Discrete Time Signal Processing

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Lecture – 36
Bilinear Transformation

Quickly let us take a recap of what we did last time. We are suppose, we are giving the prototype low pass filter, or prototype filter, analog filter (Refer Time 00:30), in a rational form, like this.

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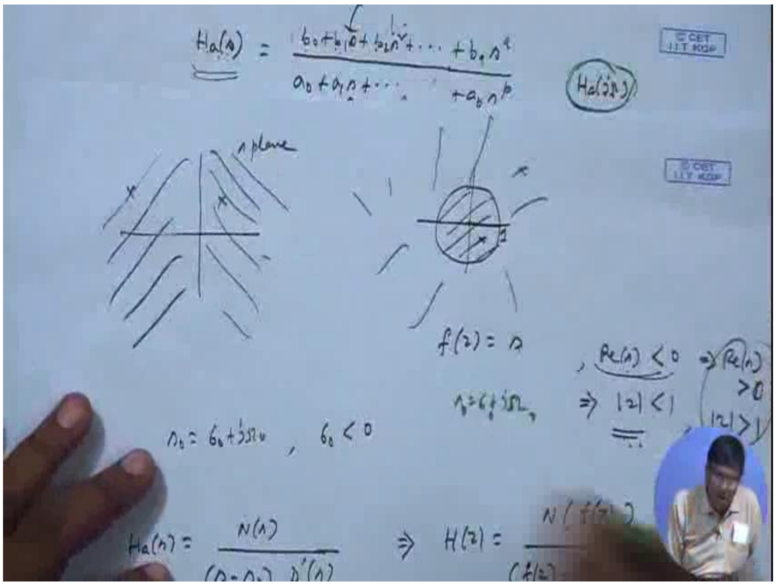

One way to obtain a digital transform function is, to deter, is to replace s by, some appropriate function of z. This function is your choice, but this fz should satisfy some properties, because I want some conditions to be valid, that is, if h a s is given to be stable and causal, h z also should be stable and causal, if h a s is rational in a s, h z should be rational in z inverse, or z so and so. And there is 1 more condition that we look at later. So, s to be replaced by fz, so; that means, H z is nothing, but b0, plus b1 fz, plus b2 f square z, plus, dot, dot, b cube, f to the power q z and denominator also a 0 plus a1 z and dot, dot, dot.

Now, my claim is fz itself, should be rational function in z inverse. that is it should be equal to some N z, or fz inverse, by D z dot D z inverse, if you put that back here, b0 plus b1 N z by D z, plus b2 n square z by D z square, then dot, dot, dot, divide by a 0 plus a1, N z by D z and dot, dot, dot, if you simplify, final it will be a numerator polynomial in terms of, z inverse, power may not be cube, power may be higher than cube. Alpha 0, alpha 1 z inverse, dot, dot, dot, alpha, something you know, cube prime, z inverse cube prime, divide by beta 0 plus, beta 1, z inverse dot, dot, dot beta p prime, z to the power minus p prime, (Refer Time: 01:59) p prime p prime (Refer Time: 01:59).

As an example, we consider h a s to be rational function of that form, b0 plus b1s, a 0 plus a1s, plus a 2 is (Refer time: 02:10) square, numerator is a polynomial in this, denominator polynomial in this, and s was in placed by fz, but f z was taken to be taken itself of rational function in z inverse, like 1 minus z inverse by 1 plus z inverse, if that be then so, numerator polynomial in z inverse, denominator polynomial in z inverse. If z is again rational function of z inverse itself, if that f z if I put here in place of s, and I get b 0 plus b 1 this, this is s instead of s I am putting this, and a 0 plus a 1 this plus a 2 square, and if you now simplify, b 0 into 1 plus z minus z inverse, whole square, plus b 0 1 minus z inverse 1 plus z inverse, and simply, denominator also. We did all this, and you can simplify it, it will be a function of numerator, will be a function of, will be polynomial in terms of z inverse, denominator will be polynomial inters of z inverse; it will be a rational function.

Therefore, whenever you replace s by a function f z, make sure the function f z itself is a rational function in z inverse, that is, of the form N z by D z, and if you put that N z by D z everywhere here, you can simplify, you will get and what all numerator polynomial in z inverse, by overall denominator polynomial in z inverse. So, it will be a rational function. So, number 1 condition was that, f z should be a rational function in z inverse itself. Next was this, that suppose, just give a minute, yeah, suppose h a s has lot of poles, one of the pole is at s 0. So, that factor is minus s 0, I take out separately; all the other factors are put together in 1 polynomial of degree 1 less, and call it d prime s, numerator as before N s. Now if you replace s by f z, numerator will be N, a function of f z, instead of this will be function of f z; denominator will be in place of case f z. then minus s 0 as it is d prime function of f z.

Now, this factor will now, we not s minus s naught, but f z minus s naught. So, what will be the pole in terms of z? You solve for this equation, f z is equal to s naught, and let the solution be z naught. So, there will be a pole, of H z and z equal to z naught. Now if the original analog function is causal and stable, then I know the real part of s is, less than 0.

There is if you put, s is equal to sigma plus j Omega, j Omega, and is suppose real part of s is, s 0, like this then s 0 will be, less than 0. In that case if I put that s 0 here, and solve for z 0 that, that z 0 also should be a new pole, but it should lie within unit circle. Therefore, if the analog is system is causal and stable, that is if the real part of s 0, real part of the pole is on the left hand side of the z Omega axis, that is negative, if you solve for f z equal to s 0, the z 0, there is a new pole that comes in terms of z, in terms of z, the new pole is of z 0, that z 0 also should lie within unit circle, that is mod of z 0 should be less than 1. Conversely, if a sigma 0 is positive, there is, the pole has real part to the, pole is to the right of to this 0 Omega axis, of course to which it is the non causal and non stable, I mean cannot be stable and stable and together.

In that case, corresponding z also corresponding solution z 0 to be such, they should lie outside the unit circle, that is mod z 0 greater than 1. Therefore, if you solve for, in general, f z equal to s, then as long as real part of s is negative, corresponding z should be within unit circle; that means, left up plane, left of part of the l h p of this plane, should map to within unit circle in the z plane, and if real part of s is greater than 0, that is, is positive, that should map to outside the unit circle.

That is the mapping, and third things was also very important, that suppose, I consider h a j capital Omega, that it is frequency response; that means, real part of s I will put to be 0, and s will be just j capital Omega, for that s, if I solve for f z equal to s, and s is j Omega, then I want the corresponding z also should be having of this kind of form with mod 1, there is, that that pole, or that z, should lie on unit circle with z plane, why? Because, suppose, I have got H z, I want to find out its frequency response at any e to the power j Omega; so I should put alpha 0, plus alpha 1, e to the power minus j Omega, plus, dot, dot, dot, plus alpha k, e to the power minus j Omega cube, dot, dot, dot, but this was obtained by simplifying this original function.

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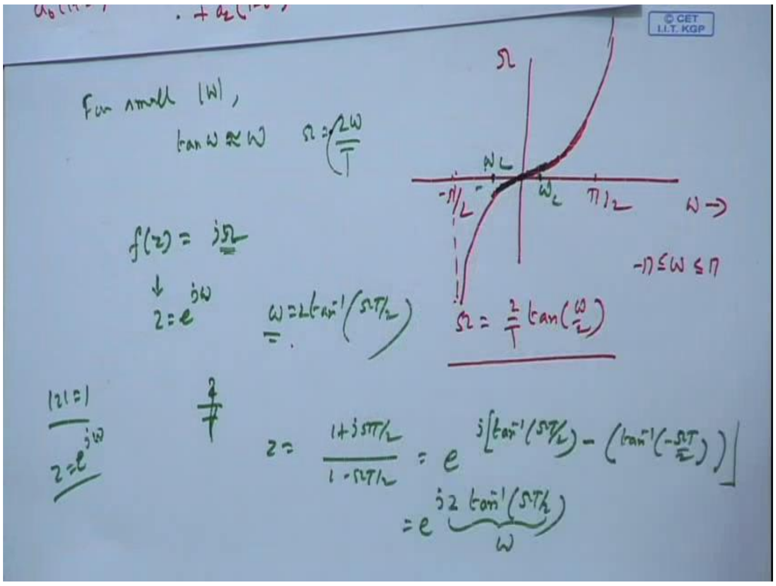

So, I can, as well put, e to the power j Omega here, that is b 0 plus b1, f of e to the power j Omega, plus b 2, f square of e to t he power j Omega, dot, dot, dot, but f to the power j Omega, by this solution is, a pure complex is, even j, number j capital Omega. So, it will be b0 plus, b0 plus, b1 capital B1 j capital J Omega, there is here b1 j capital Omega, b 2 j capital Omega whole square, b 2 j capital Omega whole square, and dot, dot, dot; that means, its value will be same, as the value as, will of h a s at s equal to j Omega, therefore, at digital frequency is small Omega, what may be the value of this h z? That is what will be the frequency response? Whatever is the digital frequency response At any digital frequency Omega It will be of the same value, as the analog frequency response, at another analog frequency capital Omega, which are related by this equation, that is f of e to the power j small Omega, is j capital Omega.

Therefore, if you solve for, this requirement if you solve for an equation like this, f of z equal to, s, when s real part is 0 purely s, is purely j Omega, j capital Omega, that z should also becomes of this form. e to the power j Omega which magnitude 1, then 1 frequency response is mapped to the other. With this, I consider one particular transform which is called bilinear transform, of this form, this was f z equal to 2 by capital T, capital T was just a constant, could be 1, 1 minus Z inverse, by 1 plus Z inverse. So, first the f z is rational, in z inverse, because numerator is a polynomial z inverse, so it is denominator. So, that condition is satisfied. Now suppose we are solving for, f z equal to s, and s is sigma plus j capital Omega, and then I will put s sigma to be negative, I will see whether z lies with unit circle, or not and vice versa. So, then we simplify, it was 1 minus z inverse, by 1 plus z inverse, was is equal to s T by 2, and then if you do 1 plus s

T by 2, very simple add 1 to both, it will be 2 by 1 z inverse, subtract s T by 2, subtract left hand side and right hand side from 1.

So, it was 1 minus s T by 2, and 1 minus this, that will be raised to 2 z inverse, by 1 plus z inverse. And now you take the ratio of this by this that will give raise to Z, very simple you can check it, and this will 1 plus s T by 2, 2 by 1 minus s T by 2. Now you replace s by sigma plus j Omega. So, real part is 1 plus sigma T by 2, plus j imaginary part, Omega T by 2, denominator 1 minus sigma T by 2, minus j Omega T by 2. So, what is mod z? Square root mod z means, if z is a ratio of 2 polynomial, ratio of 2 complex number, mod z is the ratio of the magnitude of the numerator, by magnitude of the denominator, we have seen. That is, if z is z 1 by z 2, mod z is mod z 1 by mod z 2. So, here numerator is a complex number, denominator is a complex number. So, mod z, mod z was a square root of the 1 plus sigma by T by 2 whole square, that is, square of this real part, square of the imaginary part, Omega square, T square by 4, denominator also like that.

Now, here if you see, sigma is negative, that is real part of s is negative, then 1, will have something minus, sigma T by 2, and here it will be 1 something plus, because sigma is negative, other terms are same, which means mod z will be less than 1. So, therefore, z will be within unit circle. So, if real part of s is negative, corresponding solution z will be within unit circle. So, left of part of explain will map within unit circle. On the other hand, if real part of s sigma is positive, this is 1 plus something, and this is 1 minus something, so numerator greater than 1. So, denominator greater than 1, which is we right up plane, will map to outside the z unit circle. So, causality stability condition is satisfied. And lastly sigma equal to 0. When you put sigma equal to 0, then what happens? That is what we will see today. If you put sigma equal to 0 here, that is I am evaluating, we say, the analog frequency response. So, s is just j capital Omega, if you solve for it, then if you solve for 2 by T, like here, if sigma is 0. So, it is just 1 square, plus Omega square T square by 4, again 1 square plus Omega square T square by 4.

In that case mod z mod, equal to 1, which means corresponding z lies on the unit circle which is what you wanted. There is, z will be of the form of sum, e to the power j, sum, sum Omega, for that capital j, capital Omega, solve for it, you get 1 small Omega, as a function of that capital Omega. You can make directly now, evaluate like here, 1 plus j Omega by 2, because sigma will be 0. So, Z will be 1, plus j, we have seen this magnitude, magnitude, numerator magnitude is 1, plus Omega square T square by 2 so is the denominator. So, this magnitude and magnitude cancel. You are left with this angle. e to the power, j minus. So, tan inverse minus theta, is minus tan inverse theta, and minus, minus plus, and it will become e to the power j, twice tan inverse, Omega T by 2. So, see this is, this you can call small Omega. So, at that small Omega, e to the power j, that small Omega, is equal to, e to the power j, small Omega right? So, if you put that small Omega construct, z equal to 2 to the power, that small Omega, j Omega, then for that, e to the power j Omega, f of z will be this j capital Omega, again.

There is, if you solve for f of z, if we are already seen the form, f of z equal to, suppose, we are solving for j capital Omega, in this case you already seen corresponding f of z. If you put the f of z form here, and solve for z, this a general form of z, if you solve for f z equal to s, general form of solution z is this, we have already seen. Now in a particular, s sigma is 0. So, sigma means 0. So, z is 1 plus j, Omega T by 2, divide by 1 minus j Omega t by 2, that is the z you get, and that z has magnitude 1. So, the solution will be of the form z, in e to the power some small Omega, where Omega is this. This Omega will be function of this, for a given capital Omega.

We will get small Omega of this form; that means, analog frequency response at capital Omega, what may be the value, at this Z equal to e to the power corresponding j into this Omega, will be have the same value. Because e to the power j, this Omega if you put here, then f of z becomes j capital Omega, and in the analog response, you get the analog frequency response, are capital Omega, alright? We have seen this already, that is, once again you put this small Omega equal to this, Z equal to e to the power j small Omega, if you put Z equal to that e to the power j small Omega, is equivalent to putting b 1 f e to the power j Omega, b 2 f square e to the power j Omega, but e to the power j this small Omega, is equal to j capital Omega. So, it becomes analog frequency response, at capital Omega.

So, at this small Omega, digital frequency response is same as the analog frequency response, at this capital Omega. So, this is the relation. Thereby one frequency response plot, can be, you know, mapped to another response plot, if you plot this, how does it look like? You can proof the other way also, Omega by 2, 2 by t, is capital Omega, if you put capital Omega, because plotting tan is easier than tan inverse, that is why I am making it tan, tan and Omega I know, its range is pi 2 minus pi, because frequency response of the digital transform function, I will see only over this range. So, if Omega is pi, this stand pi by 2, I mean Omega is minus pi, it is minus pi by 2. We all know how tan (Refer Time 16:05) behaves, it is like this. Now suppose you are (Refer Time: 16:27) low pass filter, now this part if, suppose, Omega is small, in this small range Omega, I know, sin Omega is Omega, and cos Omega is 1, approximately. So, tan Omega is approx Omega.

So, for small Omega, small value of mod Omega, on either side; tan Omega will be, approximate equal to Omega. So, it will be linear. 2 by t and capital Omega will be, 2 by T into small Omega. So, it will be linear curve, which slope to 2 by 3. So, if it is linear. So, what is small range? Will be Omega c and minus Omega c.

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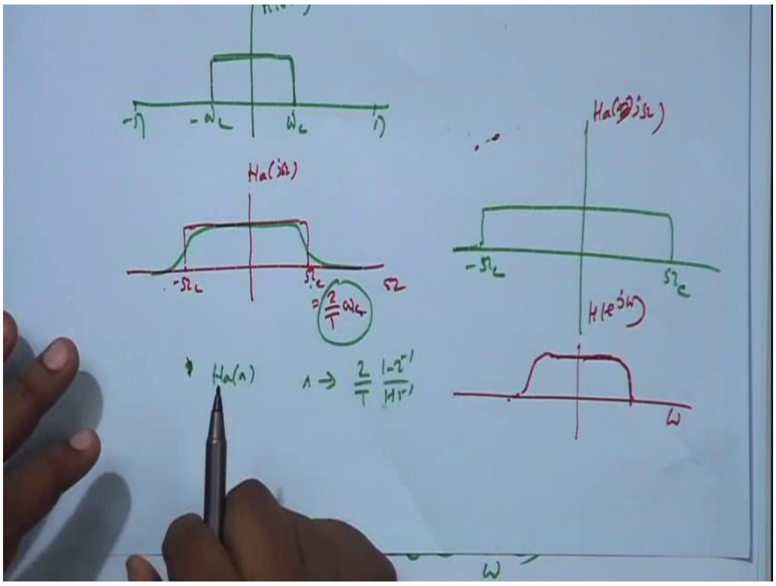

Then if you have a filter and if you are looking for a digital filter of this type, again, I am drawing only the whole plot but actually one should draw the magnitude plot and other plot also. Phase plot also, but suppose these are thing then since this is linear, corresponding analog map will be just linear, linear conversion. It will be like this, capital Omega. So, whatever I have at in a small Omega, 2 by T times that, why is that, 2 by T times that, will be the analog frequency over the same value will go.

So, the be capital Omega c, this is 2 by T, and likewise, all values will go, and this is the analog response. So, if analog filter is a low pass filter, digital filter also will be the same, same. Only frequency which are changed, but if the range increases, suppose you design what analog low pass filter, but this is not over as zone, it is over a bigger zone say, then it will not be a linear map, there will be a frequency warping, this will be cramped, but the lower base, there will be linear map, but after that, this small Omega, this was H a s, h j capital Omega, after that what is happening, this is not linearly mapped, you this cramped, because small increase in small Omega, but s is increasing very much. So, it will be, there is some non-linear, if you something like this, but otherwise, if, if T is large, 2 by T, into Omega, then what happens, sorry, if T small, if T small then I will give you what is larger range, t small I will lot larger range, this linear zone, and if analog filter is a good low pass filter, digital also will be a good low pass filter, but after wards there will be a frequency, this thing you know, warping, there will be cramping effect, even if we increase this, this will not increase, this will kind of get cramped.

But then last question is, how did you get this? I started with this transformation, is a good transformation, (Refer Time: 20:28) much. That is, you first suppose, you want to design a Butterworth filter, you start, you take a T. T should be as small as possible then 2 by T 1 minus z inverse, by 1 plus z inverse, is the function. So, first you design an analog Butterworth filter, of large order, to very high sort of everything, but part of everything, but cut off frequency we design like this, from here, it digital cut off frequency is given to you Omega c, taking it to 2 by T Omega c, that is a capital Omega c, for that you design this Butterworth filter. See what is the H a s? This H a s, will be rational ns, there you replace s by, this 2 by T, 1 minus z inverse by 1, plus z inverse. If T is small, and this filter is the, there will be very little frequency, or warping effect, so digital transform function will be like this, will be kind of replica of analog function, as much as possible, but otherwise, there will be non-linear effects. You need to learn perfectly, Butterworth, it could be like this, you know, it will be non-linearly distorted, on the boundaries.

And how did you get this? This is a question. So, this is a good transformation and it solved our purpose, but how did you get this? So, there are ways of getting such transformation, one is this, that suppose, how did you, how you can get bilinear transformation? You can, I will tell you. So, you get a motivation, you can generate your own transformation by, you know, working on different ways.

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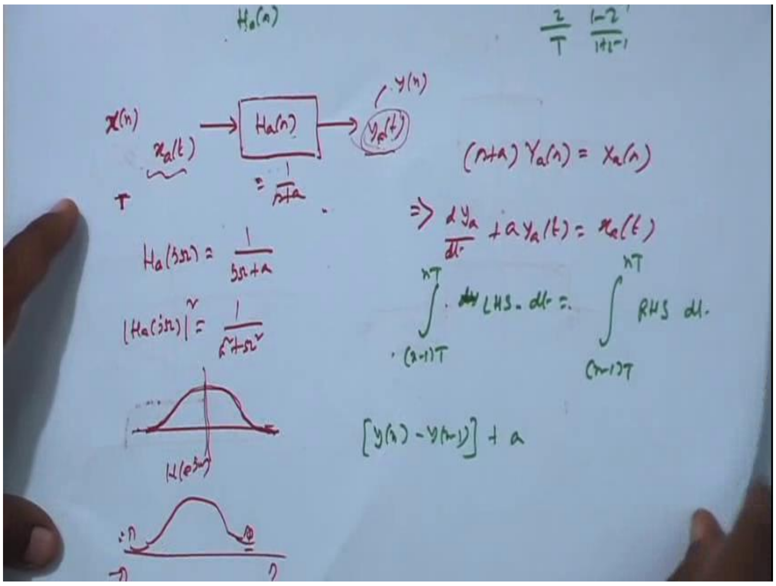

Suppose there is an analog system, H a s equal to just 1 by s, by a first order, very simple system. This is x a t, y a t, suppose this will low pass filter. So, s plus a, Y a s, is X a s, which means, s into this, and we assume that system is switched on, at some point of time and initial conditions are 0, that is why it is linear and shift invariant, and if the that is why x, y a s by x s which is 1 plus s by (Refer Time 22:25) 1 by s, there is also impulse, the transform function, and this s into Y a s, that corresponds to d y a d t, because initial condition 0 means Laplace transform will be just s into Y a s, and a into y a t, is equal to x a t zero initial condition. So, ratio output Laplace transform, by input Laplace transform, will be the transformation function, and this is again the Laplace transform, of the impulse response analog impulse function.

Suppose this given to you, and this is 1 by s plus a, is a good low pass filter, that is I am happy with this transform function, say. So, is magnitude response will be, now suppose I am happy with this, is a kind of low pass filter; obviously, happy with this, its a some response, some kind of response say, as Omega increases goes down to 0, as Omega become 0, is a constant. So, do some kind of response (Refer Time 23:46). Now, suppose, I sample this, at a very small rate with period T, high sampling rate, same with high sampling rate. So, they are (Refer Time 23:59) of something of (Refer Time :23:59) So; obviously, this will give raise to y n, this will give raise to x n, where y n is y a n, into capital T, x n is x a n into capital T, but discrete time Fourier transform y n will be replica of just periodic repetition, of the analog Fourier transform of this, because we I am, way of micro state, same will be here.

So, if I now take the ratio of, discrete time Fourier transform of output to or by, discrete time of Fourier transform of input, then that ratio, will be a replica, that is periodic repetition, of, the analog response. Because from minus pi to pi this zone, what about I have the DTFT of, of y n, and what about I have for the DTFT of x n, they are nothing, but replicas of capital Y a j Omega, and capital X a j Omega. So, if I divide by them, that was will be replica of capital h a j Omega, and (Refer Time: 24:55) purely repetition of that.

So, suppose I, I take this filter, this filters, search me well. So, I sample the input and output, at a very small period, at a very high rate, and take y z, take x z, take the ratio of them, that ratio will be my digital filter. What will be the transform function? What will be frequency response? j equal to e to the power j Omega, means DTFT, that is output DTFT by input DTFT, but that I told you, that will be replica of the analog DTFT, analog function, which is what I wanted. I wanted replica of this, in the regional domain.

So, if I sample the output and input at very high rate, we have a micro state, then the I take z transform output, by z transform of input, I get suppose rational function, and I am happy with the digital filter, stability causality of those are things suppose satisfied, but frequency response wise, I will be happy because, if z is e to the power j Omega, it will become output of t y e to the power j Omega, divide by input of x e to the power j Omega, and then ratio will be nothing, but if the from the range minus pi to pi, I mean Y capital Y to the power j Omega is nothing, but, replica of analog Fourier transform of Y a t, capital X to the power j Omega will be the replica of, capital X a e to the j capital Omega that is a analog Fourier transform x a t.

So, if I take of the ratio of that 2 DTFTs, that ratio will be a replica of the analog frequency response, because that is analog frequency response H a capital j Omega, is the ratio of analog frequency Fourier transform of output, by analog Fourier transform of the input, which means capital H e to the power j Omega, which is the ratio 2 DTFTs, that will be of the of similar kind of form. And therefore, by objective of getting a filter of that kind in frequency domain is realised, but question is, the h z that I will get by this process, will it be rational? Will it be causal and stable to find, so, how to get that? Now here, suppose I apply, I want to discretize it. So, I will be integrating left hand side, from say n minus 1 T, to n T, T is the sampling period, we just went to T. So, this L H S, d t and this will be R H S , d t right? Now d y, d t, d t, if you integrate it will become y a. So, y a n T that is y n, minus y a, n minus 1 T, that is y, n minus 1. This part I get as it is, but this part a into y a t, integrate from minus, n minus 1 t, to n t. So, I have to find the area, area under what?

So, suppose this is my y a t, this is n minus 1 t, this is n t, to find out the area in this right y a t.

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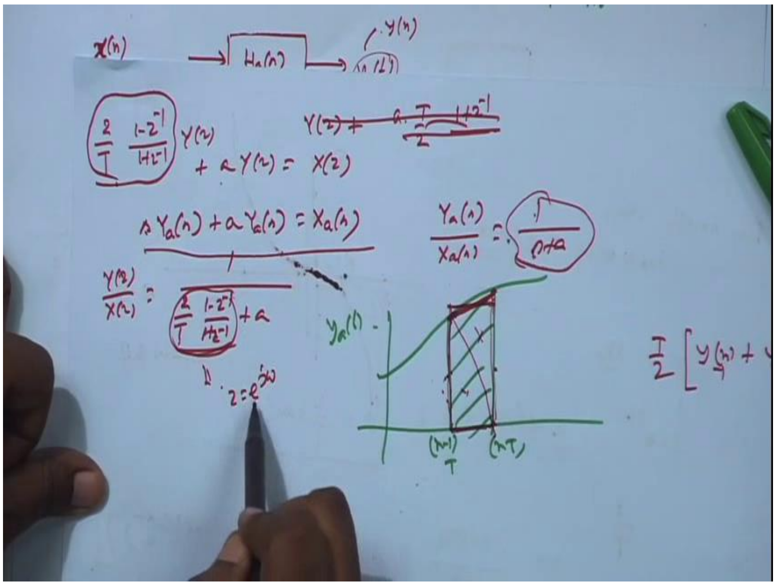

This integral means what? y a t integrated from n minus 1 to n T means, area under this curve, within this zone. This we can approximate by a, c, trapezoidal rule, if you draw a line straight line, this trapezoid, area will be almost equal to the area of this original strip, if the width is very small, but area of the trapezoid is what? Half of the sum of the 2 sides, into this T, T by 2, very simple is summation of these 2 areas, this triangle and this triangle. So, that we can easily see; T by 2 into this side, plus this side, half of this side plus this side means what? It is like the rectangle, and this, any way, this is very simple, T by 2, this is area of the trapezium, I think all of us studying in school, let us not spend time on this. y, this side plus this side by 2. So, by 2 is here, and this is T, T by 2, this side is y, a, n minus 1 T, which means y in terms of discrete signal, y n, and this side is y a n T, y a n T, is in terms of the discrete signal, y n, it is y n.

So, it is y n minus 1, and y n. So, if we put that here, a into something here, x a t, I have to find out its area also, under the same zone. So, it will be t by 2, and x a n t is, discrete sequence X at n, and similarly x n minus 1. Now if you take z transform of both sides, it will be 1 minus, 1 minus z inverse, y z, this 1 minus z inverse will come only where, we have got derivative. So, perfect y a comes, y n minus y n minus 1 comes. So, there is a minus sign, 1 minus z inverse. In all other fellows, there is, you have to apply trapezoidal rule, to find out the area whether we are here or here. So, you have summation on the 2 sides, y n, n minus 1, x n, x n minus 1.

They come if you take z transform; it will be 1 plus z inverse, 1 plus z inverse. So, it will be a, t by 2, 1 plus z inverse, y z, is equal to T by 2, 1 plus z inverse, x z. Which means, now from this quantity, which obtain from the derivative, integral derivative, and therefore here got in directly, y n minus, y n minus 1, and therefore, 1 minus z inverse y z, you suppose, you get this factor 1 minus z inverse, in all other cases you get 1 plus z inverse. So, suppose I divide both side by 1 minus z inverse, to make this free. What we have here is, y z, plus a, T by 2, or may be the other way, 1 plus z inverse, I have on this sides, where I have got trapezoidal rule applied.

So, suppose I divide left hand side and right hand side by this, 1 plus z inverse. So, in that, and T by 2, T by 2, 1 plus z inverse, that is common. Whenever we have applied, I applied trapezoidal rule, T by 2 into 1 plus z inverse, was common. So, divide left hand side right hand side by that. So, you have 2 by T, plus a Y z, equal to X z. And what was my original analog function? Analog, analog thing was? s, in terms of s, that is z domain digital, and analog s domain. You see here, a into capital Y, here a also into capital Y, Y a here, Y here, capital X, here X a, here s into Y a s, Y a s is becomes Y z in this case, but s is replaced by this, that is why, if you now take the ratio of Y z by X z from top, Y z by X z this plus a will come below, and from here 1 will come, and if you, from here if you do the same thing, plus a will come below, 1 will come above.

So, you see, in this analog transform function, if I just replace, s by this, 2 by 3 1 minus z inverse, 1 minus z inverse by 1 plus z inverse, I get the digital transform function, and what is the digital transform function? If T is very small, I apply very small T to high frequency sampling about (Refer Time 33:41).

So, Y z, by X z, Y z by X z, and Y z by X z, in terms of frequency response, it will be capital Y, e to the power j Omega, it will be X, e to the power j Omega, but they will be replica of analog response, within the minus pi to pi zone. So, the ratio will be same as replica of analog, your transform function. So, digital transform function from minus pi to pi, minus pi to pi ,will be kind of replica of analog Fourier transform. So, my filter response will be, I mean whatever I wanted, that will be satisfied, and how I am obtaining? By choosing a very small t, I am replacing this by this. Has it satisfied rationality? I have already seen this, if you now have any Y z by X z, obtained by replacing s by r, rational function, this is itself rational,1 by s by s. So, numerator is a polynomial, with only 1, denominator is a polynomial in, s plus a (Refer Time: 34:39) higher order polynomial, I took the simplest case.

Another I told you if s is replaced by rational number in z inverse, these function also rational in z inverse (Refer Time: 34:49) Also we have seen, this stability and causality, and also, if s is j capital Omega, then this amounts to sum, z equal to sum, e to the power j Omega. So, that, if you put that z here, f of, that is f of z, there is f of e to the power j Omega, will give me us to that j capital Omega; that means, at capital Omega, what will be the response, same thing will have at z equal to e to the power z j Omega, obtain by solving this equal to j capital Omega. This you have dealt with. So, this is the origin, this how people first derived it, and then analysed. So, that is about this, bi-linear transformation.

Thank you very much.