

Signal Processing Techniques and Its Applications
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Lecture - 20
Properties of Discrete Fourier Transform

So, we have discussed about how you compute the DFT; given an example, a real-life example is also given. Now, today, we will talk about the properties of DFT, properties of DFT, properties of Discrete Fourier Transform. There are several properties of discrete Fourier transform, any transform.

So, by the way, what is a transform? We said Fourier transform, Z transform, and Laplace transform. So, transform means the change of the orientation, or I can say the signal is there. So, you want to change it to look at the inside of those things. So, suppose I have this one, this object. Let us say it is this, or this portion is visible to me, but this portion is not visible to me. Now, this portion I cannot see it.

So, let us say similarly, I have a signal; let us I can see the signal in this space. So, I want to know the signal to see other spaces also where the signal exists. So, I want to see this glow in this object and surface. So, what do I have to do? I have to make a transformation. What transform? I have to rotate it.

So, this rotation is a transformation. So, in mathematics, I can say I have a signal in one coordinate, one system or one dimension. I have to change the signal in another dimension that is transformed right. For example, I have a signal in the time domain; I have to transform the signal in the Z plane, Z domain, and Z dimension, which is a complex Z transform dimension.

So, I have a signal in the time domain and have transformed it to the Fourier domain. So, that is the Fourier transform. So, every transformation is nothing but a you can as system. I can apply a system. I give an input; I take transform output. So, I have a system. Transformation is nothing but a system. So, I have the properties of that system. So, what are those properties?

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Periodicity

Periodicity: if $x[n]$ and $X[k]$ are an N -point DFT then (N)

$x[n+N] = x[n]$ for all n

$X[rN+k] = X[k]$ for all k where r is an integer

$$X(rN+k) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi n(rN+k)/N}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j2\pi nr} e^{-j2\pi nk/N}$$

$$= X(k)$$

$e^{-j2\pi} = 1$

$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}$

$r = \lfloor \frac{rN+k}{N} \rfloor$

$k = rN + k$

The first thing that comes to mind is properties are called periodicity, and the transform domain is also the periodicity that exists. So, what is the transform domain periodicity? So, if I have a signal $x[n]$. So, I have a signal $x[n]$. I have taken N point DFT, and I got $X[k]$. Then, I said that this length of the DFT is nothing but a definition of the period, so that means $x[n+N]$ should be equal to $x[n]$ for all n .

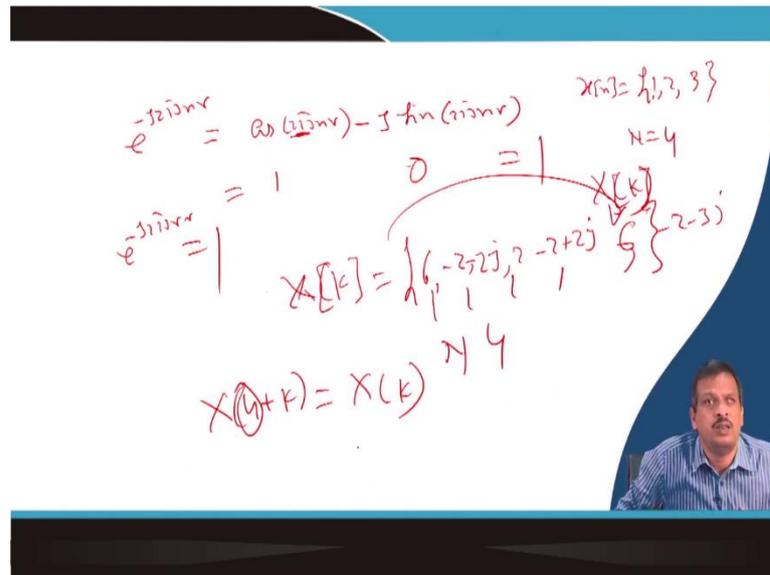
So, N is the period. Similarly, $X[rN+k]$; r is an integer. So, I will take that $X[k]$. So, $X[rN+k]$ should be $X[k]$; r equal to 1; that means $X[n+k]$ should be equal to $X[k]$; that means N is periodic, N is the period of both $x[n]$ and $X[k]$. How do I prove it? Very simple. So, $X[rN+k]$ is nothing but an $x[n]$. So, I change the dimension rN plus k is replaced by r of capital N plus k . So, you know that $X[k]$ is

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j2\pi nk/N}$$

Here, instead of k , I replace it with rN plus k . So, k is replaced by rN plus k . Now, if I do it. So, it is nothing but a. So, it is $j2\pi n$ into r and N into $e^{j2\pi nk/N}$. So, this product is this part, and this product is this part because $e^{-j \frac{2\pi rN \cdot n}{N}}$, capital N will be cancel. So, it is nothing but an $e^{j2\pi nr}$. Now, if you look at n as an integer and r as an integer, if I say

$$e^{-j2\pi} = \cos(2\pi) - j \sin(2\pi)$$

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So, what I said? I said, what is the value of $e^{-j2\pi nr}$; $e^{j2\pi nr}$ where n is an integer and r is an integer? It is nothing but a $\cos(2\pi nr) - j\sin(2\pi nr)$. If n and r are integers, then I can say it is a cos integer multiple of 2π . So, I can say this is nothing but a 1, and this is nothing but a 0.

So, it is nothing but a 1. So, I can say e to the power minus $j2\pi nr$ is equal to 1. So, I put it here $e^{-j2\pi nr}$ is equal to 1. So, I can say it is n equal to 0 to n minus 1, $e^{-j2\pi nk/N}$, which is nothing but $X[k]$. So, I can say if N is the N point DFT, n is the length of the DFT, then N is the period of $X[k]$ and also signal $x[n]$. For a practical example, we have seen that if I take a signal, $x[n]$ is equal to 1, 2, 3, and if I compute N equal to 4 points DFT, I get $X[k]$. You can see that $x[n]$ is repeating.

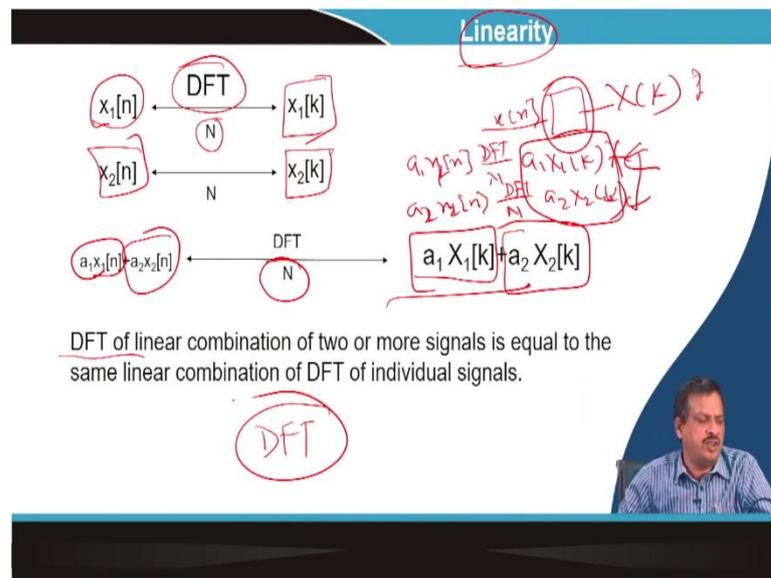
So, if I say that X by $X[k]$, X of $4k$, we have already seen it $X[k]$ we have seen for this one, 6 $-2-2j$, 2 then $-2+2j$ then again comes 6. So, 0, 1, 2, 3. So, this is 4 points, N equal to 4. And again, it will be repeated to the 6 I have shown you. You can do it also. Take this and see if the 6 will come, then minus 2, minus $2j$ will come. So, I can say at n $X[k]$, 4 plus k is equal to again $X[k]$; then I can say that 4 is the period N equal to 4 is the period. You can do it. Do it yourself.

You can compute it and see whether this exists or not mathematically. This is proof here, which I have already explained. So, the first property is the periodicity. What is the meaning? This means that if I compute N point DFT, then N becomes my period of the

frequency domain signal and also the period of the time domain signal. If I put paired up with 0. So, the 0 will be included, and it becomes the periodic signal with capital N.

So, what about the signal we have taken? We will assume that while we are calculating DFT in the frequency domain, once you do the DFT, the N becomes the period for both the signal, time domain signal and frequency domain signal. So that is called periodicity.

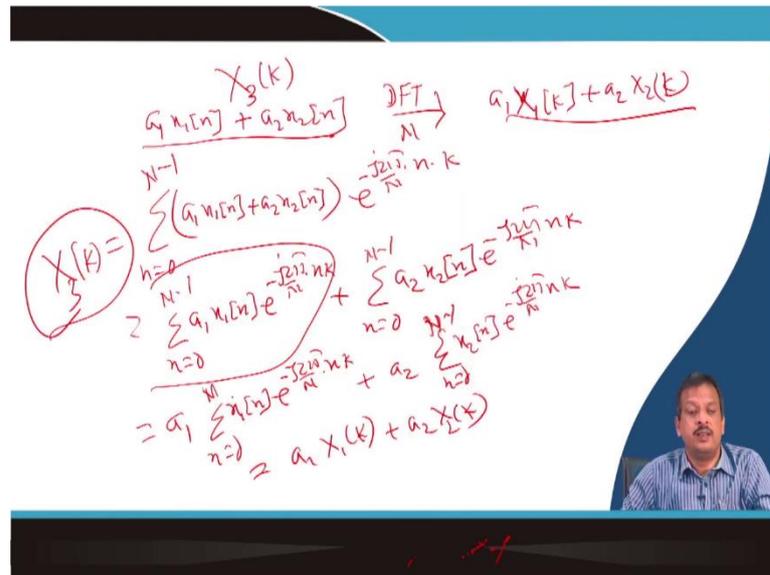
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Now, we can go for another property, which is linearity. Again, DFT is a transformation, so I can say transformation means nothing but a system. So, whether I applied $x[n]$, I got $X[k]$. So, it is nothing but a system. So, a system has a property. The first property I have said is periodicity, and the second property is linearity, whether the DFT transform is linear or not. Forget about the slide. If I told you whether a system is linear or not, how do we test it?

When I say the system is linear, that means it supports the superposition principle. So, if a system supports the superposition principle, then I said the system is linear. So, let us say I have an $x_1[n]$, whose N point DFT is $x_1[k]$, and $x_2[n]$, N point DFT is $x_2[k]$. Now, if I apply a signal that is $a_1x_1[n]$ plus $a_2x_2[n]$, And if I take the same N point DFT, then I should get $a_1X_1[k]$ plus $a_2X_2[k]$. If it is I get, then I said it supports the superposition principle, so that means I have taken $a_1x_1[n]$ DFT of N point, I get $a_1X_1[k]$, I taken a $a_2x_2[n]$ $x_2[n]$, I have taken N point DFT, I get $a_2X_2[k]$.

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Now, if I take a signal that is $a_1x_1[n]$ plus $a_2x_2[n]$, should I take the same DFT of N point? Should I get $a_1X_1[k]$ plus $a_2X_2[k]$? If I get it, then I can say this is equal to this plus this. So, it supports the superposition principle. So, what is the DFT of this signal? Let us calculate. So, $X_3[k]$.

$$X_3[k] = \sum_{n=0}^{N-1} (a_1x_1[n] + a_2x_2[n]) e^{-j\frac{2\pi}{N}nk}$$

This is the signal multiplied by $e^{-j\frac{2\pi}{N}nk}$ ok. Now, I can say this is

$$X_3[k] = \sum_{n=0}^{N-1} a_1x_1[n] e^{-j\frac{2\pi}{N}nk} + \sum_{n=0}^{N-1} a_2x_2[n] e^{-j\frac{2\pi}{N}nk}$$

So, your a 1 is a constant a 1 is come outside.

So, I can say n equal to 0 to N, $x_1[n] e^{-j\frac{2\pi}{N}nk}$ plus a 2 comes outside n equal to 0 to n minus 1 $x_2[n] e^{-j\frac{2\pi}{N}nk}$. So, if this is nothing but $a_1X_1[k]$ plus $a_2X_2[k]$. So, I can say this is the x_3 . So, x_3 is nothing but a summation of individual Fourier transform. So, it supports the superposition principle. So, I can say DFT is a linear transformation. Discrete Fourier transform is a linear transformation. So, the linearity property is whole. So, it is not a non-linear transformation. It is a linear transformation.

So, what property have I proved? Periodicity and Linearity. So, what is the application of linearity property? What is the application of linearity property? So, suppose I have a two

signal, I have a two signal, the addition of the if I take the addition of the two signal, then take the DFT. And if an individual signal makes a DFT and then adds up, both will be the same. I have two signals; if I add up, then take the DFT, and if I have an individual signal, I take two times. If I compute the DFT and then add it up, both will be the same.

So, that is the linearity property, ok. I can use it, and periodicity. I know I have to know only up to n. Once I know x[n], they start just periodic, just repeating, ok? So, that is the periodicity.

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DFT symmetry

Circular Symmetries of a sequence

N-point DFT of a finite duration sequence $x(n)$ of length $L \leq N$ is equivalent to the N-point DFT of a periodic sequence of period N, which is obtained by periodically extending $x(n)$

The shifted the signal k units $\rightarrow x_p(n-k) = \sum_{m=-\infty}^{\infty} x(n-k-lN)$

New finite-duration sequence $x_s(n) = \begin{cases} x(n-k) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$ which is related to the original sequence $x(n)$ by a circular shift

$x_s = x(n-k, \text{ modulo } N) = x((n-k)_N)$

For the circular shift of $x_s(n)$ with $k=2$ and $N=4$, we obtain $x_s(n) = x((n-2)_4)$ and, thus,

$x_s(0) = x((-2)_4) = x(2)$ $x_s(1) = x((-1)_4) = x(3)$
 $x_s(2) = x((0)_4) = x(0)$ $x_s(3) = x((1)_4) = x(1)$

$x_s(n)$ is simply $x(n)$ shifted circularly by two units in time

Handwritten notes: $x(n) = 1, 2, 3, 3$, $N=4$, $L \leq N$, $x(n+N) = x(n)$, $1, 2, 3, 0, 1, 2, 3, 0$, $0, 2, 3, 0$, $x_s(n) = 1, 2, 3, 0$



The next property is DF symmetry, which is a very important property of the DFT. So, if I say N point DFT of a finite duration sequence on x[n] of length L is less than N, what do I say? I have a signal x[n], let's say, which is nothing but a 1, 2, 3, 3, and I want to compute DFT length N equal to 4.

So, the length of the signal is L. L is less than equal to N, which I have discussed, and is equivalent to the N point DFT of a periodic sequence whose period is N, which is obtained by periodically extending x n; that means how do I mathematically write this sentence? x p n is nothing but a l equal to minus infinity to infinity x[n] minus lN. What is the physical meaning of it?.

Physical meaning is that, as you know, the periodicity of the DFT $x[n+N]$ is equal to $x[n]$, and we have proved N is that N point DFT. Suppose I have a signal length of 3; I want to

make it 100 in length, or I want to make it, let us say, a length of 9 or length of 12 I want to make it. How do I make it? It says to consider $x[p]$, whose length is 4, and repeat that part again and again and again.

So, I can say $x[p]$ plus $x[p]$ and $x[p]$. I get 12 samples. So, I have 4 samples: 1, 2, 3, and 0 because my l is less than, and so, I have to put a 0 there. Now, I can repeat it 1, 2, 3, 0. I can repeat it. So, for 1, 2, 3, and 0, I got 12 samples $x[n]$. So, $x[n]$ is nothing but a repeating of every l . It can be infinite also, so minus infinity to plus infinity with the integer multiple of capital N .

Now, interesting things; there we said circular symmetry, no. So, now, interestingly, if I say if I shift, the signal is k unit. So, let us see this signal is shifted $x[p]$ is equal to n minus k . So, if it is n minus k . So, n is replaced by n minus k . So, $x[n]$ minus lN . Let us say I have a finite duration sequence, which is $x[s]$ is equal to $x[s]P$ 0 to N minus 1; otherwise, 0; $x[s] n$.

So, $x[s] n$ is nothing but a $x[n-1]$ modulo in $x p, x p, x p, x p, x p$ repeating. So, it is nothing but a $x[n]$ minus k modulo N . For example, let us say for a circular shift $x 3 n$ with k equal to 2, N equal to 4. So, $x 3 n$ if I see $x 3 n$. So, $x[s] n$ is nothing but a $x[n-2]$ into 4. So, capital N is equal to 4 here, and the shift is 2.

So, shift L is k equal to 2 and N equal to 4. So, this is my equation. So, $x s 0$, small n equal to 0 is equal to x of minus 2 modulo 4 is equal to $x 2$; $x[1]$ is x of minus 1 modulo 4 is equal to $x 3$, N equal to 2, $x[2]$ is equal to $x[0]$ which is nothing but a $x 0$ and $x[3]$ is nothing but a $x 1$. Now, if you look at how it is circularly happening,

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An N-point sequence is called circularly even if it is symmetric about the point zero on the circle

$$x(N-n) = x(n)$$

An N-point sequence is called circularly odd if it is antisymmetric about the point zero on the circle.

$$x(N-n) = -x(n)$$

The time reversal of an N-point sequence is attained by reversing its samples about the point zero on the circle.

Circularly Even Periodic: $x_p(n) = x_p(-n) = x_p(N-n)$

Circularly Odd Periodic: $x_p(n) = -x_p(-n) = -x_p(N-n)$

Conjugate Even Periodic: $x_p(n) = x_p^*(N-n)$

Conjugate Odd Periodic: $x_p(n) = -x_p^*(N-n)$

$f(n) = f(-n)$

x_p x_p^*

So, if I say I have a circle, ok. Let us say I explain it by taking an extra slide here.

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$x_s[0] = x[(0-2) \bmod 4] = x[2]$

$x_s[1] = x[(1-2) \bmod 4] = x[3]$

$x_s[2] = x[(2-2) \bmod 4] = x[0]$

$x_s[3] = x[(3-2) \bmod 4] = x[1]$

$x_s[0] = x[2]$

$x_s[1] = x[3]$

$x_s[2] = x[0]$

$x_s[3] = x[1]$

So, what am I saying? I take a circle. So, this is $x[0]$, this is $x[1]$, this is $x[2]$, this is $x[4]$. Now, I said $x[s]$ modulo 4. So, $x[s] \bmod 4$, $x[s] \bmod 4$ is equal to $x[n-2]$, shift is 2, 2 time shifting modulo 4 ok. Now, I said, what will be the signal $x[0]$? n equal to 0. So, n equal to 0 means $x[0-2] \bmod 4$. So, it is nothing but a x of minus 2 mod 4.

So, I have only 4 sample x of minus 2 mod 4 x of minus 2 mod 4 is nothing but a $x[2]$. So, I shifted the signal. So, if I circularly shifted 2 times the signal $x[0]$ the $x[s]$ 0, the shifted

signal first sample will be the second sample. So, it is anti-clockwise shifted 2 times. Let us say I want to shift in 1 times $x[s]$ n is equal to x of I shift only 1-time $x[n-1] \bmod 4$.

So, once I say, shifted signal $x[s]$ 0 is equal to x of minus 1 mod 4. So, I want 1 time shift. So, it is nothing but a $x[3]$ sorry 1, 2, 3 sorry $x[1]$ will become 1 time shift, x of this will become plus 1 minus 1 mod minus 1 mod means $x[3]$ here, $x[3]$. In this case, if I say what should be the $x[0]$, $x[1]$? $x[s]$ 1, $x[s]$ 1 is equal to $x[1-2] \bmod n \bmod 4$.

So, it is nothing but a x of minus 1 mod 4, which is nothing but a $x[3]$, $x[3]$. So, again if I shift it. So, if I shift 2 times, this sample will come to here, this direction 2 times. So, this direction 2 times I shifted, then x_0 becomes x_2 , x_1 becomes x_3 , then x_2 becomes what? What is $x[s]$ 2? It is nothing but a $x[2-2] \bmod 4$, which is nothing but a $x[0] \bmod 4$, which is nothing but a what? $x[2]$. So, this will be 2 times shifted to $x_0, 1, 2$. So, anti-clockwise 2 time circular sheet.

So, if I say that circular symmetric open sequence if I have a sequence $x[n]$, if capital N is my DFT, then I can generate the sequence by just circular shifting. Shift 2 means circularly anti-clockwise 2 times shifting, shift 3 means anti-clockwise 3 times shifting, shift 1 anti-clockwise 1 time shifting. So, that is called the circular symmetry of a sequence; then I will come to the circular symmetry of the frequency domain signal. So, those are the properties. So, circularly, even periodic $x[p]$ n is equal to x . You know that even function and odd function.

So, even if f of n is equal to f of minus n , then I said it is an even function. So, $x[p]$ n $x[p]$ minus n equals x p N minus 1, n ; circularly odd periodic odd signal minus conjugate even conjugate signal. If x p is a real complex signal. So, what are the conjugate signals? x p star and conjugate odd periodic so that you can prove it also ok. So, that is the third property symmetry property of the sequence. Then, there will be a symmetry property of the frequency domain signal, which I will discuss next week.

Thank you.