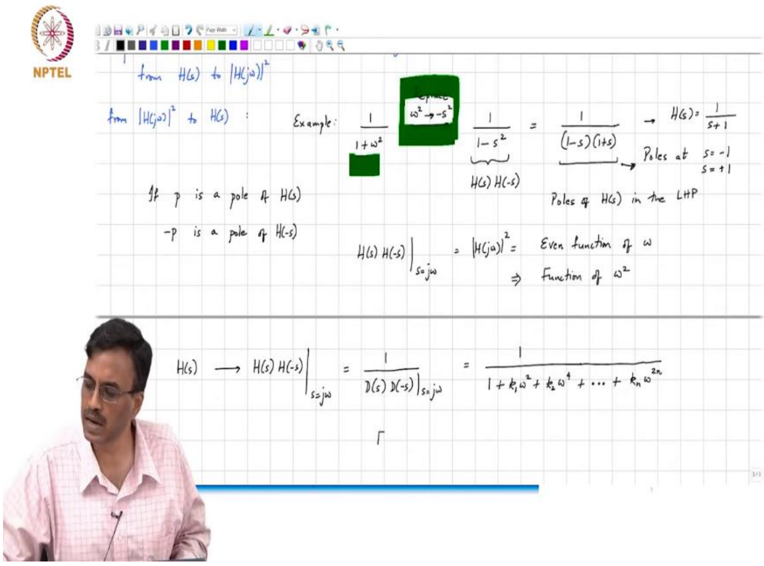

Alright so, very good. So, let us see what we have learnt so far. So, we have an all-pole filter H of s then from this we get the magnitude. How do we get the magnitude? We get H of s time H of minus s evaluated at s is equal to j omega, which is basically 1 over D of s times D of minus s evaluated at s is equal to j omega, which will therefore, be a ratio of polynomials in omega square right, where the numerator polynomial is still 1 the denominator polynomial is of the form some k 1 omega square plus k 2, what are the next term? Omega to the power 4 … all the way up to Omega raised to the 2 a.

$$H(s) \rightarrow H(s)H(-s)|_{s=j\omega} = \frac{1}{D(s)D(-s)|_{s=j\omega}} = \frac{1}{1 + k_1\omega^2 + k_2\omega^4 + \cdots + k_n\omega^{2n}}$$

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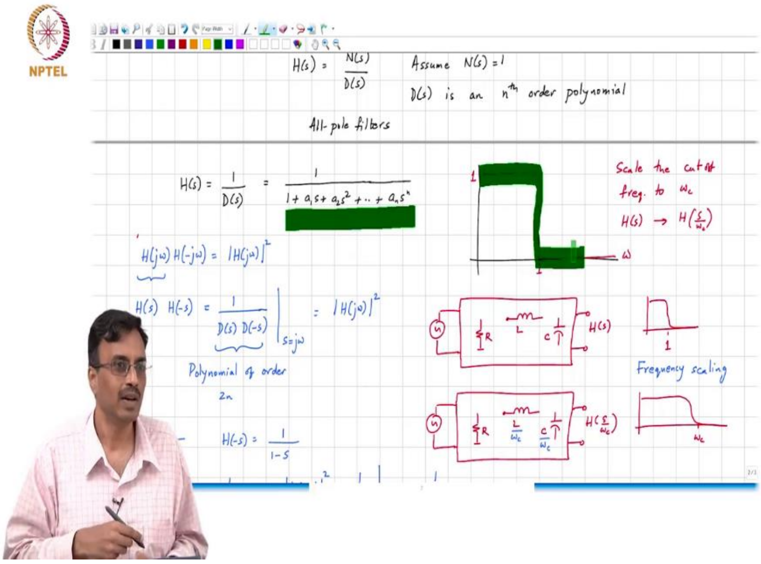

k n omega raised to the 2 a correct. And what is our job find remember that this k 1 through k n should not be confused with a 1 through a n they are still I mean the k 1 through k n are; obviously, related to the a s, but they are not the same right. I mean it is obvious, but I just wanted to point it out alright.

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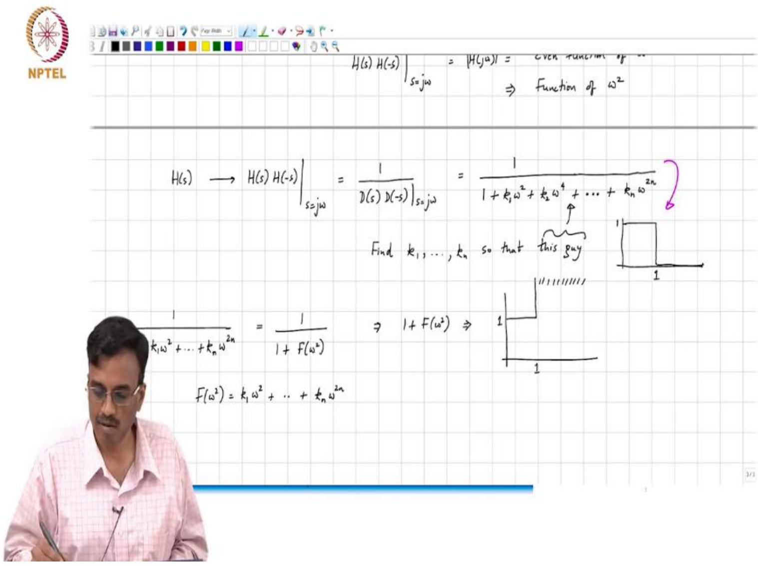

So, what is the job now? What is that problem statement? Find $k_1, \ldots, k_n$, so, that this guy attempts to look like this right, small observation that I have taken a small liberty here.

And basically, said that well you know if the magnitude approximates a brick wall it must follow that the squared magnitude wall also approximately brick wall.

So, rather than working with square roots of this polynomial you might as well say I will make the square of this approximate the brick wall if I take the square root, it will also approximate the brick wall correct. So, in other words, I need to make the squared magnitude response which is this function approximate the brick wall.

And of course, now, we have a ratio of polynomials and all the action is happening in the denominator right. So, making this approximate the brick wall I am going to write this as 1 plus k 1 omega square all the way to k n omega to the 2 n is I am going to write this as 1 by 1 plus some function of omega square right.

$$\frac{1}{1 + k_1\omega^2 + \cdots + k_n\omega^{2n}} = \frac{1}{1 + F(\omega^2)}$$

Where that function of omega square, I mean all are unknowns in that function correct and that function F of omega square is k 1 omega square plus k n omega to the 2 n alright and this must approximate the brick wall ok.

$$F(\omega^2) = k_1\omega^2 + \cdots + k_n\omega^{2n}$$

So, if this must approximate the brick wall what comment can we make about the denominator function. If $\frac{1}{1+F(\omega^2)}$ must approximate the brick wall, what comment can we make about $1 + F(\omega^2)$, the denominator itself. What must it approximate so, that its reciprocal approximates the brick wall? Well, this must approximate. This must be 1 in the range 0 to 1 and must go to infinity beyond alright, well there is I mean 1 is the same as 1 on for large values of 1. So, there is no point in carrying 1 on the 1 plus F of omega square and then you know you have 1 on the in the picture so, this is equivalent to saying. Find F of omega square so, that what does it do, what is supposed to do?

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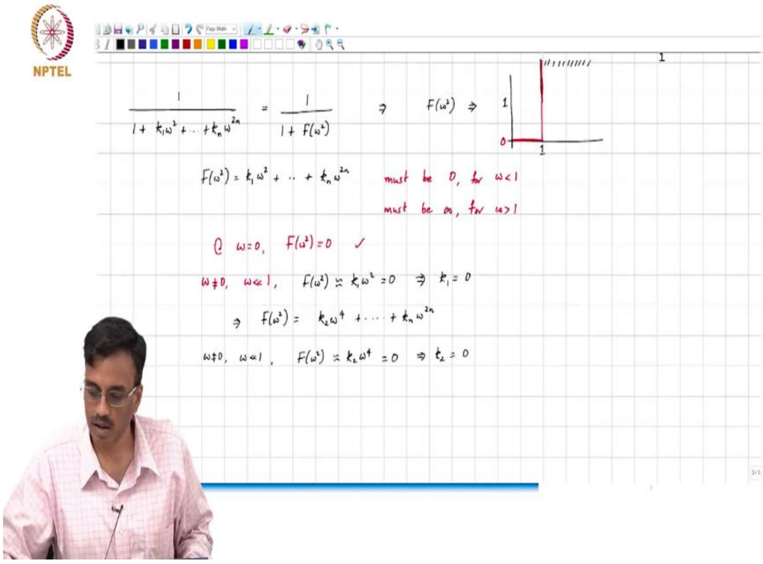

Very good, right so, basically, we say F of omega square must be 0 in the range 0 to 1 and must go to infinity beyond is that clear alright. So, now a problem statement is finding a polynomial F of omega square, which we have already tabulated here which basically is 0 for $\omega < 1$ right, must be infinite for $\omega > 1$.

I mean clearly, I mean do you think we can actually do it exactly I mean can we find a polynomial which does this we; obviously, cannot right ok. So, the only thing we can do is you can only find a good approximation to it correct. Now, let us see what we can do well at Dc at omega equal to 0, what is F omega square.

Well that satisfies our specification evidently right because we want $F(\omega^2) = 0$ at D c it is 0 we are happy right at frequencies. So, at D c, F of omega square equals 0 we like that. For frequencies slightly away from D c so, in other words omega not equal to 0, but omega much much smaller than 1, which of these terms will dominate Well, F of omega square is approximately k 1 times omega square right and what do we want this to do. We want it to be 0 right.

$$\omega = 0, \qquad F(\omega^2) = 0$$

$$\omega \neq 0, \;\; \omega \ll 1, \;\; F(\omega^2) \approx k_1 \omega^2$$

Remember that F of omega square must be 0 in the interval 0 to 1. So, we will say well, for frequency slightly removed from D c we would like this function to be 0. So, we want it to be 0, which implies k 1 must be equal to k 1 times omega square wanted to be 0. Omega is not 0 so, what is the conclusion? k 1 must be 0 right. So, our within quotes what we have our function now is F of omega square is k 2 Omega raised to 4 plus blah plus k n omega raise to 2 n correct.

$$\Rightarrow k_1 = 0$$

$$\Rightarrow F(\omega^2) = k_2\omega^4 + \cdots + k_n\omega^{2n}$$

So, for small frequency omega not equal to 0 omega much smaller than 1 F of omega square is now approximately. Very good, it s k 2 omega to the power 4, which you want to be 0. So, what do you get? k 2 equal to 0 correct alright, yeah this is looking too simple right. So, we keep going all the way right and then.

$$\omega \neq 0, \quad \omega \ll 1, \; F(\omega^2) \approx k_2\omega^4 = 0$$

$$\Rightarrow k_2 = 0$$

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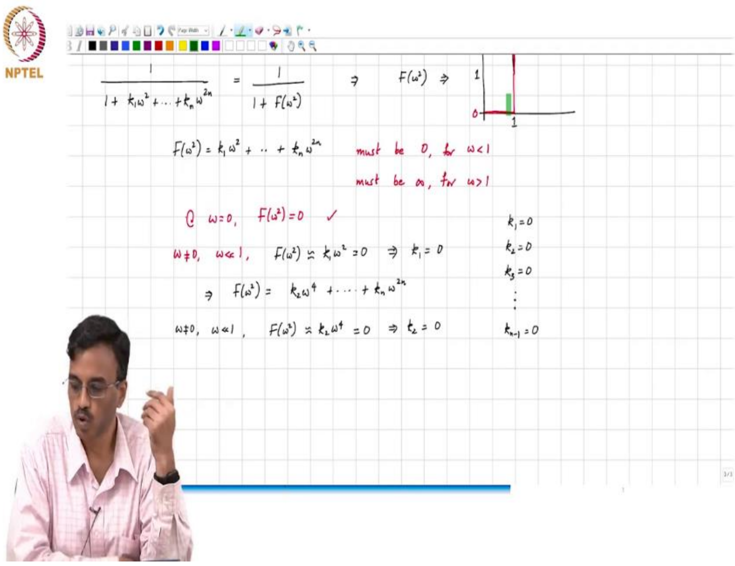

So, therefore, say k 1 is 0 k 2 is 0 similarly k 3 is 0 alight all the way k n minus 1 equal to 0 alight. And then now you say last step let me make k n also equal to, k n also equal to 0 can we do that? Why? Yeah. So, basically if you make k n equal to 0 then it is true that well in the pass band the gain will be perfectly flat right, but the whole idea in having a filter is to reject the stop band right. So, therefore, you cannot make k n equal to 0.

$$k_1 = 0; \quad k_2 = 0;$$

$$k_3 = 0; \; \dots \; k_{n-1} = 0;$$

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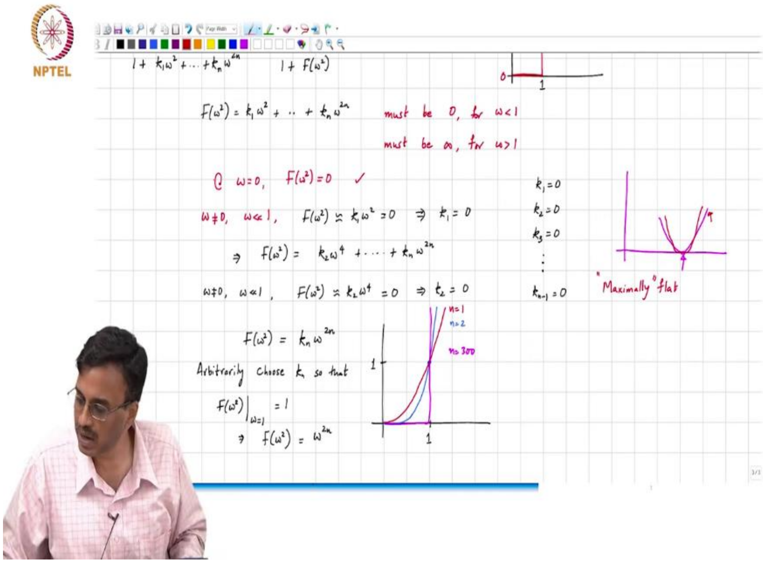

So, the transfer function I mean the polynomial, therefore, must be ah so, F of omega square must therefore, be equal to k n times omega to the power ok.

$$F(\omega^2) = k_n \omega^{2n}$$

Now, the question is what must k n. So, how does this look like? Let me draw a picture. So, this is 1 let us say and this is also 1.

So, as right so, we just normal I mean arbitrarily choose k n. So, that F of omega square at omega equal to 1 is 1 ok.

$$F(\omega^2)|_{\omega=1} = 1$$

It is a this is a quite an arbitrary seemingly arbitrary condition and there is good reason why people did this as we will see going forward. So, therefore, what comment can we make about k n, well its 1. So, F of omega square is nothing but omega to the 2 n.

$$F(\omega^2) = \omega^{2n}$$

So, if n is 1 how do you how will you see this how does this look like, is simply nothing but a come-on folks.

It keeps doing that right and well this is a parabola and this is nowhere close to brick wall, but you can see that you know what you call as I said you know with approximation you know beauty lies in the eyes of the beholder right. So, you know if your you know or as I say beauty lies in the eyes of the beer holder right. So, if you are sufficiently drunk, I mean anything looks like anything else right.

So, you can say oh well this looks like a if we look at from sufficiently far away it looks like a brick wall right. Now, you work a little harder for n equal to 2 how does it look like omega to the power 4 how will that look like at D c will it be flatter than omega square or will it be you know be flatter at omega equal to 1 it will be the same. Beyond omega equal to 1 will it be steeper or less steep than? It will be steeper. So, you basically have something like that ok. So, this is giving you a path to I mean. So, it is clearly doing a better job of approximating that brick wall alright. Now, if we have you know n equal to 300, what do you think you will get?

Well, you will get you will get something like this right. For all practical purposes you know you will get something like ok, it is till the polynomial except that you are now not able to right. Because there this F of omega square is becoming flatter and flatter and flatter as you know n increases so, n is equal to say 300 ok.

It makes sense I mean if we have a higher order as you know filter it polynomial it is reasonable that you are able to a better job of approximating what you want alright. So, what you call and a couple of comments see remember that if you have you know a curve which does this at that point it is well the curve is doing this what at that point it is.

It is flat right ok. Now, what comment can you make about that now at that point? The red curve is flatter right ok and mathematically what do you think I mean in the magenta curve mathematically you see that the first derivative is 0 correct. Alright if you have something which is even flatter like the red curve what comment can you make?

The first derivative is 0. The second derivative is also 0 and within quotes its flatter right. So, if I make the first derivative second derivative third derivative forth derivative all I mean all the way except. The highest derivative 0 right the curve becomes flatter and flatter and flatter and flatter right ok. Now, if I make the highest derivative also 0 what will happen I mean it is you know as flat is getting, but then it does not do anything else right it is 0 correct so, that is not useful.

So, if you make all derivatives except the highest order-th one is 0 it cannot get any flatter than that right the curve cannot get any flatter than that right. So, that such a curve is called maximally flat, right and that clearly depends not only in the order of the derivatives, but at which point which is a point of flatness the point at which the derivatives go to 0 is the point of flatness correct. So, this that we have done namely F of omega square equal to omega to the power 2 n is maximally flat at what frequency?

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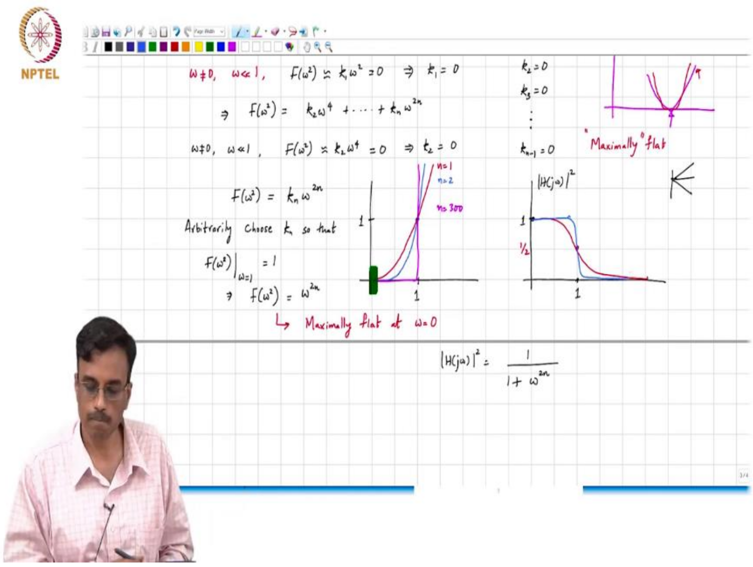

Flat at D c or omega equal to 0 right. So, if F of omega square is doing this now, I think it is pretty straightforward to figure out what the magnitude response of the filter itself is doing what is H of j omega square doing, at D c it is, what is the value at D c? 1 right around D c it is. It is a I mean its it is close to 1 alright, but it can be something can be close to 1 in many directions right what how does the curve go at D c does it go like this or does it goes sloping downwards or.

Around D c does the magnitude response go sloping upwards or going go sloping downwards or does it stay flat. For example, so, the magnitude response therefore, is nothing, but mode H of j omega the whole square is nothing but 1 by 1 plus omega to the 2 n right.

$$|H(j\omega)|^2 = \frac{1}{1 + \omega^{2n}}$$

So, how does this look like? Around D c. At D c it is 1 around D c right what I am asking you is this is D c does it do this does it do that or does it do that what does F of omega square do.

Around D c what does F of omega square do? Is it flat or is it going upwards or is it going downwards? It has flat as it can get correct. So, if F of square omega square is flat at D c what comment can you make about 1 plus F of omega square is at flat. It is flat so, what comment can you make about 1 by 1 plus F of omega square. It will also be as flat it as it can correct. So, therefore, the frequency response therefore will basically do this which is, what is the value at 1?

This is going to be half. So, the response is going to do say this if you keep increasing the order you will get flatter and flatter at the origin right.

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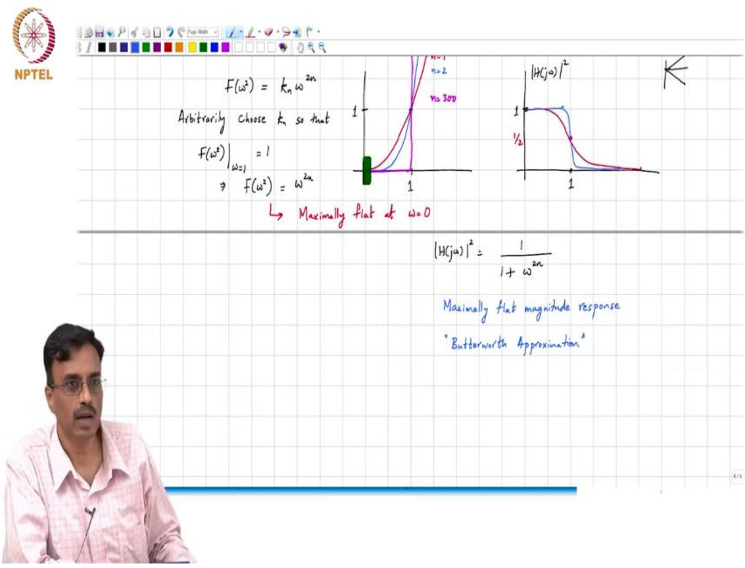

So, this response is called the maximally flat magnitude response and after the guy who discovered it this basically called the Butterworth response, the Butterworth approximation I must say. As you can see this is a simple nice way of approximating the ideal brick wall alright.

So, in the next class we will do the remaining stuff that needs to be done namely I mean now we have found a H of j omega square which approximates the brick wall, but what do we actually need. We need H of s, but we know how to do that. So, we will do that in the is this clear.