Discrete Time Signal Processing
Prof. Mrityunjoy Chakraborty
Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture – 15
Properties of z-transform

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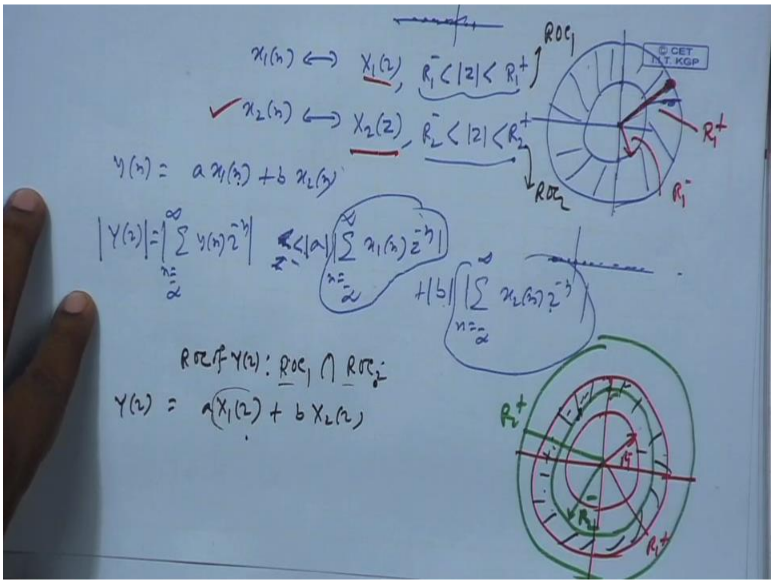

We studied z transform, now you are studying some of their properties. Suppose you are given one sequence x 1 n, and it has got z transform x 1 z. In general r o c will be between two circles; one circle at another circle. There is if your sequence is both sided, then r o c, I am assuming that z transform exist. So, therefore, there is some region in the z plane where X 1 z exists, or in that case that region is called r o c region of convergence. So, that we have seen will be typically the region between two circles. And if the sequence becomes fully left sided, there is anti causal sequence, inner circles 6 to 0. Origin may be included may not be included, if the left sided sequence starts from the left of origin and goes further to the left as you have seen last time then origin is no problem, but if the left sided sequence starts somewhere from right of the origin and then goes backward, then origin on we included.

Similarly, if the outer circle goes out then; that means, the sequence will be right sided. If that includes infinity; that means, the sequence starts at either zero or further to the right and goes to the right. So, that z equal to infinity on (Refer Time: 01:41), but if the sequence starts at any point to the left of origin, then that goes right ward, then z equal to infinity will not be included. These are all discussed in the previous mod you can see. So, that is of saying this is given and we denote the r o c to be mod z there is r radius or there is a mod z less than some outer radius, let us call it outer circle radius, that is inner circle radius. Outer circle radius you call r 0 1 plus this r 0 1 minus, one because it is x 1, so these r o c. Similarly another sequence is given x 2 n it as got another z transform, its r o c is this origin of convergence, sum r 2 r. There is another region of this kind that is a kind of ring, annular ring between two circles, this here outer radius was r 1 plus inner radius was r 1 minus, if this time r 2 plus r 2 minus.

Then if I give you a summation of the two (Refer Time: 03:00) sequence; the linear combination a times x i n b times x 2 n and you call it y n what is the z transform y n. So, y z will be summation y n get 2 the power minus n and which is nothing, but a time. So, for y this two exist mod of this is mod of this, which is less than equal to mod of this is mod of this is less than equal to mod of this entire thing, which is nothing, but mod of a times mod of this, and then mod of this into mod of this, less than equal to because of the (Refer Time: 03:57) than equality there was a sum.

So, if you apply mod on the sum, mod on the sum is less than equal to, the sum of two mods, and these mods where you can apply mod of a product is product of mods. So, earlier if you had a time this summation. So, if you take mod of the total thing, is mod a into mod of this summation. So, mod of this summation, this should be finite, this should be finite, then right hand side is finite then z transform y z is bound to exist. It is a sufficient condition I agree, but in terms of sufficient condition also only, what will be the new r o c, it should be such that both these becomes finite, this remains finite; mod a mod b they already finite. So, if it is finite if it is finite for any z, then this is finite this is finite summation finite mod y z for your chosen z, is less than equal to something finite which means mod is finite, so y z exist. So, this magnitude is finite.

Therefore, r o c of y z should be what. it was such that this if finite this is finite, but this is what this is the mod of z transform of x 1 n. this will be finite in the r o c of X 1 z, because after all this was a z transform of x i n I am taking mod of that. This we have seen in the last class, this mod will be finite, if I am picking z from the r o c of X 1 z, where it is defined. And this will be finite if I am picking z from the r o c of x 2 z, but for this right hand side there is only one common z; therefore, if this r o c and this r o c, they have some intersection that is overlap common area. Then from that I should pick z for that z, this will be finite, this will be finite, because that z is part of the r o c of this part of the r o c of this. So, and then right hand side will be finite for instance if there is an overlap like that then, this is one annular ring this may be r 1 minus this may be r 1 plus. And suppose you have got another case, this may be r 2 minus this may be r 2 plus.

So, see in one case you have got this green between these two green circles; that is one region, that is the r o c for this guy this at between the two red circles r 1 minus r 1 plus you have the r o c of this guy, but there is an intersection, there is a overlap common area between the two, and that common area is, from this inner green circle to the outer red circle this, this is the intersection between the two r o cs. So, if I call this as r o c 1 if I call this as r o c 2, r o c 1 is the area between the two red circles, r o c 2 is the area between the two green circles. Now if they have some intersection, the overlap that intersection inside (Refer Time: 07:22) notation we denote as intersection r o c 2. If there is an area of intersection, then this will be the r o c of y z, if it exist, and in that case y z will be nothing, but a times x 1 z and b times x 2 z, because if you know that y z will exist, there is this mod will be exist finite. In that case if you take j transform y n z to the power minus n summation; that is same as a x i n z to the power minus n summation which is a X 1 z plus b x 2 n z to the power minus n summation which is x 2 z for b x 2 z, because x 1 z also is existing x 2 z is existing, because I am picking z from the intersection between the two r o cs therefore, for that z this is existing this is existing.

So, that summation exists and that is your y z. So, r o c of y z will be intersection of the two r o cs and in that case, if this intersection exist, if there is no overlap then there is no r o c for y z. So, z transform does not exist for y n. So, overlap such overlap must occur. If there is a overlap area between the two r o cs r o c 1 r o c 2 then that will be the new r o c of y z you should pick z, from there for that z x 1 z also exists x 2 z also exist. Therefore, if you take the z transform y n there is summation y n z to the power minus n sum, it will be a into summation x i n z to the power minus n that will be existing. So, you can called it x 1 z plus b into x 2 n z to the power minus n, that will be existing its mod will be finite

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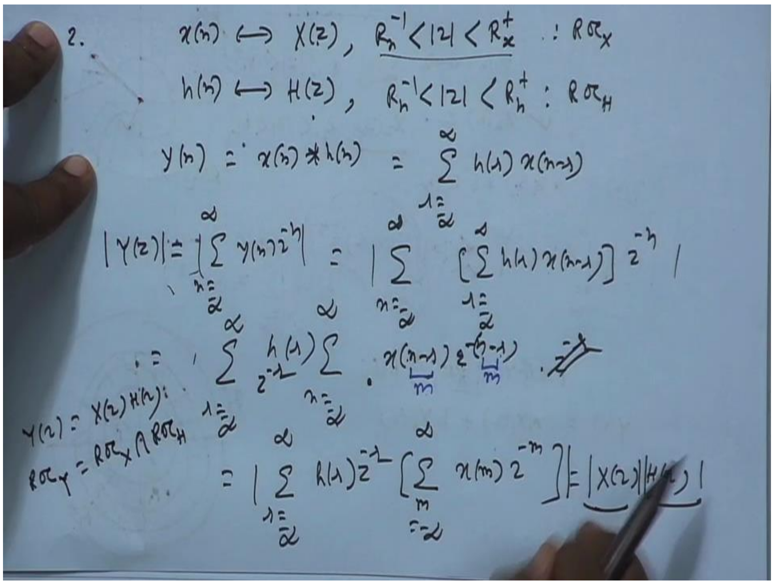

So, it will just summation of the two; this one property. Number two, convolution suppose x n has got x transform some r x, because it is connect to x r x plus r x minus this I call r o c x for x n. And there is another sequence h n, it is z transform h z its r o c is, general r o c r h plus r h minus, which I call r o c for h. now suppose I have got a convolution y n as I told you I always prefer to write convolution this away, h r x n minus r, it does not matter here though, either it is finite. So, z transform of y n there is y z as it did not exist, it will be assuming it exits, y n z to the power minus n n minus infinity to infinity, y z you replace by that.

Let be this h r x n minus r r from minus infinity to infinity z to the power minus n, then two summations interesting the two summations that is a next step, r summation goes out summation over n goes in. So, all quantity which is the function of n they should be inside. So, x n minus r this as n, so this would be inside z to the power minus n it should be inside; h r can be common as outside z to the power minus n. Suppose I make it minus n minus r, because r is every time you pick up one r from outside sum h of r then into you run this summation for that r, n varies from minus infinity to infinity, but r is constant in this sum. So, z to the power minus n minus r if I bring; that means, I brought down extra term z to the power minus minus plus; that is z to the power plus r. So, I should consider it by here z to the power minus r.

But again this does not depend on n, this depends only on (Refer Time: 11:40) this can go out of the summation as a common term. So, I can push it here. So, this summation now n minus r you can call m, which means this will be. Now m minus r is m, r is fixed. So, if n goes to plus infinity m goes to plus infinity, if n goes to minus infinity m goes to minus infinity. So, x of m z to the power minus m, now this summation, if this exist, so this is the z transform of x. So, this you can take common and it goes outside the summation, and then there is Z 0 will go out here, and then have inner summation this h r z to the power minus r this again z transform of h as you being it exits. So, it will be X z.

So, if z transform here exist exits, they will be just be product x z h z, but for that this should exist, and this should exist; that is mod of this. In fact, if you take now mod, mod of this, everywhere mod of this, mod of this, you know finally, mod here mod of a product, and therefore, mod of this and mod of a product is mod of the. So, this to be finite this to be finite, this would be finite in the r o c of x z this is finite in the r o c of h z. if the two r o cs have some overlap like the previous example, then that overlap area will be the common r o c for y z.

I should pick z from their and their the z transform of y n Y z will be product of the two z transforms; that is y z as we have seen before applying mod here it was x z h z, and r o c will be, r o c of y will be intersection between the two r o c x as before, because this must be existing, but this exist only here, and this also must be existing there is mod this will be finite, this will be finite, this is finite here.

So, both to be satisfied means this also to be satisfied, this also to be satisfied. So, the two r o cs if they overlap there is an intersection, like my previous diagram if the intersection common area, between two r o cs there is between two red circle one area between two green circle one area 1 r o cs and their common area is this shaded part. If such overlap or intersection exists, then that will be the r o c of y and there if I pick z from there, then y z will be exist with h z. But if there is no such overlap then y n does not happen a valid z transforms alright. We are assuming that such an r o c exist; that is the two r o cs overlap, and therefore, y z will be x z into h z, is the intersection part of the two r o cs, this is one property.

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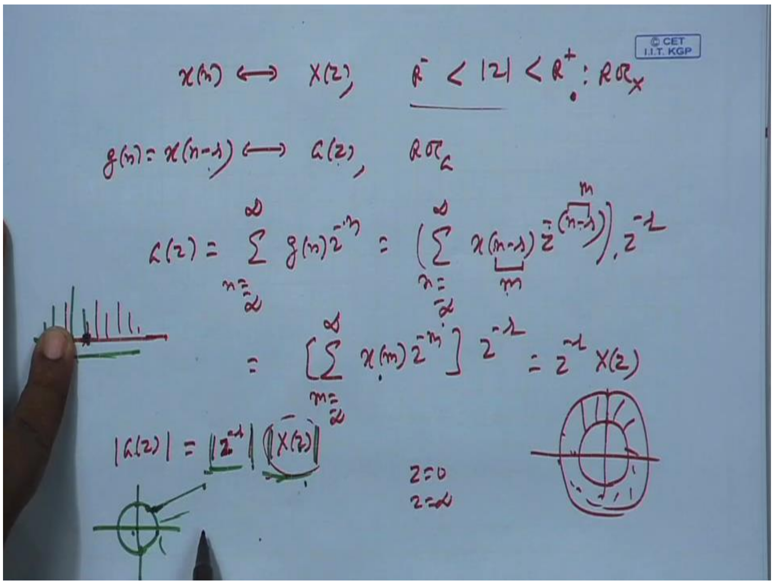

Then suppose x n has got z transform x z as before, for mod z say r plus r minus in general. In general it is between two concentric circles, the annular ring between two concentric circles. Inner circle you can go down to zero or outer circle you can go down to infinity as special cases that you have seen, this is given. Now if I have x n minus r is a linear sequence, you can call it g n, what will be g z and what will be the r o c, what will be the r o c of g, this r o c of x, what will be the r o c of g. So, fine suppose g z exist there is this bound for some z in the z frame g z exist, there is mod of g z is finite.

Suppose in that case I can write g z is this summation, same formula z to the power minus n, and now bring back g n is equal to x n minus r, sorry z to the power minus n, but I make it z to the power minus n minus r that is I bring an extra term z to the power plus r, to cancel it I have z to the power minus r, but summation is about n, and this does not depend on n. So, this entire summation can be here and this can go out as a common. and now you call it m, you call it m, if n goes to plus infinity m goes to plus infinity, because n minus r is m same sign if n goes to minus infinity, m goes to minus infinity, which should be a summation, r was the delay. So, this is nothing, but z to the power minus r x z, but only thing is I am assuming this is existing, there is its mod will be finite. For the mod to be finite it is a product mod of the product is mod or z to the power minus r into mod of x z. So, this should be finite, but this is finite what this r o c x, and this to be finite.

Now, if suppose the sequence is most I did from minus infinity to infinity, then you know that r o c will be between two circles. So, the case of z equal to 0 or z equal to infinity will not arise, because it is between two circles. So, z to the power minus r; suppose r is positive. So, z to the power minus r, if z is 0 0 to the power minus there would be a problem if it do not exist, but z equal to 0 does not occur in this case. Similarly if r is negative, there is g n is x n plus something. So, it will be z to the power minus r, but r is negative, so it is a positive power of z. In that case z equal to infinity would have created a problem, because this would have gone to infinity; that means, this would not have existed, but that is ruled out in this case, because z equal to infinity is not allowed only this area is allowed. So, here, but this kind of cases r o c remains as before, r o c of g z is same as the r o c of x, but suppose I have a sequence which is purely right sided. Suppose you start from here somewhere here, purely right sided.

In the case, hence you are starting from here in that case, we have seen r o c will be r o c of x will be like this, outside the circle and it will go up to infinity, it will go up to infinity. So, in that case also r o c will be r o c of g z for will be where if the area where mod x z is finite; that means, r o c of x which is starting from this circle, some circle are going up to infinity and this also is finite. So, if this sequence starts at some point to the right (Refer Time: 19:50) it goes to the right z equal to infinity is not a problem, at z equal to 0 will not occur, because I am going outside the circle. So, this will be existing, because even if I put z equal to infinity. If r is positive suppose then the positive, then negative power of z, and even if z is infinity this will exist, and z equal to 0 will not occur here, because I am starting from a circle and going out. So, in this case again g z will have the same r o c x z.

Suppose it is right side z, but you are starting from left of origin, and going to the right and r is a positive, this are various cases. In that case we know r o c of x z will be starting at circle, some circle and going up going up to infinity, but not in putting infinity, there is the r o c of x. And if you do not input infinity then this thing, this fellow z to the power minus r, this will not create any problem in any case r is positive even, if r is negative this will not create any problem, because infinity is not allowed. So, likewise you will see that r o c will be same as r o c of x. Here if this is the sequence r o c of x z will start from circle and go up to infinity, but not including infinity. So, that is r o c x. In that region, so r o c of g z should be such which we overlap of the r o c of this, and r o c of this now r o c of this is starting from a circle going up to infinity, but including infinity. In that zone, if I pick up any z from that zone, there is neither infinity not zero, and this will have no problem even if r is negative r is negative.

So, z to the power plus. So, positive power, even there is no problem because z equal to 0 z equal to infinity it is not one coming. So, r o c of g z will be same as this r o c of this. Any z you pick up from the r o c of x z this also the finite here, for this sequence. earlier also you have seen if you start from here, and you start from here and r is a positive, then again this will be finite ,even if you put z equal to infinity, and this r o c will go up to infinity will into the infinity. So, this is the finite, because even you put z equal to infinity this is negative power of z no problem with that, and z cannot become 0 because you are starting from one circle. So, a whole r o c of x z, if I pick up any z from there, this also will exist there. Therefore, there also r o c of g z will be same as that of z.

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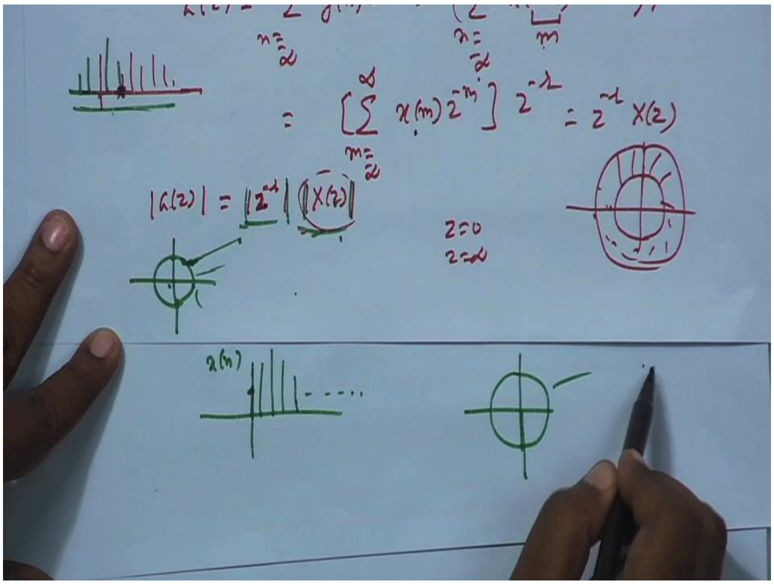

But suppose we consider this case, you start like this, and this is x n, only right sided, but its r o c will be again outside a circle, and is going up to infinity. If r the shift is negative, if shift is negative, it will be shifted to this side. So, in that case, again to find out the r o c of g z, you have to first see the r o c of x z and the region where this exist and intersection of other two. Now r o c of x z is outside a circle going up to infinity. And once again if I pick any z from there, bring it here up to infinity. If the problem is infinity cannot be include their now, because if r is negative to the power positive power that would have come. In this case I am taking up to infinity, but z to the power minus r if r is negative, then z to the power minus means z to the power positive r, some positive value. So, z equal to infinity will not be allowed, even if r o c of x z includes infinity. So, these are the things that we you know same depending on where the sequence starts and all that.

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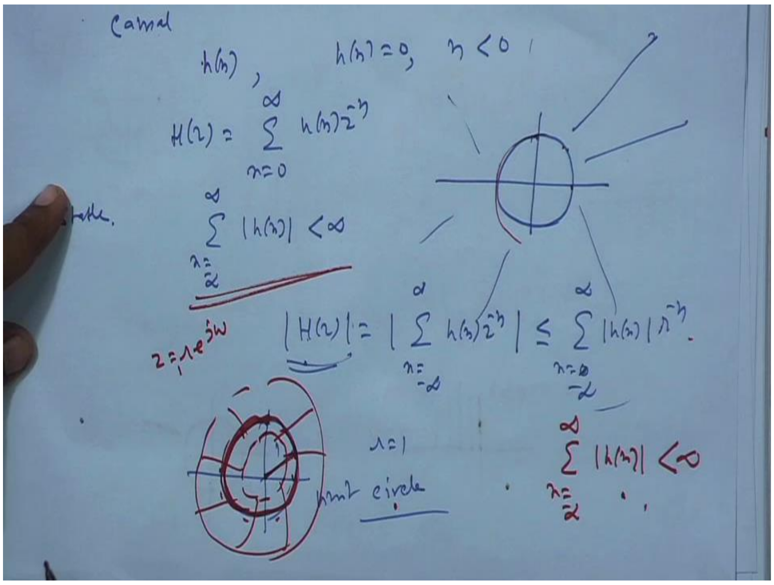

Now, so we have see convolution, we have seen delay shift, and now suppose you got a rational system; before that something more. Suppose there is a causal sequence h n, causal; that means, h n equal to 0 for n less than 0, h z. So, instead it is purely right sided sequence. So, h z will be we have said for purely right sided sequences r o c will be outside the circle and you will go up to infinity. So, we have causal sequence z transform. We will have causal sequence will be, I mean region of convergence, starting from a circle and then going up to infinity. Stable, now what is a stability condition, if h n is the (Refer Time: 25:54) linear shift invariant system, it is summation, but for z transform h z to exist, there is mod of which is less than equal to, mod of a summation less than equal to summation of mod, and mod of a product is product of the mod, and if you take that this it is r to the power minus n, so minus infinity to infinity, not necessary causal, this thing.

So, suppose r is 1; that is r equal to 1 for that also if it is finite; that means, I will say r o c of h z includes r equal to 1 that is r equal to 1 means, a circle of radius 1, it is called unit circle in the z plane. So, if the r o c of h z includes r o c of h z as such is, if it is both sided it will be between two circles, but if it is like this one is inner one is outer, and this is the r o c of h z, but it includes this r equal to this unit circle, for which r is 1. Then what will happen? this right hand sum just becomes and I am wanting, if it is part of r equal to 1 is part of r o c; that is this circle is part of r o c. if you put r equal to 1 this summation also should be finite, which is same as this condition. This as the stability condition, and here I was just approaching r o c region of convergence of h z. So, I took out the summation of h z applied mod over that, and mod of this is less than equal to summation mod h n up to the power minus n.

So, if h z is to exist, mod of h z is to exist there is, if you are considering the r o c region of convergence of h z, where is I am picking z from the r o c of h z, where z is in general r into e to the power j omega, then for that r if I have this summation this would be finite. Now I say suppose r equal to 1, is part of that r o c. if r equal to 1 this summation becomes this, but r equal to 1 means in the z plate it is a circle of radius 1, which is called unit circle. So, if unit circle is part of the r o c of h z; that means, on the unit circle also, this mod of this, I mean this will be finite, but on the unit circle r is 1. So, this will be finite, but this condition and this condition are same. Therefore, the system will be stable, if r o c of h z includes unit circle, if it is causal its starts from r, r o c starts from some circle and goes outside up to infinity, including infinity. If it is stable r o c where ever this should include unit circle.

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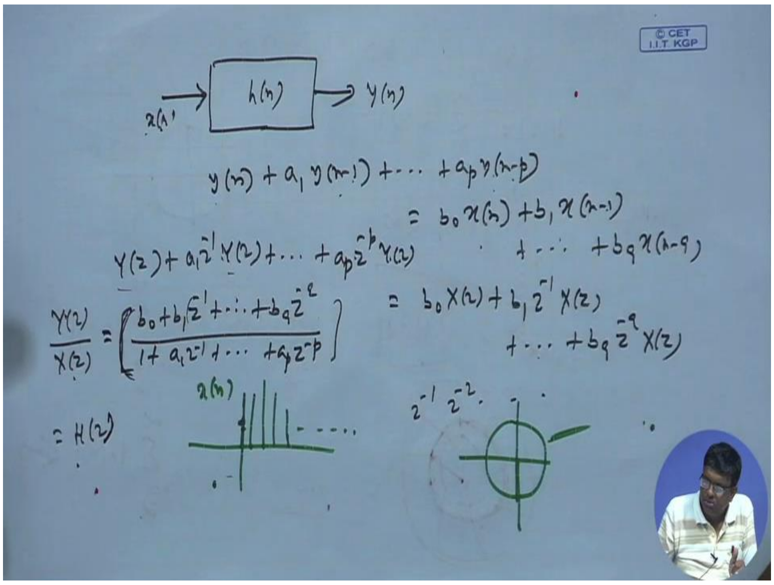

Now, there is a relation of this with, if you go for a linear shift invariant system impulse response h n, but this was rational, it is given like this; y n. Suppose we do not know whether it is linear shift invariant, just this equation is given. if we take z transform on the left hand side, we take z transform on the right hand side, this will be equal this will give raise to y z then a 1 z inverse y z. I am assuming that z transform of all of them and all of them have some common area, some common area of intersection, the r o c of y z r o c of y z minus 1, r o c of the z transform of y n minus 1, r o c of the z transform of y n minus p for x. All have some you know overlap area I am picking z from there. So, all of them exist. So, it is z z inverse. So, it is z inverse p, and this side is b 0 x z plus b 1. I am seen already if it is x n minus r, then z transform becomes z to the power minus r into original z transform r is 1. So, z to the power minus 1 original z transform x z, like what I have done here.

Y n at z transform Y z y n minus r r is 1. So, z transform if it exist, it will be z to the power minus r there is z to the power minus 1 into original y z. If it is n minus p it will be z to the power minus p original y z, similarly here. And then if you take y z common x z common, this ratio will be. Now if initial conditions are given to be zero for this equation, then I know this system will be linear and shift invariant and in that case output y n and input x n, they also related by a convolution relation between x n and h n, and z even y z is x z into h z or y z by x z in that case will be h z. If the system is linear and shift invariant that is this equation has been given with 0 initial conditions. So, in that case h z become like this. ratio of two polynomials; numerator in terms of z inverse powers of z inverse, a polynomials in terms of power of z inverse; z inverse 1 z inverse 2 dot, dot, dot (Refer Time: 32:33) also. From this transform function, this called a transform function, concept of pole and 0 will come.

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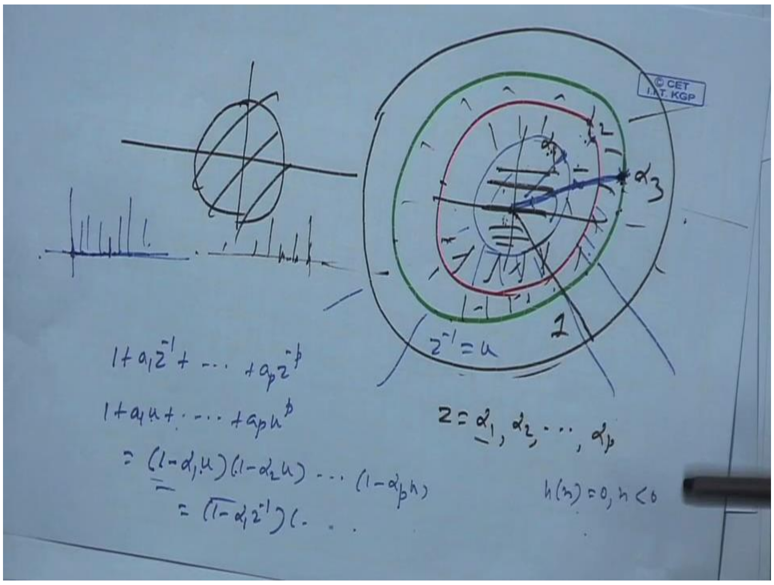

Suppose this numerator denominator polynomial, z to the power minus 1 suppose I call u. So, it is 1 plus a 1 u plus dot, dot, dot a p u to the power p. You can factorise it in terms of you know in this manner; one may be minus some coefficient say alpha 1 u 1 minus alpha 2 u dot, dot, dot, 1 minus alpha p u, all first order factors this is fundamentally in algebra. So, if you multiply z 1 into 1 into 1 you will get 1 that will be some term, like u with some coefficient another term u square like that you can factorise, and then u means z inverse, so 1 minus alpha 1 z inverse dot, dot, dot, dot. Same you can do in the numerator. So, these you can write as, but numerator I leave it as it is, numerator polynomial I call it n z, n for numerator is a function of z. So, instead of writing the full expression I just write n z, but denominator i factorise. So, it will be alpha 1 u. Now what is a root of this u equal to 1 by alpha 1? What is a root of this, u is equal to 1 by alpha 2, dot, dot, dot, dot.

So, this function is that denominator they have got p factors, every factor gives us just one root, if you take u equal to that this factor becomes 0. So, there will be division by zero this means this will suit up to infinity. So, this function will not exist. So, therefore, r o c of h z, but otherwise if you take u at u is z inverse, you can directly write now. So, z inverse was alpha 1, u is at what point, at what value of this will be 0, alpha 1 into z inverse equal to 1, alpha 1 z inverse equal to 1, which means z equal if you take this z inverse is 1 by z, if you take that to the right hand side alpha 1. So, at z equal to alpha 1 alpha 1 into alpha 1 inverse cancels become 1 1 minus 1 0. So, at z equal to alpha 1 this function, this factor is become 0 and overall function does not exist, because it amounts division by 0. Then z equal to alpha 1 will be equal to a pole.

Similarly, z equal to alpha 2 will be called a pole, because at z equal to alpha 2 also this factor become 0, and the function overall thing does not exist. Similarly at z equal to alpha 2. So, I am got p poles z equal to alpha 1 alpha 2, dot, dot, dot alpha p. So, if z takes values other than them, then each of the factor exist n z also exist, the whole function exist. So, if you are giving a form like this, a transform function you are giving a form like this and numerator polynomial in z, numerator polynomial in z. Then you find out, you do carry out factorisation in first order factors of the denominator, you get this pole, alpha 1 alpha 2 alpha p. You now know this poles cannot be part of the r o c of h z. see if I am trying to construct, trying to have an r o c of h z in the z plane that r o c must not include alpha 1 alpha 2 up to alpha p; that is suppose you have got this alpha 1 alpha 2 alpha 3, this alpha 1, this alpha 2, this alpha 3.

Suppose we draw circles through this and the circle through this and another circle through this. Then you see this region can be 1 r o c, because neither alpha 1 nor alpha 2 nor alpha 3 exist here. So, z is neither equal to alpha 1 within the circle inside the circle, neither equal to alpha 2 nor equal to alpha 3. So, for any z here these factors, none of these factors will become 0 3 factors in this example. So, this overall function h z will exist. So, this qualifies for 1 r o c, then this region, this neither carries I mean neither has alpha 1 nor alpha 2, because if the region between the two circles it does not include the circles. So, z is neither equal to alpha 1 nor equal to alpha 2 nor equal to alpha 3, there is one possibility. And again this is another possibility, and outside the circle is another possibility. Now if it is this first possibility inside a circle. Now if it is inside a circle going up to zero, you remember what should be the corresponding sequence in j time domain, it will be a left sided sequence, and that is starts either at zero or goes starts from left side.

So, z equal to 0 is not a problem. So, either from here or from here or form here and goes down like this and this side zero. So, we will call it anti causal sequence. If it is here then it is annular ring between two circles, so; that means that will be both sided. If it is here, if I take r o c to be this there is another valid continued for r o c, because z is neither alpha 2 nor alpha 3 nor alpha 1 within this, there is one valid r o c. So, in that r o c again it is a region between two circles for annular ring, there is region between two concentric circles, with centre or original centre. So, again that will be another both sided sequence, and if I take this outside this green circle area up to infinity, that will again another valid r o c, but this time it will amount to a sequence, which is purely right sided, because I am going outside a circle and going up to infinity. So, it can neither start for here, or it can start further from right from right of the origin, either like this, this, this, this or you can start from here at right sided, you cannot start I means, so its causal. Now if the system governs by this transform function, is to be causal and stable.

For stable we know r o c must, for causal it should start at a pole for this pole, from the origin for this pole in terms of some areas magnitude, if you take this much is the magnitude of alpha 3, alpha 3 is a complex value, this is the magnitude this is the angle. So, this much is mod alpha 3, this much is mod alpha 1, that much is mod alpha 2. So, you take that pole which are the maximum magnitude that that pole will be furthest from the origin. Now draw a circle through this go outside. So, this outside region cannot contain any of the poles. If I take that to be r o c of h z, then that will be the amount to a causal sequence in time domain, because it is starting from one circle and going outside up to infinity.

So, in that case I will get a causal sequence. not for I take the r o c between this, because then the sequence will be both sided, not causal, not when I take it here, then again will be both sided. Not when I take it here, where h n will go to the left side. For causality h n should be 0 for n less than 0, it should start at n equal to 0 and go up to infinity, then its causal, but the corresponding r o c will be outside a circle, that is possible only if you take this pole farthest from the origin and draw a circle, and then go out. Then this region cannot have any pole, that is another valid r o c, but that is outside a circle of infinity that is only way.

So, if I take that r o c my this thing, sequence in time domain, that is if I take the inverse transform, inverse z transform that if I get h n that will be causal. Stability, for stability I have told that r o c should include unit circle; that is circular radius one. So, this r o c outside the circle should contain a unit circle. Unit circle means radius should be one. So, this r o c will starts from the green circle and goes outside up to infinity, this should include this unit circle, which means every pole should be within unit circle, and then you take the one the pole which is furthest from the origin, draw circle through that, and take the r o c to be the area outside the circle. So, that will be outside a circle of infinity.

So, the corresponding sequence will be causal right sided, and that includes unit circle therefore, that will be stable, which means for stability and causality, poles of a rational system must lie, all the poles within a unit circle. That is all for today.

Thank you very much.