Discrete Time Signal Processing
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Lecture – 24
Factor – of-M Polyphase Decomposition of Sequences

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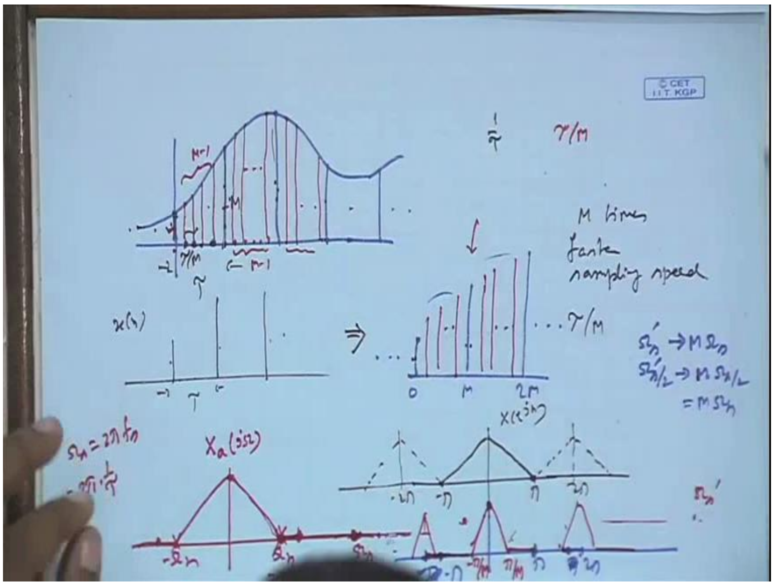

Interpolation means I have got these loaded sequence; this sample, this sample, this sample, this sample like that and this red color samples are to be recovered from this loaded sequence exactly without any error I am told one thing that originally sampling frequency that sampling period was tau sampling frequencies the frequency has 1 by tau, but that was the of the nyquist rate. So, there was no aliasing originally therefore, if in increasing the sampling rate further, no question of any aliasing.

So, now 1 thing that suppose originally you have got this sequence x n and what is you have to it is given that analog Fourier transform again I am drawing analog I should draw magnitude and phase separating versus omega because a complex number complex we cannot be described by 1 plot, but never the less just for explaining my point as you already say now. So, again 1 plot only suppose it is giving like this is a band limited function maybe something like these are band limiting frequency to sampling frequency in order to avoid aliasing should be at least twice omega h or more.

So, I will start with the extend case I mean model line case with this omega s; omega s is 2 pi into sampling frequency fs and fs is 1 by tau. So, that the omega s I am taking that case for omega s is just twice these. So, value has been say; omega s could be to the right and omega s I mean if omega s is here it is twice omega h. So, omega s by 2 that is half sampling frequency will be here with omega h, if omega is omega s is further to the right omega s by 2 will be some ever here. So, there be 0. So, in that case we can assume that original analog fourier transform is up to this is takes a step like these up to these.

So, this is like the band limiting frequency now instead of these and omega s by 2 with these, but in our case I am just taking the example where omega s is here and this is equal to omega s by 2 these point all right omega s by 2 no question of aliasing because omega s is twice omega h the band limiting frequency this original Fourier transform is 0 here has by which would 0. So, sampling period sampling frequency I am taking to be twice that band limiting frequency.

So, nyquist rate no problem no aliasing all right that. So, in that case if I sample what will be the dtft you have seen dtft will be no aliasing it will be replica like these this half sampling frequency always go to pi and minus pi, but since half sampling frequencies same as band limiting frequency. So, band limiting frequency will also go to pi and minus pi like that and then if there will be periodic repetition dot, dot, dot right that is the dtft of this guy, that that is original x e to the power j omega where x it is low sampling rates sequence.

But, suppose I had sampled the original analog wave from at a faster sampling rate. So, smaller sampling period sampling period is now tau by m earlier these was tau now tau by m or coherently sampling speed sampling frequents sampling rate has gone up by m m is an integer, that is why between 2 samples original samples I now have m minus 1 sample see this was tau now this tau by m. So, these becomes mth sample all right then for the sequence there is this sequence what is the dtft that also you have to see there will be no aliasing.

So, the dtft again will be replica of analog fourier transform, but it will have a different safe down what kind of safe now new omega s you can call it omega s new omega s that is m times original omega s new omega s by 2 if you call it omega s prime omega s prime that is m times original omega s by 2 now, but I know half sampling frequents whatever be the sampling frequency employed half sampling frequency will always map to pi. So, this will go to pi and this will go to minus pi, but this is now m time’s omega s by 2 and original omega s by 2 was same as omega h.

So, m times this is equivalent to m times omega h. So, these maps to pi omega h will map to pi by m. So, maybe I draw according to scale put the pi here because it was pi here and put the minus pi here. So, this is a new pi new pi means it is the mapping of omega s prime by 2 new half sampling frequency that is mapping to pi , but earlier this pi was the map of omega s by 2 there is original half sampling frequency calls to the map of omega h because they where same and there was no aliasing you see, but now new half sampling frequency.

So, omega s prime by 2 that will in any case map to pi that is m times original half sampling frequency. So, it means m times original band limiting frequency. So, if half sampling frequency in the analog domain is m times original band limiting frequency. So, map of half sampling frequency which is pi again that will be m times the map of omega h; that means, omega h will now map to pi by m and the 0s because outside they have 0 because these 0s will come these 0s because omega s. Now, going to right side m times omega s prime by 2 somewhere here that will map to pi omega s was here.

Now, if I go to the right somewhere here is your omega s prime. So, earlier omega s by 2 was here omega s pi by 2 will somewhere here, in that will map to pi and this is m times less. So, that will up to pi by m and the intermediate 0s will come here all right this is the equation the new dtft.

So, again at 2 pi around that you have got like that 2 pi plus by m 2 pi by minus pi by m here also dot, dot, dot, dot, there is a dtft of this higher sampling rate sequence interpolated sequence higher sampling rate sequence. Now, my claim is if by the processing this low sampling in this sequence some where other I give you one sequence whose dtft is these then that must be we desired interpolated sequence high rate sequence because dtft corresponds to only one time given sequence not many. So, if I can generate some sequence whose dtft is this then that must be equal to this that sequence will this only because 1 dtft means by inverse dtft you can get only 1 sequence or more than 1.

Now, I will proceeds this original low rate sequence and generate finally 1 sequence. So, dtft will be of these kinds, which means that final sequence will be nothing, but these that will be the interpolated sequence and between every 2 blue color sample it will get these red color samples how to do that I am coming. So, here I will be using this expansion.

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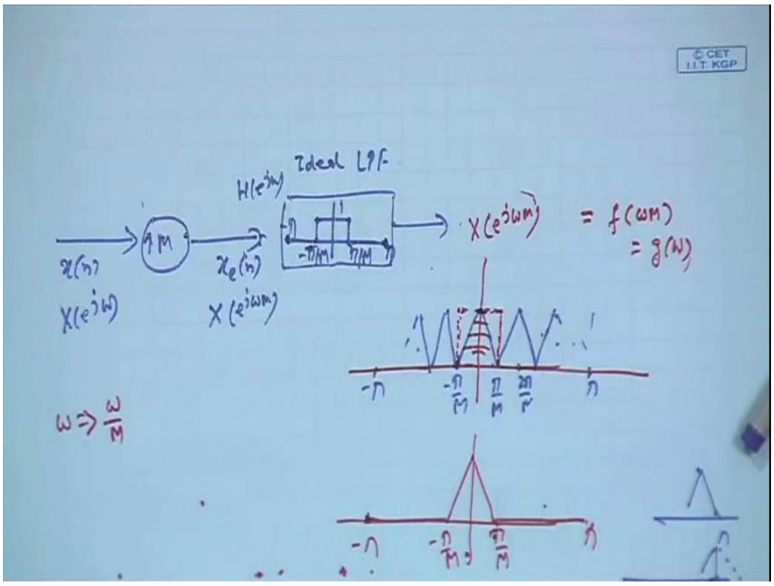

So, my first step here is expansion your x n I explained by m that means, what was expansion was you see if you have a sample x 0, x 1 and intermediate sampling points. I do not have those samples I have looking for those samples now for interpolation I do not have. So, I have I has putting zeros there m minus 1 zeros. So, sample number 1 is now sample number n there again n minus 1 0s. So, sample number 2 is not sample number 2 m dot dot. So, for this sequence this is called expansion xen and xn xe m xe 1 is 0 2 is 0 three is 0 up to m minus 1 0 xe m is x 1 xe 2 m is x 2 dot, dot, dot there I worked out the summation capital xez there is z transform of this expanded sequence which has 0, 2 samples m minus 1 zeros that is original z transform x at z to the power m.

So, complex value is not z z to the power m whatever the value that will be then z transformed at z for xez and therefore, z equal to e to the power j omega 2 for dtft and that was capital x e to the power j omega m. So, omega into m red en into m that is another red en, omega into m at that digital frequency whatever value of these dtft that will be the dtft of these expanded sequence at that omega that what we did and now I have been expanding.

By this formula here, if I have xen and if this is x e to the power j omega this will be x this e to the power j omega m because dtft of this is x equal to 0 that is what I have we derived here that is the only thing I borrow from there map shown I have taken these 2 my original xe j omega right and this has been original xe j omega this was sampled at nyqusit rate and that was the dtft that was the dtft xe j omega.

So, what is xe j omega m if this is my xe j omega what is xe j omega m, I plot it now as before xe j omega e to the power j omega is a function of omega right you call it f of omega then xe e the e to the power j omega m it is nothing but f not omega, but omega n to m because wherever omega was coming that is replaced by omega m and you will call it some function g omega I have to plot g omega versus omega I have to plot g omega verses omega right g omega is these guy which is f omega m which is g omega I am calling lets giving a name g omega, how to plot g omega now here you see at g omega at say omega equal to some omega by m what will be the value if you put omega by m m mm cancels f omega and what is f omega original x g omega.

So, original x e to the power j omega at any omega whatever value you have that will come to g omega at omega by m that is any g omega at any omega by m is same as f omega, that is capital x e to the power j omega. So, capital x e to the power j omega at any omega whatever the value that will come to this g omega at omega by m there is for this function at omega by m which means.

Which means pi will come to pi by m minus pi will come to minus pi m 2 pi will come to

2 pi by m this will be like this dot, dot, dot, dot final from pi to minus pi. So, either it will go like these or it will go like it will be either if it is pi it will be either end like this or it will if it is pi it will there is you know go like this on this side depending on whether capital m is odd or even all those things, but you have this kind of staph, but that means, if I expand the dtft will be these, but for interpolated function for this interpolated function where you do not have 0s.

Here, I would have x x are samples interpolated function, what will be the dtft of that that we have already worked out? This is the dtft then I told you by processing extend I already processed extend generate the xen by further processing if I can generate another sequence whose dtft is this there I am done because dtft these means in time domain these will be the sequence, because 1 dtft corresponds to only 1 sequence now have gone up to this if you compared these 2 if that you see this minus pi by m to pi by m these part its common, but other part is 0 from minus pi to pi, which means if I now pass this sequent to an ideal low pass filter ideal LPF low pass filter of these kind pi by m minus pi by m and then 0 up to pi 0 up to minus pi these a dtft H e to the power j omega right it looks like this pi by m minus pi by m 1. So, whatever goes comes in this it passes and then 0 up to pi 0 up to minus pi.

If I pass this xen through this filter then what will happened this sequence as this dtft, but if passes through that low pass filter from minus pi by m to pi by m. So, only this much this is the band width pass band of that low pass filter and then it is 0s. So, we will get only these much, other parts will be 0 and this is what we wanted pi by m to pi 0 pi by m to pi 0 minus pi by m to minus pi 0 and then again of this. So, this is the way you can generate this interpolated sequence, you have got ideal filter if it is not ideal filter they have some part from right and left will filtering will and they will create some error aliasing and all that. So, these 2 are very important topics interpol decimation interpolation.

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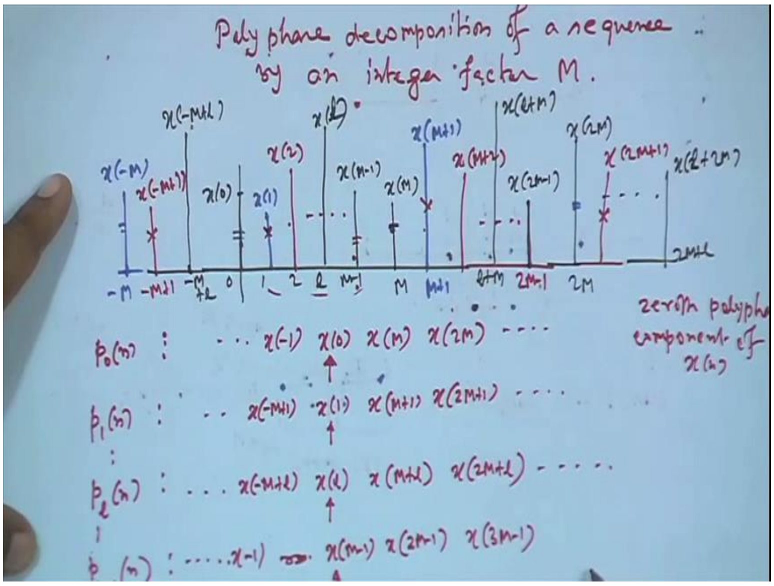

Now, I will go to another important concept in multi rate signal processing that is called Poly phase decomposition. Suppose, I have a sequence I have x 0 then x 1 dot, dot, dot, dot, x m minus 1. So, 0th first second up to m minus 1 then again I have at mth point xm then m plus 1. So, 0 to m minus 1 m to 2 m minus 1 will be just m at m plus 0 m plus 1 m plus 2 up to m minus 1. So, this is your will draw by black ink then again that 2 m x 2 m dot, dot, dot, dot nothing new I am just drawing the samples as it is on these side also you can draw.

Now, suppose I pick up the 0th guy, then mth guy 2 mth guy minus mth all that that is I am decimate it by a factor m to zeroth sample then throw away the next n minus 1 samples then the next guy mth then again forgot the intermediate samples m minus 1 samples take the 2 mth like that if I do like that I get a sequence which I called poly phase component zero th poly phase component, zero th poly phase component of xn that will be like these I can just write in terms samples x 0 that will be the we sometimes do like this you may stop showing it by a diagram like these the point like these this is the origin x 0 sample number 1 is a next that will be your if it is origin forgot about these next m minus 1 samples.

I will take these 1 say then x to m from here in this x minus 1 dot, dot, dot, dot this is the origin these called the p 0 n p 0 n means 0th poly phase component of xn then suppose I have everything is fine I take next 1 p 1 n I take this sample x 1 x m plus 1 that is at a gap m x 2 m plus 1 to this cross you have see x 1 m plus 1 2 m plus 1 at minus m plus 1 these guys and form a sequence again x 1, but that if this is the sample number 1 here this is my origin 0th sample.

I am constructing the sub sequence p 1 n where x what will be the 0th sample. So, then x m plus 1 will be the sample number 1 like into a sample number 1 you understand the sampling rate has gone down here after 0 I am jumping, much of time then only xm then. so much of time. So, like that 1 then m plus 1 2 m plus 1 dot dot dot dot all right in general p e of n will be what you might take 1 lth sample l x l from 0 to m minus 1 then from m to 2 m minus 1 again there will be this l, l means l plus m now jumping from here.

So, x e plus m the somewhere here is x l plus l plus m l plus 2 m for 2 m plus l whichever way all right 2 m plus l in this point here also x minus m plus l there general case l could have been 0 I am here p 0 l could have been one. So, I think x 1 then here x m plus 1 like that l general l xl. So, xl m plus l or l plus m 2 m plus l minus plus l like that, general case. So, again I construct a sequence using them, but xl will not be the lth sample here again I will put it at the origin in my construction at lastly you see you start with 0 m 2 m like that 1 m plus 1 2 m plus 1 like that finally, you will end with m minus 1 then m minus 1 plus if there is 2 m minus 1, 3 m minus 1 like that he do not go to m again because there is always taking care of in this xm is already represent here all right xm x 0. So, there is already p zero n.

So, I will be stopping here at this x these origin this instead of l it is m minus 1. So, it is 2 m minus 1 it is 3 m minus 1 just l is replaced by m minus 1 here l was 0 l was 1 l is m minus 1, everywhere minis m plus m minus 1. So, x minus 1 dot, dot, dot, dot, dot, dot, dot this dot, dot, dot, dot. So, this called zero th this called fast poly phase component lth poly phase component these now using them how to get back these you will see if I expand these p 0 m there between every pair of samples I bringing m minus 1 0s.

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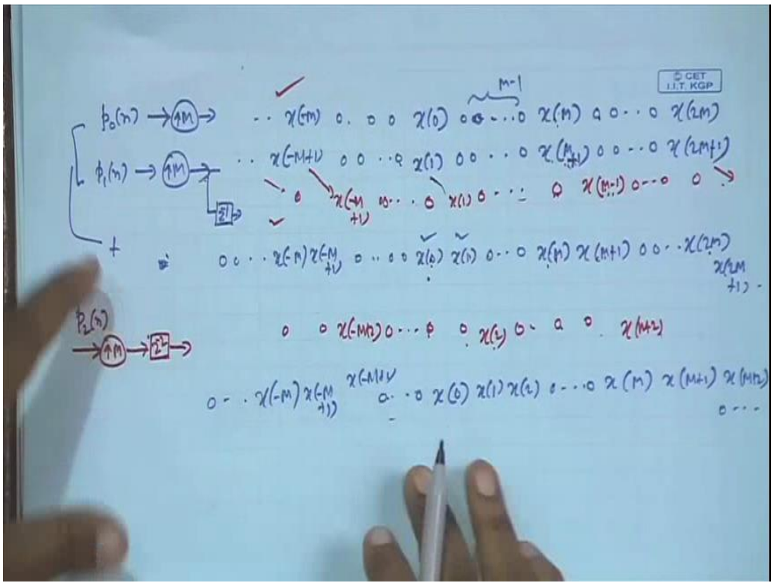

That is I will expand p 0 n what will I get x 0 then 0s m minus 1 0s then x 1, sorry xm this guy again m minus 1 0s again 0 0 dot, dot 0s x. So, this is not these wrongly I x minus m dot dot all right then again suppose p 1 n this also expand what will I get x 1 as it is then zeros then x m minus 1 x 2 m minus 1 x minus m plus 1 dot dot.

Suppose I delay this this I delay this output this was here I if I delay it what will happened this will go to the next this will go to the next. So, you will have this 0 will come here. So, you will have 0 x 1 then the 0s will move below these there will be 0 x m minus 1 this will come here then again 0 0 0 0 will move here x 2 m minus 1 will go there like that this will come here then 0s 0 from here if I delay by 1. Now, if you add this and this if we had what will have x 0 plus 0. So, x 0 x 1 plus 0 x 1 then of course, 0 will 0 0s then xm now below then the 0 has come this has move to right, xm plus 0 xm xm minus 1 plus 0 xm minus 1.

So, this way if you see you will have if you add that 2 then you will get x 0 coming here x 1 going here then some 0s then xm m minus has gone here then some 0s obviously, it will be 1 less 0 earlier it was m minus 1 now it was m minus 2 because you are adding xm with 1 0 or xm minus 1 which another 0s. So, 0s are getting their this m plus 1 again I am it is m plus 1 see again in case you have doubt x 1 this guy at 1 then 1 plus m you jump to the right by m xm plus 1 jumped to the right by m again, 2 m plus 1 there is why x 1 m plus 1 2 m plus 1 jumped to the left by m, minus m plus 1 like that.

So, it is actually m plus 1, 2 m plus 1 then again zeros x minus m x minus m plus 1, zeros dot dot dot dot. So, you see if you compare x 0 it has come up x 1 has come up, but then we have zeros then again xm has come up xm plus 1 has come up, but then again all zeros then again x 2 m 2 plus 1 have come up then again zeros like that. So, then if you see if you expand next 1 p 2 n you will have x 2 just expansion will be x 2, x m plus 2, x 2 m plus 2 like that and if you delay it by 2 times.

Now, if you delay it by 2 times then x 2 will go here xm then jump to the right by m x m plus 2 will go here, x minus m plus 2 will go here and there will be 0s there will be 0s between below them also like that the zeros if you add these plus 0 these plus 0 these plus 0 these plus 0 these this like that. So, will that will be further constructed now in you will have x 0 followed by x 1 followed by x 2 there some 0s then again xm m plus 1 m plus 2 dot, dot, dot, dot there 0s and all that this side also 0 0 0 then you have got x minus m x minus m plus 1, x minus m plus 2 dot, dot, dot, dot.

So, again x 0 will there x 1 will be there x 3. This 3 will be there this three will be there this will be there these three will be like that you know. So, as you go and doing it finally, you are going to add this 0s will go and then entire sequence will be built up.

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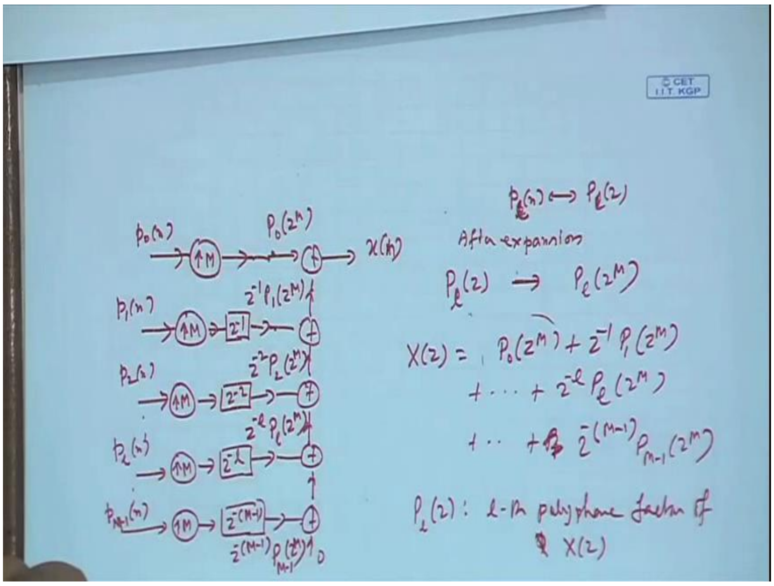

So, you will see that you can get by if you try yourself will see get xn by this method that is p zero n then p 1 n as I have done, but I will shifted by only 1, p 2 n I will shift it after expansion I will now shift it by 2 in general pl n z to the power minus l dot, dot, dot p n minus 1 n sorry m minus 1 m minus 1 n and if I add these this with this this with this and with this they.

So, if you go on adding this of course, is nothing these is not required any way because this is a 0 this will be your xn if p 0 n has got z transform capital p 0 z what is p 0 n we have already seen 0th guy mth guy m 2 mth guy like that p 1 n first guy at the 0th position m plus 1 th guy at the first position 2 m plus 1 th guy second position like that p 2 n means second guy as 0th position m plus 2 th guy first position like that. So, if this is a pln for that matter has got plz in general pl by a sequences plz after expansion after expansion means what I am plugging in 0s m minus 1 0s between every pair of sample.

We have seen expansion if you do this if the original z transform is a plz it will becomes pl z to the power m we have seen expansion today that is between every pair of sample if I bringing m minus 1 additional 0s is high speed sequence only intermediate sample values are 0s. So, the real thing sequence will have a z transform p l z sequence will have a z transform which will be original pl, but instead of z, z to the power m. So, this will be p 0 z to the power m this will be this will be p 0 z to the power m followed by delay. So, z inverse p 1, sorry not p 0 p 1 z to the power m here it will be capital p 2 z to the power m after expansion then z inverse 2. So, it will be z inverse 2 p 2 here it is again capital pl z to the power m then z to the power minus l.

So, z to the power l and lastly here again p m minus 1 capital p m minus 1 z to the power m and then these delay. So, it will be. So, if I add all of them you get z transform of xz which also shows xz is p 0 poly phase component 0th poly phase components, but expand z. So, there is z transform then 1 shifted version of you take the second poly phase components, but expand it and then 1 shift dot, dot, dot z to the power l minus l pl z to the power m that is you take the lth poly phase component expand it, but you have to shift it shifted by l to the right that is why these and lastly, sorry.

If the z domain poly phase decomposition by a factor m each of this is called not z to the power m each plz is called that lth poly phase factor of xz not p pl z to the power m you make it z to the power in this summation this plz is called lth poly phase factor and this is how these decomposition is called mth order or factor of m poly phase decomposition of I am giving z transform z we conclude here today we will then combined this decimation interpolation expansion and this poly phase decomposition to builds up efficient structures in the next class.

Thank you very much.