Circuit Analysis for Analog Designers
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Lecture - 10
Review of Modified Nodal Analysis (MNA) of linear networks

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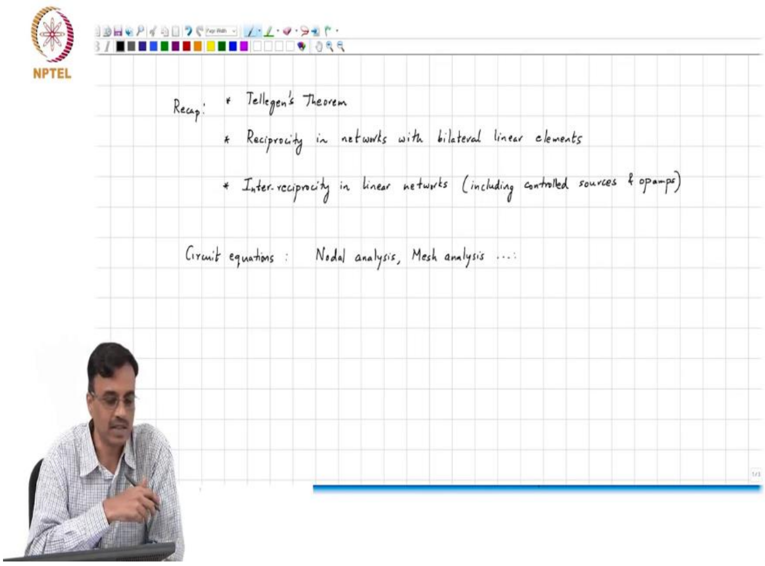

Lecture 5, a quick recap of what we have done in this course so far. So, we have learnt about Tellegen's theorem. We have learnt how Tellegen's theorem was used to prove reciprocity networks with linear bilateral elements and unfortunately, we found that well while reciprocity is great and very useful in practice we found that it is not applicable to a large class of networks that we often deal with namely those with control sources.

And, then we said well there is a fix and that is by using the concept of inter reciprocity where the idea is that while you cannot interchange the location of the excitation and the response in the original network. It is always possible to come up with another network which is called the inter reciprocal network or the adjoint network, where you can interchange the role of the excitation port and the response port, right.

And, this is immensely useful in practice simply because you have you are now in the position to be able to find multiple transfer functions to a single output as a voltage or current in one shot without having to rerun the computation again and again. And, as I was mentioning yesterday this is the this is routinely done in all circuit simulators whether they are doing noise analysis or whether you are doing transfer function analysis, good.

So, having seen that, now let us go and move forward and look at or rather refresh our memories about ways of writing circuit equations. In your earlier classes you likely seen you have probably seen many ways of doing this you know the nodal analysis is probably the first one that comes to mind then there is mesh analysis etcetera.

And, you know for good reason nodal analysis the basic framework of nodal analysis is very popular and the reason is that it is often involves more work to find meshes. And, then you know find though I mean and then once you found the mesh currents then you do some more operations to get the branch currents.

On the other hand, nodal analysis gives you in one shot gives you all the node voltages and since the branch relationships are known the branch currents can also be found in one shot. So, let us go through a quick refresher.

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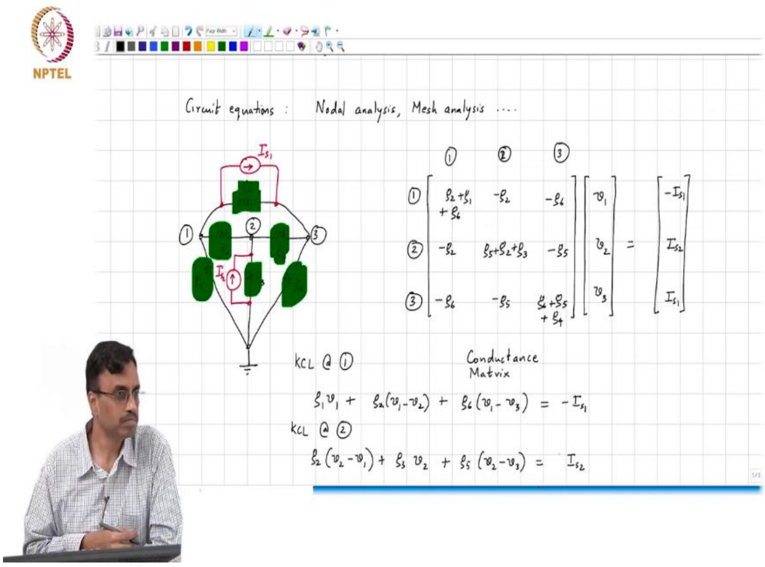

So, let us start with networks which only consists of resistors or conductors and current sources and again I am going to take this ice cream cone network as an example. So, I am going to choose some reference some node as the reference voltage or ground. And, we will have the conductance's are g_1, g_2, g_3, g_4, g_5 and g_6, alright and the nodes are numbered 1, 2, 3 and perhaps we have some sources say I_{s_1} and we have another current source exciting the network that is I_{s_2}, right.

And, how do you write the nodal equations? Well, it is very straightforward as you have seen in past. The nodal equations are simply expressing Kirchhoff's current law at a node, right and once you see that it is not necessary to write equations you know one by one carefully staring at the network. We have already seen this in your basic electric circuits and networks class. What are the unknowns that we are trying to solve for?

The node voltage vector which is v 1, v 2 and v 3, right and if you write a Kirchhoff's law at in node 1 you know you can go through it the long form or recognize that you can do it element by element as you have seen in your earlier classes, right. So, I will since it is a long time that you saw this let me just write Kirchhoff's law at node 1 to give you an in a quick refresher on why we do, what we do. So, if you write KCL at node 1 what do we see is g 1 times v 1 plus g 2 times v 1 minus v 2 right plus g 6 times v 1 minus v 3 equals that is the sum total of all the current going out of the node and that must be equal to the current going into the node and that must be equal to in this case minus I s 1 right, ok.

$$g_1 v_1 + g_2(v_1 - v_2) + g_6(v_1 - v_3) = -I_{s_1}$$

And likewise, KCL at node 2 gives us g 2 times v 2 minus v 1 plus g 3 times v 2 plus g 5 times v 2 minus v 3 equals. No, at node 2 what is the sum total of all the current flowing of the source exciting? Sorry, that is I s 2 indeed.

$$g_2(v_2 - v_1) + g_3 v_2 + g_5(v_2 - v_3) = I_{s_2}$$

Thank you, alright ok. So, now you stare at this and then say well. So, g 2 this particular conductance appears only in KCL for node 1 and 2, correct and therefore, and how does it appear in this matrix in the conductance matrix. This is the conductance matrix. You can arrange these equations in matrix form and the reason is called the conductance matrix is because every coefficient corresponds to the conductance of some element to the other and therefore, as far as g 2 is concerned for instance, where do you think which, rows will g 2 appear in? Well, g 2 goes between nodes? 1 and 2. So, it must appear in the first two rows right and g 2 relates which two node voltages? v 1 and v 2 and therefore, it must also appear in the? It must also appear in the first two columns. So, between 1 and 2 basically how and how does it appear in the first row? How does it appear?

What ok can you fill up the entries of the matrix can you tell me which all of those entries in the matrix will correspond to g 2 , g 2 yeah. So, these are three rows, these are the three columns corresponding to v 1, v 2 and v 3. So, can you help me write the nodal equations? What will happen to g 2?

It will appear in the you know first two rows and in the first two columns and as you can see it must be g 2 minus g 2 minus g 2 and g 2. Right and since g 2 does not appear anywhere else we can now forget about g 2 we are done, right. What about g 1?

First row first column and what about g 3? Second row? Second column, very good. What about g 5? Between 2 and 3. So, that is the g 5 minus g 5 minus g 5 and g 5, that is done. What about g 6? Between 1 and 3. So, that is basically that is done then what about g 4? Well, at that only appears in the third row third column, correct and this must be equal to. Where does I s 1 appear? It must appear between nodes 1 and 3. So, in which of these on the right-hand side which are those rows will it appear in? So, it is minus I s 1 because as the current flowing into node 1 and this will be plus I s 1 and likewise I s 2.

Correct?

$$\begin{bmatrix} g_2+g_1+g_6 & -g_2 & -g_6 \\ -g_2 & g_5+g_2+g_3 & -g_5 \\ -g_6 & -g_5 & g_6+g_5+g_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -I_{s_1} \\ I_{s_2} \\ I_{s_1} \end{bmatrix}$$

So, once we have do done this once we have done this there is no need to go over the same procedure of writing the KCL at every node and then arrange then them as a matrix. You can simply look at the network and write the equations and what is the within codes algorithm?

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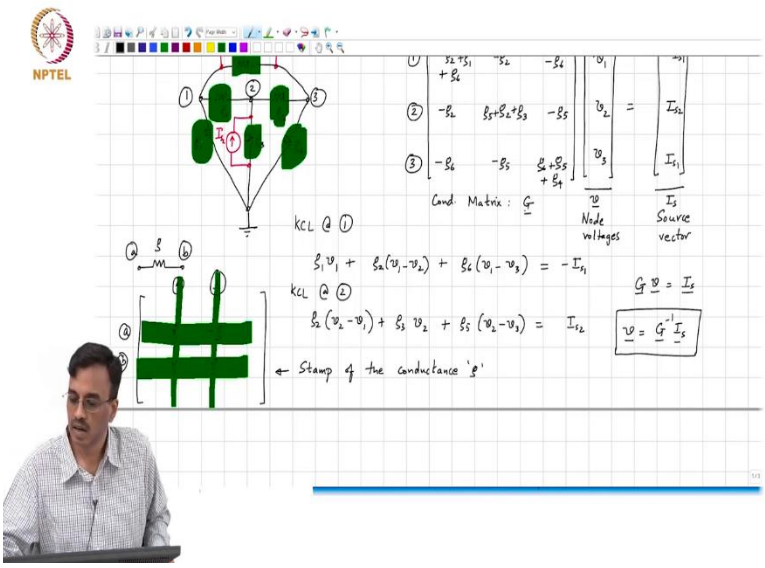

If you see a conductance between two nodes a and b and let us call this g, right. What comment can you make about the g matrix? So, we have the g matrix the conductance matrix alright it has the a-th row, the b-th row, has got the a-th column and the b-th column and how does this appear? How does this appear? It appears as g minus g minus g and g.

$$\begin{bmatrix} g & -g \\ -g & g \end{bmatrix}$$

Correct? And, likewise ok and likewise if you have a conductance only going between a node and ground it only appears on the diagonal, right and this is what is called the stamp of the conductance, right. All that this is saying is that well by looking at the number of nodes in the network you already know the size of the conductance matrix.

So, you can initiate or initialize a conductance matrix with that size and then you go element by element and replace I mean when you go to a particular element in that list of elements you have in your network. You simply plop in that appropriate stamp of that element into the matrix and then you are done with that element you do not have to worry about it anymore.

And, then you have a list of elements, so, you keep going through the list and then you know when you end the list you are done correct, ok. So, that basically this I mean you know if you can describe it so clearly it basically means that a computer can do it. There is nothing you know deeply philosophical or you know this thing about it is just very straight forward algorithm, right.

So, on the left side you have the conductance matrix; these are the node voltages which are the unknowns. On the right side, you have the source matrix or source vector I must say where you have all the independent sources, correct? Ok and once you basically find. So, let us call this node voltage this conductance matrix let us call that g and then the node voltage vector let us call that v and this is let us call that I_s for the source voltages, ok.

So, the nodal equations can be written as m G v equals I s, alright and it is straightforward to find v as simply G inverse times I s.

$$\underline{G}\ \underline{v} = \underline{I_s}$$

$$\underline{v} = \underline{G}^{-1}\ \underline{I_s}$$

Is that clear people I mean this you must have seen. So, nothing new about this I mean you must have seen this before multiple times. It is just a refresher, alright.

So, we have covered therefore in principle we have covered networks with conductance and current sources. Unfortunately, these are not the only elements that we encounter in practice.

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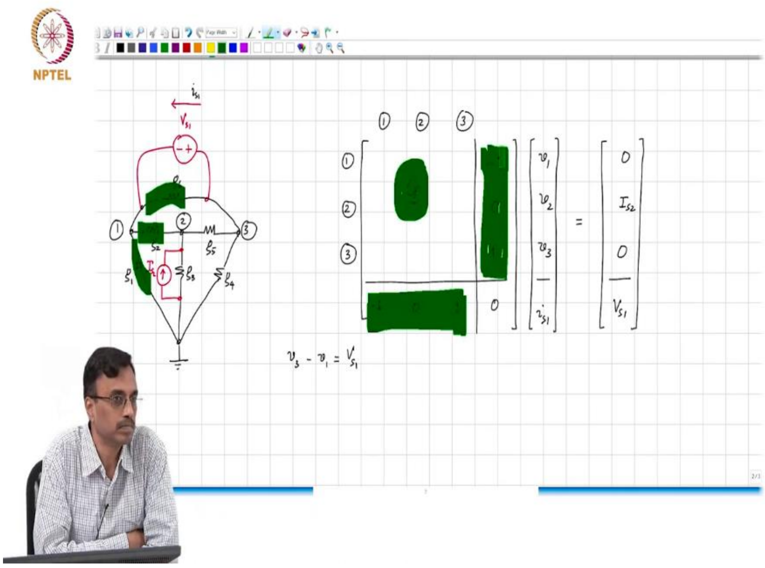

Let us see what happens for instance if we have, I am going to just simply avoid clatter I am going to remove one of these things and then replace them replace it by a voltage source, alright. And, we like to wonder you know how I mean of course, we know that if you write the equations, you know you can come up with some set of equations which eventually works right and we will be able to solve the set of equations.

But the question is you know can we come up with a clean way of describing you know how you go about writing the equation, so that it can be done on a computer you know systematic fashion without having to have intervention every time, right. So, the problem with a voltage source is that the current through it is unknown correct.

So, well then, we say well let me put this in red actually and I say oh well, we do not know the current through the voltage source. So, it is an unknown and we assign an unknown current I_{s_1} through the voltage source and this is a voltage. So, and this unknown has to be determined like just like any other unknown, ok.

So, earlier we had; earlier we had the g matrix correct with conductance I am going to not copy and paste the same thing all over again. So, I am going to call that the G matrix right which is the same as we saw earlier and again this is node 1, this is node 2 and this is node 3 and this is node 1, node 2 and node 3. We now have apart from what are the unknowns now? What are all the unknowns?

v_1, v_2, v_3 are the unknowns. We have also have an additional unknown i_{s_1}, right. So, let me just use the small i_{s_1} here simply because it is an unknown i_{s_1}, right.

And, so, if you write k c l for node one apart from all the conductance which we have already dealt with and gotten this matrix G, we now also have if you write KCL at node 1 what do you get what is the extra term you get?

Remember, now this is an extra I mean the apart from the currents flowing through these guys which we have already been accounted for in this G matrix there is also an additional current which is flowing out which is minus I s 1. So, where do you so, what do you think the equation will be in the first row what should I do; what should I do?

Where will we put the minus 1? First row, Last column, very good. So, this we put a minus 1, right and likewise whereas, do you think you will get a I mean ah. So, the voltage v s 1 is connected between nodes 1 and 3. So, where do you think what do you think will happen to how will I account for I s 1 flowing out of node 3?

Very good. So, basically you can see we have minus 1 there, we have plus 1 there and what comment can I make about the element there? Yeah, 0 because you know I s does not touch node 2, correct. So, we have one more we have one more unknown. So, we better have. One more equation and what is that equation?

Well, the extra equation is that v 3 minus v 1 equals v s 1. So, what should I do this is the how should I write that in this matrix? Fourth row Minus 1, 0, 1, 0, must be equal to what should the right hand side look like? Right hand side remember must have only the? Only the independent sources. So, where are the what are the independent sources?

The first three rows will have only? Will have only the independent current sources, so, what is what are the independent current sources flowing into node 1 is 0? I s 2. 0 and what should be there? v s 2. Sorry, v s 1. Sorry, is that clear?

$$\left(\begin{array}{ccc|c} & \underline{G} & & -1 \\ & & & 0 \\ & & & 1 \\ \hline 1 & 0 & 1 & 0 \end{array}\right)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ i_{s_1} \end{pmatrix} = \begin{pmatrix} 0 \\ I_{s_2} \\ 0 \\ V_{s_1} \end{pmatrix}$$

Alright and so, again this is the original conductance matrix augmented with something else to account for the voltage source. And, please note I would like to draw your attention to the following. This is always going to be what comment can you make about this part and this part? If you have take that part this is going to be simply the transpose of that, right and I think it is a pretty apparent how that happens, ok. So, the current flows you know between the current in the voltage source flows between the two nodes and the voltage force source forces the difference between those two nodes to be something. So, it stands to reason that this row is simply the transpose of this column, is that clear?

Now, ok now if you had any if you had another source, what do you think you will do? Let me just make life little more complicated by say for instances V s 3 for instance, ok.

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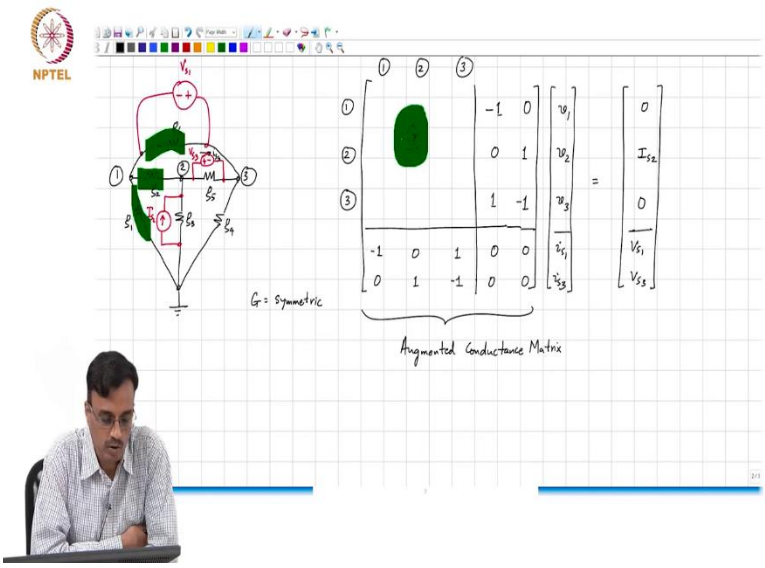

What do I do now? I mean do I need to start all over again or can I just simply you know kind of? Well, straightforward as you all point out very good. So, this current is I am going to call that I s 3, right and what do I do? Well, I add another there is an extra unknown there is an extra unknown and that is what is that saying? That is between nodes? 2 and 3 and how do I will. So, what should I do? The new unknowns are basically v 1, v 2, v 3, I s 1 and I s 3 are the unknowns. Yes, so, what should I do please? So, for every extra voltage source you augment this original conductance matrix by one extra row and one extra column, right and what happens to this the new row and new column that we need to add now, what is that?

So, V s 3 goes between nodes 2 and 3. So, it should. So, basically it must be 1 and minus 1, correct? Is that clear people? And, likewise what should this new row be? 0, 1, minus 1 and this is 0, alright and as usual on the right-hand side, well, the first three rows are basically, well, 0. I s 2. 0 like we had before and you have the independent sources V s 1 and V s 3, correct.

$$\left(\begin{array}{ccc|cc} & \underline{G} & & -1 & 0 \\ & & & 0 & 1 \\ & & & 1 & -1 \\ \hline -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right)\left(\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \hline i_{s_1} \\ i_{s_3} \end{array}\right)=\left(\begin{array}{c} 0 \\ I_{s_2} \\ 0 \\ \hline V_{s_1} \\ V_{s_3} \end{array}\right)$$

So, therefore, we are now in a position to cleanly right. So, this is basically the augmented conductance matrix, alright. What comment can we make about G? Can we make any comment about the structure of G? Well, it is symmetric because if you remove all the current sources and I mean you remove all the current sources and remove physically remove all the voltage sources, then all that you have is conductance which are obviously, if a conductance is going between two nodes that appears symmetrically across ok, alright.

And, you can therefore, solve these set of equations, in an in one shot get all the voltages and node voltages and currents through the independent sources.

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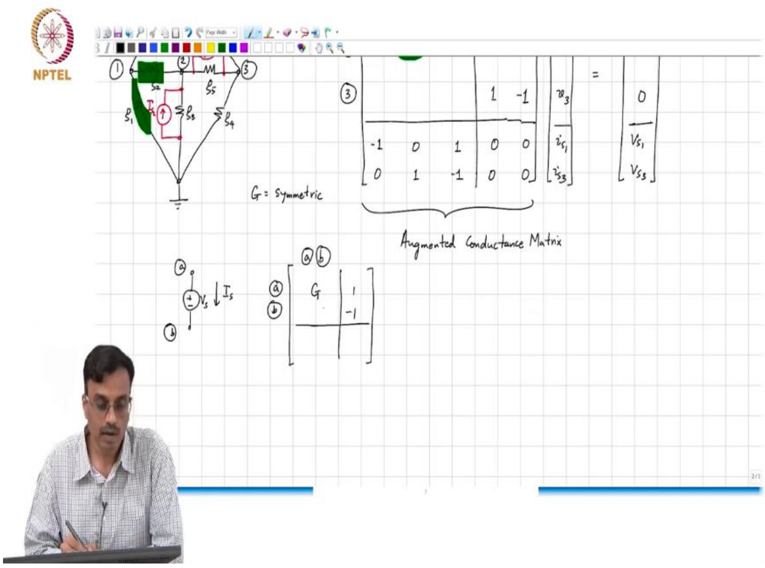

So, this is telling us what to do this you know this analysis basically telling us what to do when you have a voltage source. So, when you have a voltage source between nodes a and b and V s. You add a new unknown I s, right and what happens? You had the original G matrix, correct and then what do you do? You augment it with?

So, you have the a-th row, b-th row, a-th column, b-th column. So, what should you do? What did we do when I mean and remember I mean you know you should be familiar with this stuff so that we are able to look at a network and write down the equations without the matrix without having to go through the whole.

Very good. So, basically you must have 1 here and minus 1 here ok, in the a-th row you put a 1 and b-th row you put a minus 1.

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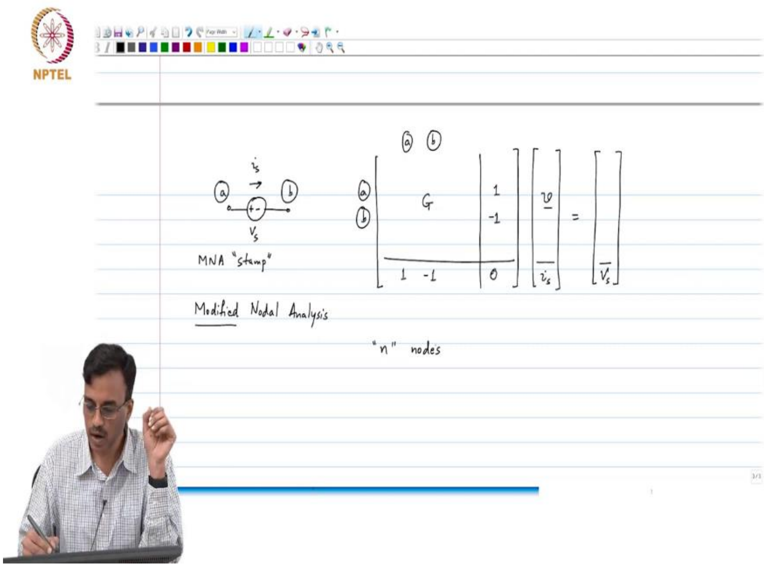

And, likewise so, that the stamp of a voltage source we had nodes a and b is the following and that is you had the a-th row and the b-th row as well as the a-th row and the b-th column and you add an extra. So, this is v s, this is the current extra variable I s and with the augmented conductance matrix what you do is, you put in a 1 and a minus 1 right between the nodes a and in the a-th and b-th rows that corresponds to the accounting for the current that is flowing through the voltage source.

And, you put in one and minus one for in the a-th and b-th columns in the last row and this matrix is going that entry is going to be 0. So, this is the node voltage vector, this is the original conductance matrix of the network stripped of all current sources' independent current sources and a voltage sources. And, then you have this new variable I s and this is equal to what do you have?

What should you have here? Here you have all the independent current sources and then now you have? V s, right. So, the way this method of accounting for voltage sources or in other words accounting for branches where the branch current cannot be expressed as a linear function of the branch voltage, right, is basically this is an extension of the nodal analysis that we know already and this is what is called the modified nodal analysis.

And, as we have seen so far you know if we have networks with conductance current sources and voltage sources, I mean this the algorithm for writing these matrices is very clear, right. You just make a you describe the net list right and in a computer, you know you have a parser which takes a circuit diagram that you draw and generates a spice net list and you keep going line by line in that net list and you look at the element and you can simply plop it is respective MNA stamp is what it is called right into the matrix.

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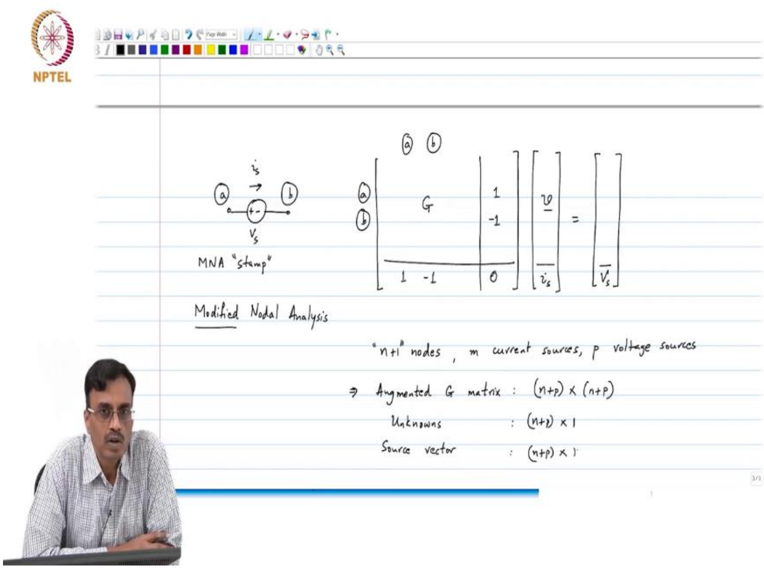

So, if you have n nodes right in the network or you let us say I will deliberately choose n plus 1 nodes in the network ok and m current sources and p voltage sources what comment can you make about the size of the augmented G matrix?

Well, you have n plus 1 nodes in the network, one node is ground we have n other nodes whose voltages you need to find. So, you will have n unknowns as far as the node voltages are concerned you have p voltage sources. So, out of p voltage sources?

The currents through the p voltage sources are all unknowns. So, the total number of unknowns is? n plus p. So, the augmented G matrix is what is the size of the matrix? n plus p cross n plus p. The unknown form a what vector how many unknowns do we have? n plus p. So, this must be an n plus p cross 1 vector and the source vector obviously, better be also. n plus p cross 1, correct and where is the source vector? The lower three I mean the lower p entries will be will correspond to? The strengths of the voltage sources.

Augmented G matrix : $(n + p) \times (n + p)$

Unknowns : $(n + p) \times 1$

Source vector : $(n + p) \times 1$