Signal Processing Techniques and Its Applications
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Lecture - 17
Discrete Fourier Transform (DFT)

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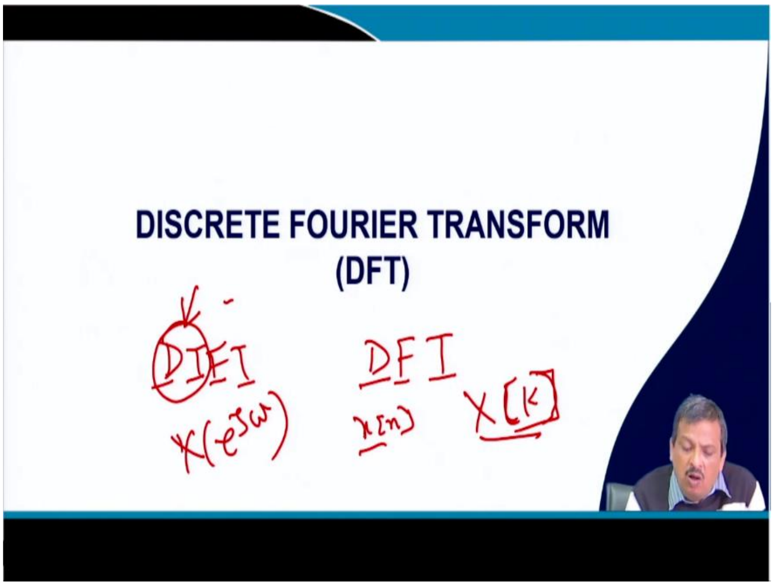

So, in the last class, we talked about the discrete-time Fourier transform. Now, we will talk about the Discrete Fourier Transform.

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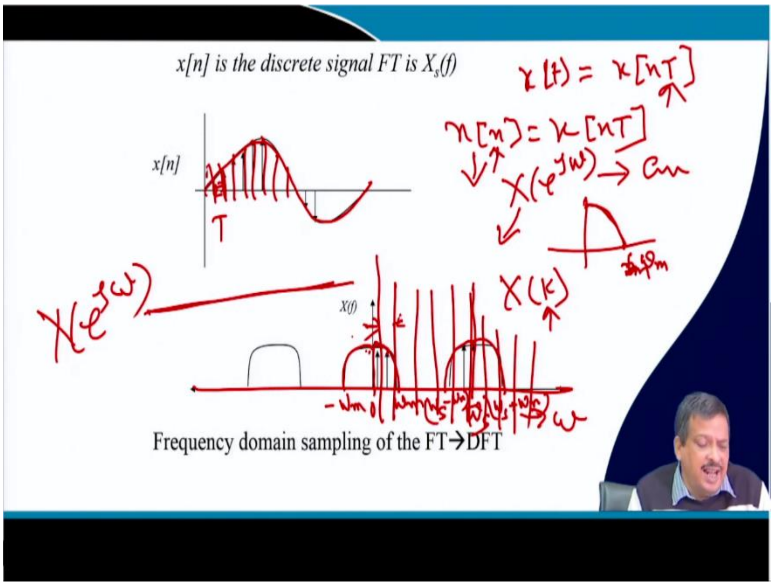

So, this class will talk about the discrete Fourier transform. So, what is the difference between DTFT and DFT? Here, I said discrete-time Fourier transform, signal in time domain signal is in discrete form, but the frequency response is h or X of $e^{j\omega}$ continuous, but here I am saying discrete Fourier transform. Here, both the signal and frequency domain representation are discrete.

So, if I say both are discrete, I know how to discretize the analog signal. So, suppose I have a signal x(t); you know that x(t) is discretized by x[nT] and that capital T is called the sampling period. So, I have a continuous signal, I discretize the time instant and the difference is called T. So, that is why I said x[n] is equal to x[nT]. Ok, the discrete signal I know, and if I take the Fourier transform of x[n], I get X of $e^{j\omega}$, which is continuous in the frequency domain.

Frequency is continuous, and frequency is not discrete. So, as you know, suppose I have a signal like this, this is my signal frequency response, let us say this is my signal frequency response, this is fm maximum frequency, let us say, or this is ω, m maximum frequency contained in the signal.

If I take the frequency response, I know it will look like this: this is ωm, and this is 0, this is minus ωm, this is ω axis. Similarly, here, let us again say this is f s ωs, which is nothing but f s is equal to 2π , ω is equal to 2π here. Then again, I get the same response: This is ωs plus ωm, and this is ωs minus ωm.

So, we get that kind of frequency response. Now, what do I want? I want discrete in frequency domain also. So, X of $e^{j\omega}$ has to be discretized with X(k). So, k is called discrete frequency like n; as I said, n is called index. So, I said I had taken the sample time. I have sampled the signal with a t interval of the time and each of the intervals is represented by an index, which is called n.

So, x[n] is called a discrete signal in the time domain. Now that I have an X of $e^{j\omega}$, I want to sample the 18 frequency axis also. So, I have an ω axis. So, this is my ω axis; I want to take the sample here also. So, how do I do that? Here, I said T is the sampling period. Here, I also have to define certain T.

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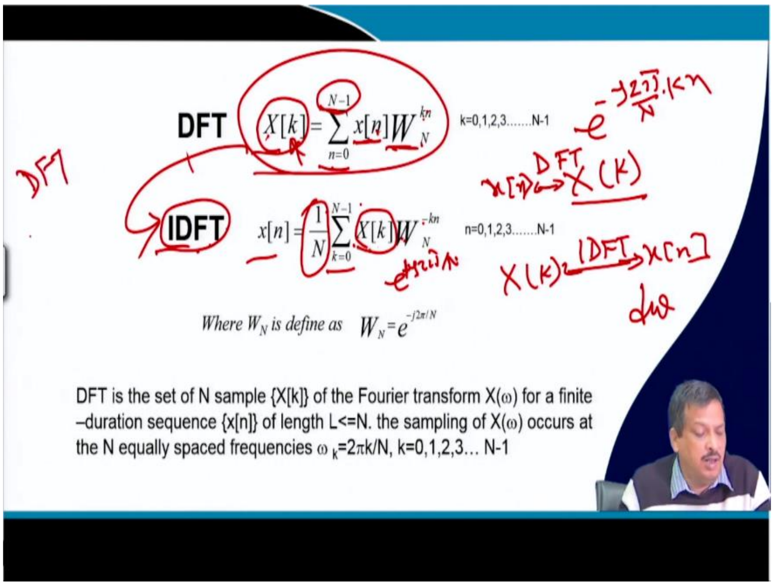

So, as I know forget about this slide.

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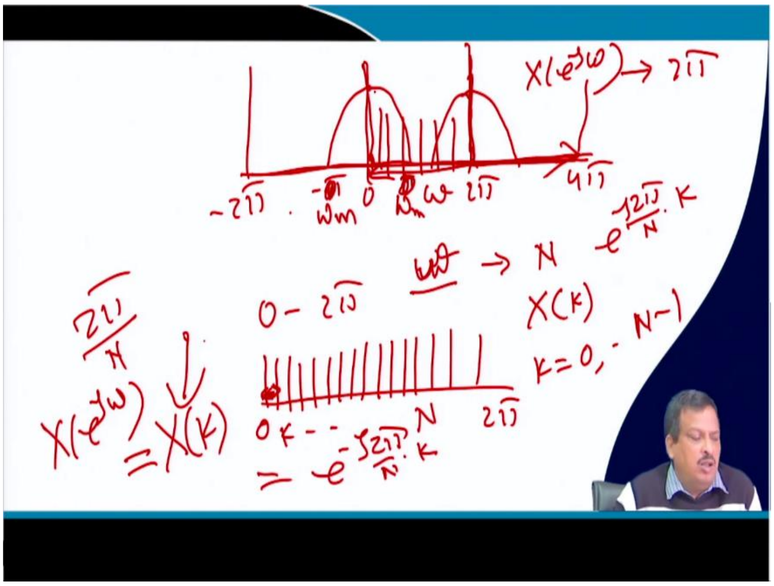

This is my ω axis, and I know if this is 0, this is 2π, and if my ωm is the maximum frequency, then the signal represented frequency is represented in this way. So, X of $e^{j\omega}$ is my. This is the X of $e^{j\omega}$. This is π, this is minus pi; let us say sorry, this is ωm and this is minus ωm and this side also there will be a minus 2π ok.

So, this is 2π. Again, it will be repeating here 2π, this is 4π. So, the signal is repeating $e^{j\omega}$ has a period of 2π. So, if it is $e^{j\omega}$ has a period of 2π, then if I want to make it discretize, I have to consider this part only sufficient and then the next part will just repeat.

So, I want to discretize this one 0 to 2π, the variation of this ω. Let us I want to discretize in an N number of discrete values. So, I have a 0 to 2π, and I divided it into N number of discrete values. So, instead of writing X to the power $e^{j\omega}$ I want to write X(k) where k varies from 0 to N minus 1, N number of division I have made. Now, what is the value of each division? The total value is 2π and N is the number of divisions I made.

So, the total length of each division is nothing but a 2π/N. So, I can say $e^{j\omega}$ is nothing, but it will be j 2π. So, 2π/N. Now, what is the index varying? k multiply by k. So, instead of writing X to the power $e^{j\omega}$, I have discretised that which is X(k), which is nothing but an $e^{-\frac{j2\pi}{N}k}$, and then I said X k is also discrete. So, I can say discrete Fourier transform. Then what should be the expression?

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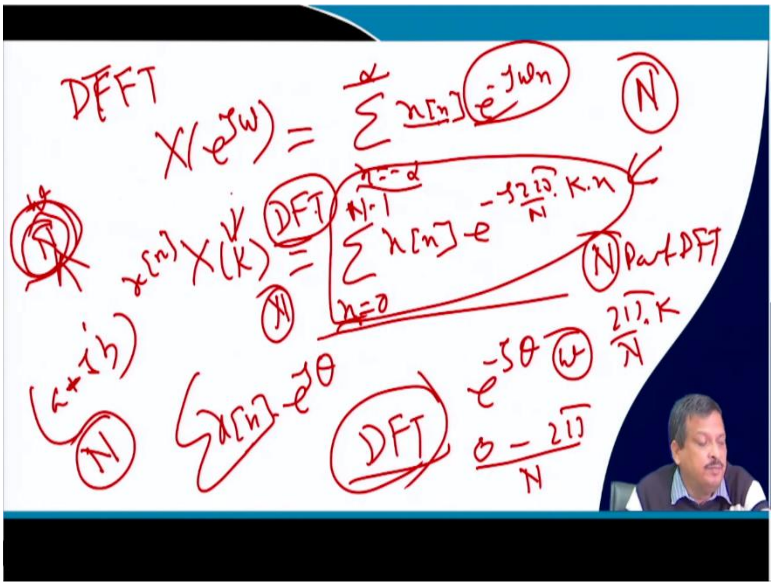

Forget about this part; so, what is the expression of DTFT, discrete-time Fourier transform? I know X of $e^{j\omega}$ is nothing but an N varies from minus infinity to infinity x[n] $e^{j\omega n}$; this is my discrete-time Fourier transform.

Now, what have I done? I have divided this ω the scale with an N number of points. So, that is called the length of DFT, the length of discrete Fourier transform. So, I can say X(k) is equal to, so I said n varies from 0 to x[n] $e^{-\frac{j2\pi}{N}kn}$, 2π/Nk is the ωn.

Now, what I am considering here? I am considering the signal to be periodic at N point because when I divided the frequency scale, I said 0 to 2π, 2π is the period. So, I am considering that in the frequency domain, the signal is forcefully periodic at 0 to 2π. So, N is the period. So, this is the expression for DFT, discrete Fourier transform. The consideration is that whatever I said, the N is the period of the signal.

Even if the signal is aperiodic, I can say forcefully N is the period of the signal. And that N is called N point DFT. So, when I say N point DFT, that means I am dividing the frequency scale into N points; N is the length of DFT or DFT length. Is it clear? So, discrete Fourier transform will be defined by X(k), n equal to 0 to say it will be N minus 1, which we got from 0 to N minus 1.

So, n is equal to 0 to N minus 1 x[n] ω N kn, which is nothing but an $e^{-\frac{j2\pi}{N}}$, N number of division into k, k is the index and multiplied by the n index. So you do not have to remember it. You can derive it using logic but do not remember it. If you remember it, you will forget. So, think about I, what is the basis of the frequency transform? This logic, you remember, is that I have a signal, and the frequency transform means the signal has to be convolved with a sinusoidal component.

So, how do I represent the sinusoidal component? $e^{j\theta}$, it has to be $e^{j\theta}$ and has to be convolved; convolved means a summation sign is required. So, that is why n is equal to minus infinity to infinity, x[n] $e^{j\omega n}$.

Now, when I say discrete Fourier transform, I say discrete Fourier transform; that means the frequency domain. also, I am discretising the signal. So, $e^{j\omega}$, ω has to be discretized. How do I discrete? I have to sample it. So, how do I sample it? I said 0 to 2π is the frequency range, and I have divided this range by N number of samples. So, I can say 2π/N is the width of each of the sample widths, and the index is k.

So, I can generate any sampling frequency using the index X(k); I just replace those things with that, and I get the discrete Fourier transform ok. Then what is the inverse discrete

Fourier transform, IDFT inverse discrete Fourier transform; because I have I get X(k) when I x[n] DFT I get x DFT I get X(k).

Now, I can say X(k) IDFT inverse discrete Fourier transform I should get x[n]. So, what is x[n] equal to 1 by N k equal to 0 to N minus 1? Now, X(k). So, it will be d ω. So, d ω is nothing but an X(k) concerning $e^{-\frac{j2\pi}{N}}$. So, if it is minus, then minus minus, it will be plus. So, it will be plus 2π N by kn.

So, in summary, what can I say? The equation of a DFT, DFT equation is ok I will take another slide.

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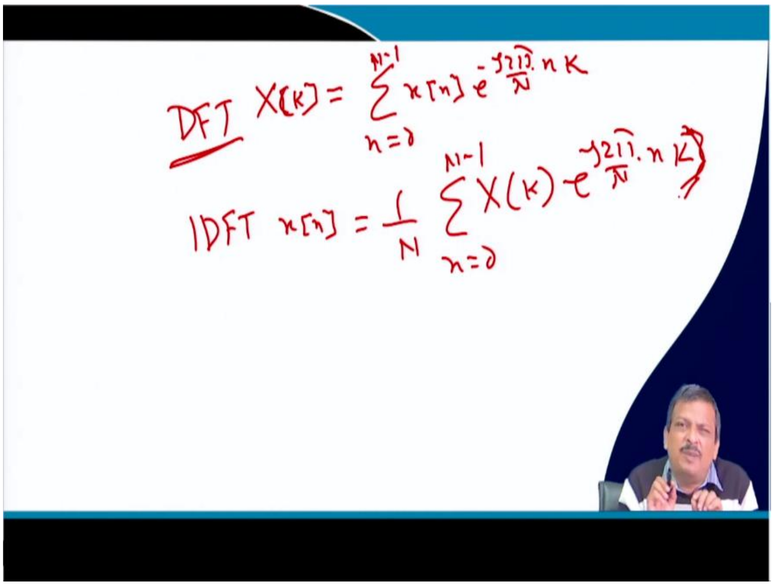

So, in summary, you have what you have to remember, is that? I said DFT equation X(k) equals n equal to 0 to N minus 1, x[n] $e^{-\frac{j2\pi}{N}kn}$. What is the IDFT? What do I get for DFT discrete time discrete Fourier transform, IDFT? I get x[n]. How do I get? 1 by n This is normalization, n equal to 0 to N minus 1 input is X(k) $e^{-\frac{j2\pi}{N}kn}$.

So, this is the IDFT, ok? So, let us give some examples, and then we will come back to the properties of things. Let us give some examples here; here, I can take some examples.

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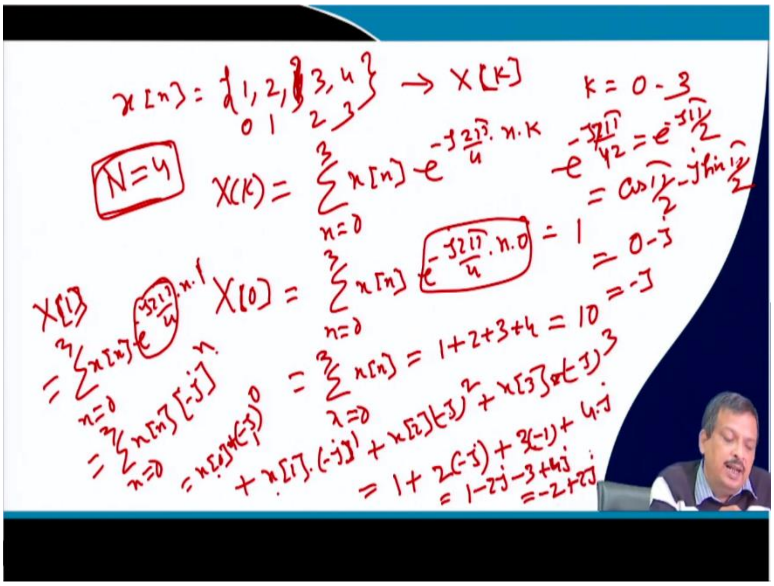

Let us say I have a signal x[n] that is equal to, let us say, 1, 2 or let us say, 4 point DFT. I said, forget about this 1, 2; let us say 3 and 4. Now, I want to take 4 point DFT, N equal to 4; I said N equal to 4; the length of DFT length is 4. So, I want to calculate X(k). So, what is the index X(k)? k varies from 0 to N minus 1. So, here N minus 1 is 3, and the signal varies from the 0th sample, 1st sample, 2nd sample, and 3rd sample, 0 to 3.

So, what is X(k)? X(k) equals n equal to 0 to 3 x[n] $e^{-\frac{j2\pi}{4}\cdot n\cdot k}$. So, I can say what X[0] is, capital X 0? First frequency component, first frequency initial component of that frequency part. So, what is there? n equals 0 to 3 x[n] $e^{-\frac{j2\pi}{4}\cdot n\cdot k}$. So, n is there, and k is equal to 0. So, I can say this whole part is equal to 1. So, it is nothing but a n equal to 0 to 3 x[n].

So, it is just a sum; you can say the 0th sample is 1 plus 2 plus 3 plus 4. So, it is nothing but a sum of the sample. What is the meaning? This means this contains the DC part of the signal; no frequency and no e to the power complex part is there. So, this is only the DC part of the signal.

So, I can say 3 3 6 plus 4; 10, then what is the X[1], is nothing but a n equal to 0 to 3 x[n] $e^{-\frac{j2\pi}{4}n}$. Now, what is e to the power? See minus j 2π by 4 equals nothing but an $e^{-\frac{j\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$

So, what is the value of cos π by 2? Cos 0 1, cos π by 2 is 0, sin π by 2 is 1. So, it is nothing but a minus j. So, I can say it is nothing but a minus j. So, instead of writing this part, I can say it is nothing but a n equal to 0 to 3 x[n] minus j to the power n, or I can say minus j to the power n k. Because when I say k equals to so, k equals to 1 here.

So, it is nothing but a

$$x[0] \cdot (-j)^0 + x[1] \cdot (-j)^1 + x[2] \cdot (-j)^2 + x[3] \cdot (-j)^3$$

So, it is 1x0, so 1 plus x1 is 2 and minus j plus x2 is 3 into minus j whole square. So, it is nothing but a minus 1 plus 4 into j whole square minus j. So, it will be a whole cube minus minus plus j. So, I can say it is 1 - 2j - 3 + 4j. So, I can say it is nothing but a - 2+ 2j. So, you can say X[1] is a complex number.

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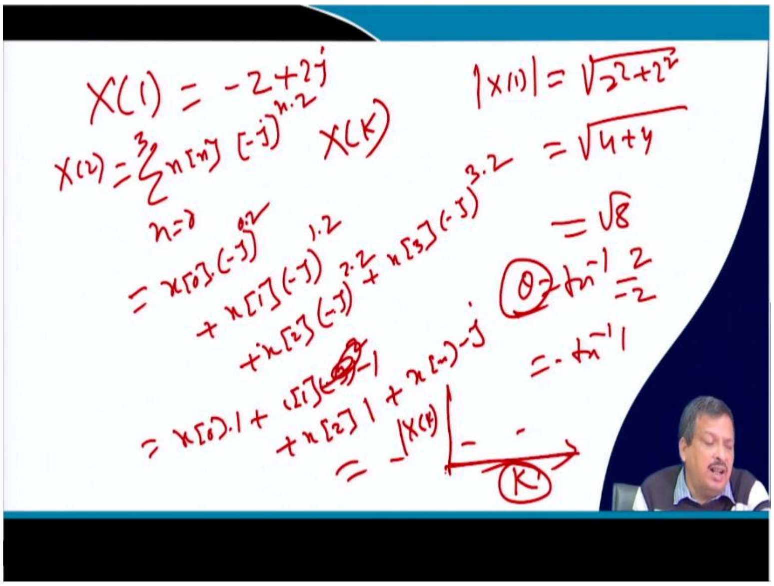

So, X[1] is a complex number, which is minus 2. Sorry, minus 2 plus minus 2 plus 2 j is a complex number. So, I can say, what is the mod of X[1], which is amplitude spectra? Root over of 2 square plus 2 square, I can say the value is 4 plus 4 root over of 8 ok. So, what is the θ? θ is equal to tan inverse 2 by minus 2 is equal to minus tan inverse 1.

Similarly, you can calculate X[1] also. What is X[1]? Again, I can say n equal to 0 to 3 x[n]. So, I replace it with minus j; minus j to the power nk n into k equal to 2. So, I can say n is equal to 0 to 3. So,

$$X[0]\cdot(-j)^{2\cdot 0}+X[1]\cdot(-j)^{1\cdot 2}+X[2]\cdot(-j)^{2\cdot 2}+X[3]\cdot(-j)^{3\cdot 2}$$

So, I can say it is nothing but an X[0] into 1 plus X[1], which means ok X[1]. Let write X[1] minus j to the square minus j square means minus 1. So, I can say it is minus 1 plus X[1] into minus j square of minus j square. So, it is nothing but a plus 1 plus X[3] into minus j square. So, that means minus j, so like that, you can calculate.

So, I can calculate X[1], I can calculate X[3], I can calculate X[3]. So, I get the sequence X(k). If I want to plot, I can plot the magnitude spectra. This axis is the k, and this axis is the mod of X(k). I can plot it. Similarly, I can plot that θ part also with respect to k. So, k is discrete frequency ok.

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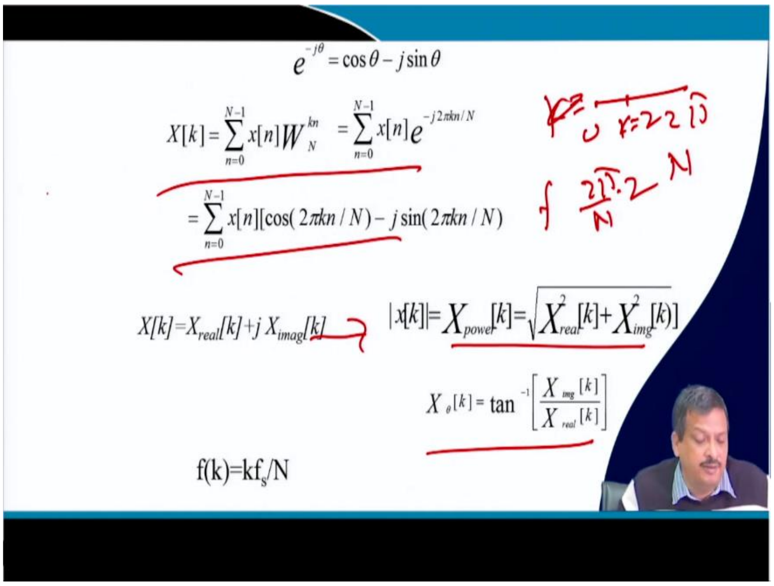

Now, what is the relationship between this discrete frequency and analog frequency? So, I can say this is that relationship. You can say this is the magnitude part. This is the θ part. So, what is the relationship? I have divided 0 to 2π in N number of points, and k is the index. So, it is k equal to 0. So, any point if k is equal to 2 here, what is the frequency?

Frequency is nothing but a 2π/N, which is the division into 2. So, suppose I told you I have a signal; let us take another slide to understand this part; you have to understand this part very clearly.

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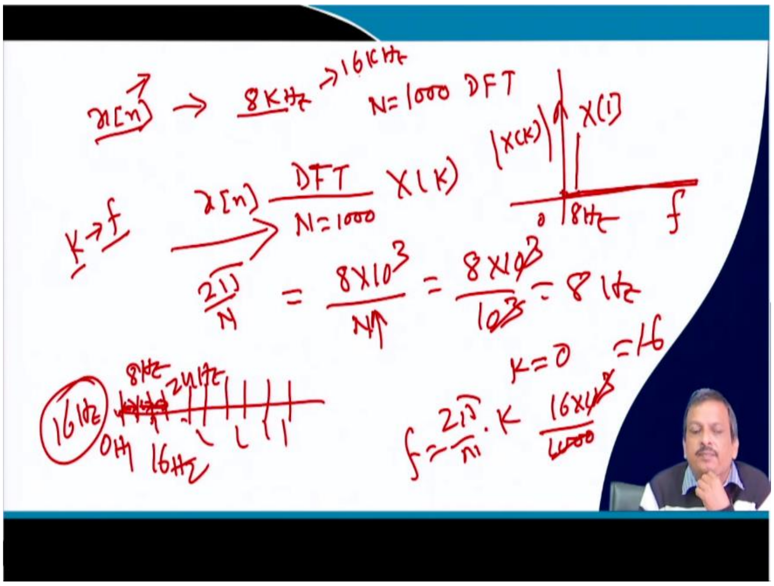

Suppose I told you a practical scenario: I have a signal x[n], which is a sample; the sampling frequency for the discrete signal is, let us say, 8 kilohertz. So, the signal's sampling frequency is 8 kilohertz, and I apply a 1000 point N equal to 1000 point DFT. So, I can say x[n] is done discrete Fourier transform, where N equals 1000, I get X(k).

Now, if I told you, can you plot this axis as the frequency f, and this axis is the mod of X k? So, what is the relationship between the k and f? Discrete frequency and the frequency component of the signal. Let us say x[n] has a frequency component; I want to know that.

So, if I say this axis is my f. So, what are the relations between the k and f? So, what I said? I have divided the maximum frequency by N number of samples. So, what is the maximum frequency? Is 2π. What is in hertz? It is nothing but a fs. So, I can say $8*10^{3/N}$ is the length of the division, the length of each division.

So, the length of each division is $8*10^{3/N}$, which is nothing but an $8*10^3$ divided by 10^3. So, it is nothing but an 8 hertz. So, 2π/N means 8 hertz. So, if I say k is equal to 0, that means 2π/N into k is equal to my f. So, k equal to 0 means 0 hertz, k equal to 1 means 8 hertz, k equal to 2 means 16 hertz, and k equal to 3 means 24 hertz.

So, I can say that I have divided the frequency scale by every interval of 8 hertz. So, when I plot k X k, k X 0, I get 0 hertz. When I plot k X, this is less; this is the plot of X[1] that is nothing but 8 hertz. So this is called frequency resolution. So, the frequency resolution of my discrete Fourier transform. So, this is my frequency resolution: 8 hertz. Let us suppose my sampling frequency is 16 kilohertz, and then what is the resolution here? 16k divided by 1000.

So, 16 hertz. So, my resolution is 16 hertz. So, as I see it, if I want to increase the resolution for a constant sampling frequency, I have to increase my N; N point DFT and the length of the DFT must be increased. Is it clear? So, this is the relations frequency resolution of a DFT discrete Fourier transform. If I ask you, suppose I said this is my DFT equation, where ω N is nothing but a collection. This one is my DFT equation. How much complex multiplication is required if the length is N?

So, $e^{j\omega n}$ is nothing but a cos 2π/N minus j. So, it is nothing but in the form of a minus jb. So, a minus jb into x[n]. So, how many? I have to n equal to 0 to capital N. So, I can say N number of complex multiplication is required for one frequency calculation. So, if k also varies in the dimension of n, I can say I am explaining a better way to take a new slide.

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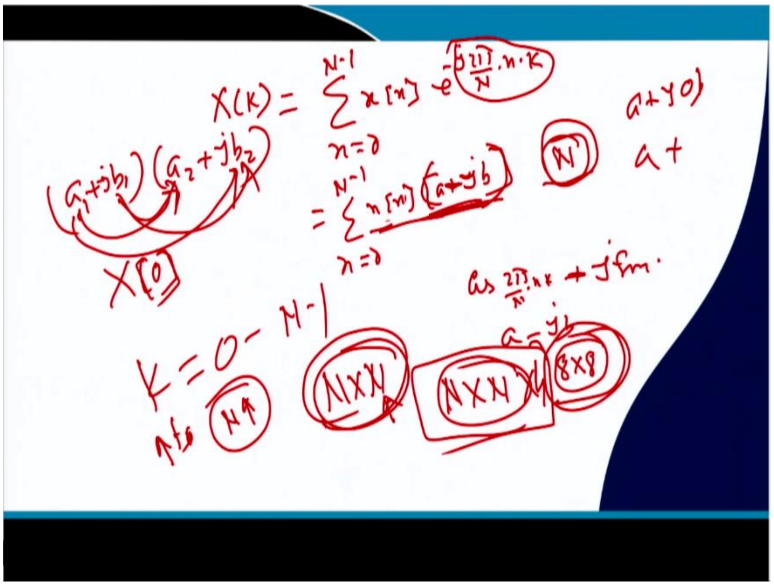

So, I am saying, suppose I said X(k) is equal to, you know, n equal to 0 to N minus 1 x[n] multiplied by $e^{-j\frac{2\pi}{N}\cdot Nk}$. You know this function is nothing but a cos 2π/N nk plus minus j sin. So, I can say it is a form of a minus j b. So, if I write down this one, n equal to 0 to N minus 1 x[n] into a plus j a minus j b.

So, I can say that to compute X for k equal to 1, let us k equal to 0; if I want to compute, then I require an N number of complex multiplication. For every k, or I can say, for every k, I require a N number of complex multiplication. x[n] may be complex or maybe real, which is a plus j0b, or does not matter. Or even x[n] can be a complex, and x[n] can also be a complex.

So, I can say whether the signal is real or complex; the number of complex multiplications for every k is N. Now, k also varies from 0 to N minus 1. So, for N number of k, I can say I required N cross N complex multiplication. So, to compute a DFT of length N, I required N cross N complex multiplication.

That is the complexity of DFT calculation. Multiplication and also summation: I have to sum it. So, that is the computational complexity of the DFT calculation. So, if it is 8 point DFT, I require 8 cross 8 complex multiplication. As you know, suppose I want to multiply a1 plus jb1 by a2 plus jb2.

So, if a multiplication of a2 complex number is required, how many real multiplications are required? a1 has to multiply with a2, a1 has to multiply with b2, b1 has to multiply with a2 and b1 has to multiply with b2. So, I required 4 multiplication and 4 real multiplication. So, I can say N cross N into 4 multiplication is required to compute a DFT. So, if it is 8 point DFT 8 into 4, complex multiplication is required, ok.

So, those are the complexities. Now, what I said? If I want to increase the frequency resolution f s or frequency resolution, I have to increase N. Now, if I increase n, The complexity will also increase. So, how do I reduce complexity to increase the frequency response and frequency resolution? Those are things we will discuss.

So, we will discuss the properties of DFT, the tradeoff between the frequency resolution and how efficiently we can calculate DFT without employing N cross N complex multiplication. That is, have you heard about that FFT, fast Fourier transform?

Fast Fourier transform is not a new transform. It is an algorithm that implements discrete Fourier transforms in an efficient manner. So, FFT is an algorithm that implements discrete Fourier transforms in an efficient manner. So, in the next class, I will talk about the properties of DFT, and I will talk about the time-frequency tradeoff between the frequency transform and the computational complexity of all those things we will discuss.

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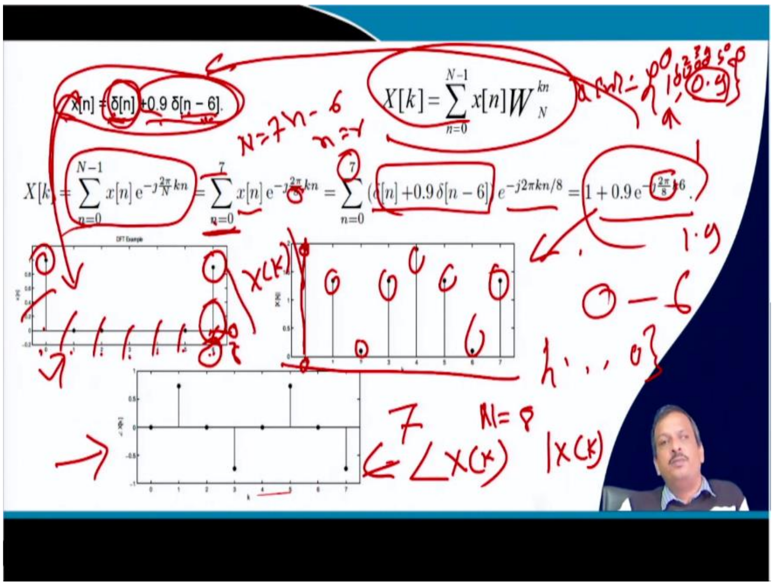

One thing, before I conclude, I just want to show you another example, which you may use in the form of like this: x[n] equal to δ[n] 0.5 δ[n]. So, instead of writing x[n] is equal to, you know, 1, 0.9, here is the 0th sample. I can also represent that in this way: δ[n]. So, the 0th sample is δ[n] 1 and n minus 6 sample. Sorry, there will be a lot of 0; 6th sample.

So, it is nothing but a 7th sample. So, it is 7 samples, n minus 6. So, n is equal to 6, it is 1. So, I can say it is 0 to 6, so 7 samples. So, I can say if it is 0th sample 1, 2, 3, 4, then 5, then 6, 0 and then the 7th sample is 0.9. So, for the DFT, I can apply the same formula again. So, here I know the length of DFT required; 7 samples are there. So, n is equal to 7. I will talk about that even if it is not an equal number, then also that can be calculated. So, next class, I will talk about 0 padding all those issues.

So, n is equal to 0 to 7 x[n]. So, I got 7 points, so I took 8 points DFT. So, if I take 8 points, the number of samples is 7, and I take N equal to 8, 8 point DFT, but here, the number of samples is 7, and the sample is there. So, to equalize it, we put 0 at the end of the sample, which is called 0 padding. We will talk about it.

So, I take the 8-point DFT. So, it is n minus 1 7, δ[n]; here, I can calculate. If I plot it, take the mod of X k; if you plot it, this is of the mod of X k, I think. No, this is of x n, this is the plot of x n, this is the plot of x n. x[n] has a1 at zeroth position and has an 1 at 6th position, 6.

So, 0, 1, 2, 3, 4, 5, 6, 7 samples; 1, 2, 3, 4, 5, 6, 7th sample it is; then I put 0 here 8 samples. Then, if I plot it a mod of X(k). So, if I calculate the mod, k equal to 0 if I calculate then if I plot it. So, this is the 0th plot. So, k equal to 0 means this is 1, so this is 1.9, almost 2. k equal to 1 means you can calculate k equal to 2, calculate k equal to 3, calculate 4, 5, 6, 7. Although the signal is up to 6 samples, here I can get 8 samples.

Similarly, you can calculate the θ also. You can get this plot. So, this is the θ plot. So, this is the angle of X(k), and this is the magnitude of X(k). So, that is nothing but a mod. So, you can calculate the mod of it.

Thank you.