Probability and Random Variables/Processes for Wireless Communication
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Module 4
Lecture No 20
PSD Application in Wireless-Bandwidth Required for Signal Transmission.

Hello, welcome to another module in this massive open online course on probability and random variables for Wireless Communications. In the previous module, we have looked at an important quantity that is the power spectral density of the random process X(t) which is basically the Fourier transform of the autocorrelation function $R_{XX}(\tau)$. Now let's look at an application of this concept of power spectral density in the context of wireless communication or in the context of communication systems.

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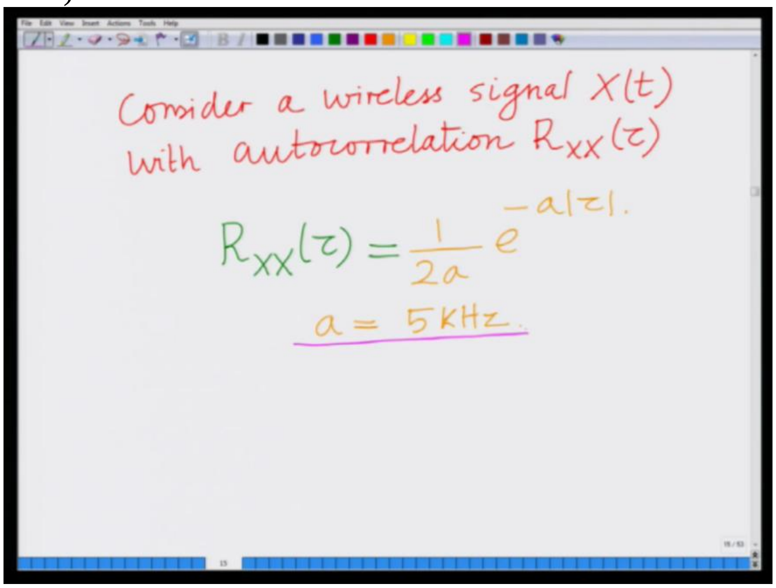

So let us look at an application of power spectral density which we have denoted by the abbreviation P(S). So we consider a wireless signal X(t) which is a random signal ~~which is naturally therefore process~~ since it is a function of time. Consider a wireless signal X(t) with autocorrelation functions, ~~with autocorrelation~~ $R_{XX}(\tau)$, which is given as –

$$R_{XX}(\tau) = \frac{1}{2a} e^{-a|\tau|}$$

Right? And further, we are given that this parameter, a is let's say equal to 5 kHz. a is given to be 5 kHz. So what we have, we have a wireless signal X(t) which is a random signal.

So random signal is naturally a random process because it is a random variable as a function of time. We have also been told that its autocorrelation $R_{XX}(\tau)$ is $\frac{1}{2a}e^{-a|\tau|}$ with the parameter a is 5 kHz. Now, $e^{|\tau|}$ is basically a decaying exponential with respect to τ on both sides of τ. That is for positive τ and negative τ because it is $e^{|\tau|}$. It looks something like this.

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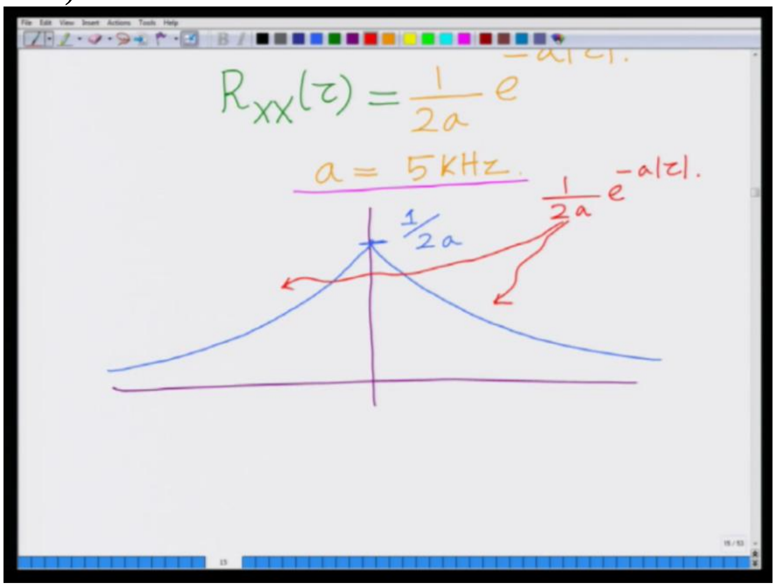

If I have to draw a decaying exponential, it looks something like this. At obviously, τ equal to 0, this is basically 1/2a and this is basically ~~your, this whole thing is basically your~~ $\frac{1}{2a}e^{-a|\tau|}$. And now we can ask a couple of questions. For instance, for this signal, what is the power?

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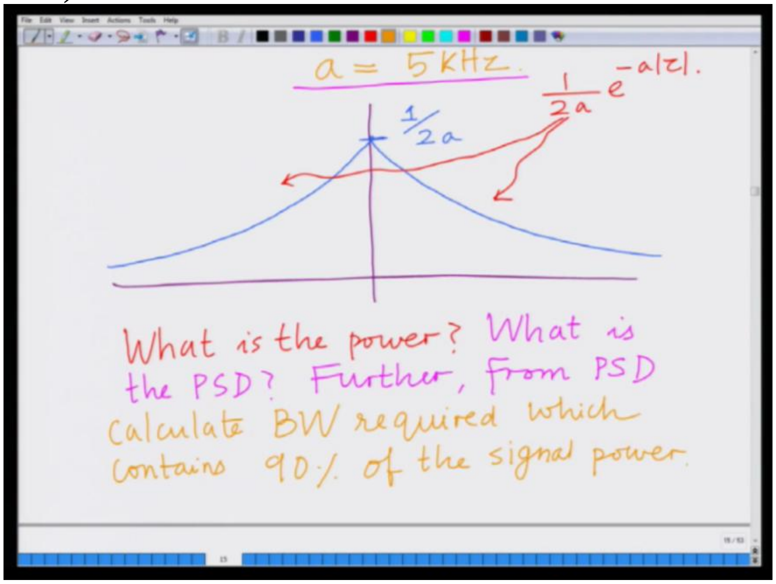

For this random signal, what is the power? For this wireless communication signal, what is the power? What is the power spectral density for the given signal X(t)? Further, another interesting aspect, from the power spectral density, calculate the bandwidth required which contains 90% of the signal power. ~~So we are being asked.~~ Basically we are being given a random signal X(t) with autocorrelation function $R_{XX}(\tau)$ is $\frac{1}{2a}e^{-a|\tau|}$. We are asked, what is the power of the random signal? Right? What is the average power of the random signal? What is the power spectral density of this random signal X(t) and more importantly, what is the bandwidth required to transmit this signal ?

And not just any bandwidth. What is the bandwidth required to transmit this signal which contains 90% of the signal power? On an average which contains 90% of the signal power. And this is an important quantity for wireless communication because we have to decide a bandwidth for transmission of the signal and ~~the signal~~ the transmitted signal cannot have an undue or cannot have a large amount of power outside the band of transmission. So therefore, most of the signal energy has to be restricted to a particular band of transmission.

Therefore we are asking the question, ~~what is the bandwidth of this~~ what is the bandwidth that is required which contains 90% of the signal power so that only 10% of the signal power or a small faction of the signal power is basically radiated outside of this band. And this is important because one has to restrict a majority of the wireless signal power to a certain band.

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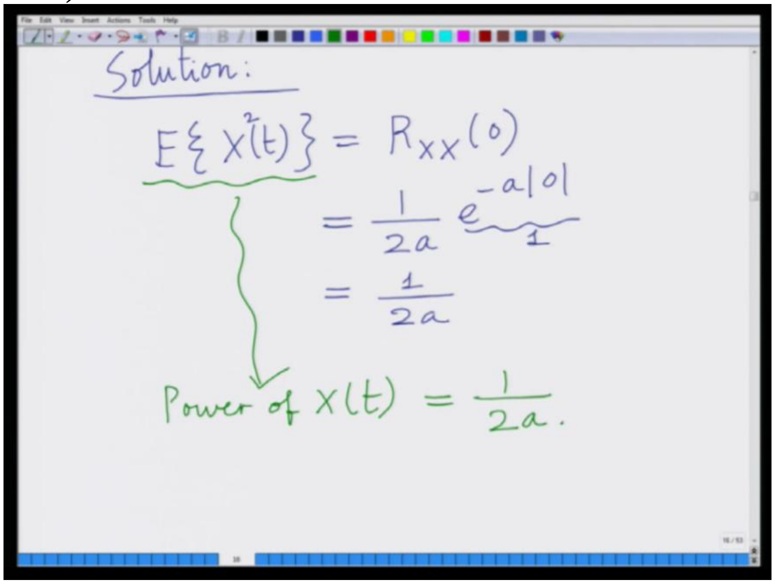

Now let us look at the solution for this. And the 1st part is simple because the power is –

$$E\{ X^2(t)\} = R_{XX}(0) = \frac{1}{2a} e^{-a|0|} = \frac{1}{2a}.1 = \frac{1}{2a}$$

So basically from the autocorrelation function, what we get is that the ~~power of the signal~~ power of the message signal the power of the signal is 1/2A.

So basically how do we obtain ~~have developed in~~ the power of the signal X(t)? Remember, $R_{XX}(0)$ that is autocorrelation corresponding to the delay, $\tau = 0$ gives the average power of the signal. By setting $\tau = 0$, we have obtained that the power of this signal is basically 1/2A. Next, we have asked, what is the power spectral density of this signal? That is the power spectral density. Remember the power spectral density is the Fourier transform of the autocorrelation function.

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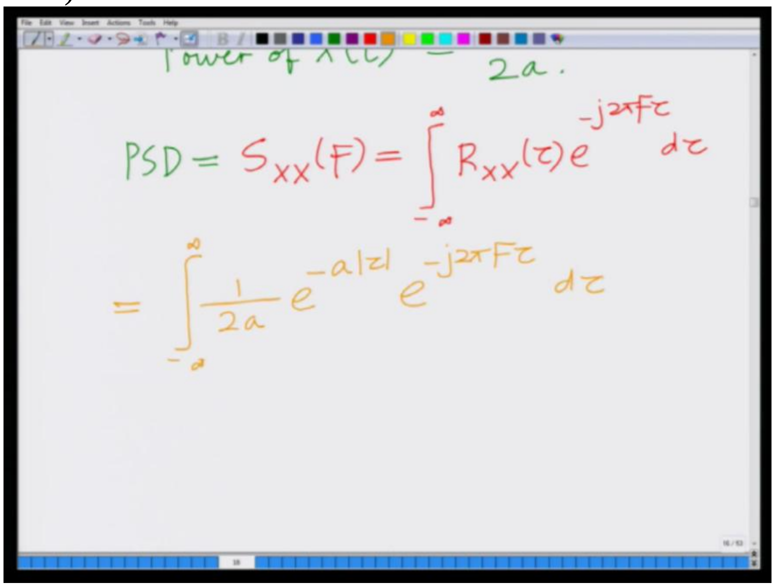

Therefore, the power spectral density can be obtained as - ~~equals RXX of Tao which is equal to basic power spectral density I'm sorry this is~~

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j2\pi f\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \frac{1}{2a} e^{-a|\tau|}\, e^{-j2\pi f\tau} d\tau$$

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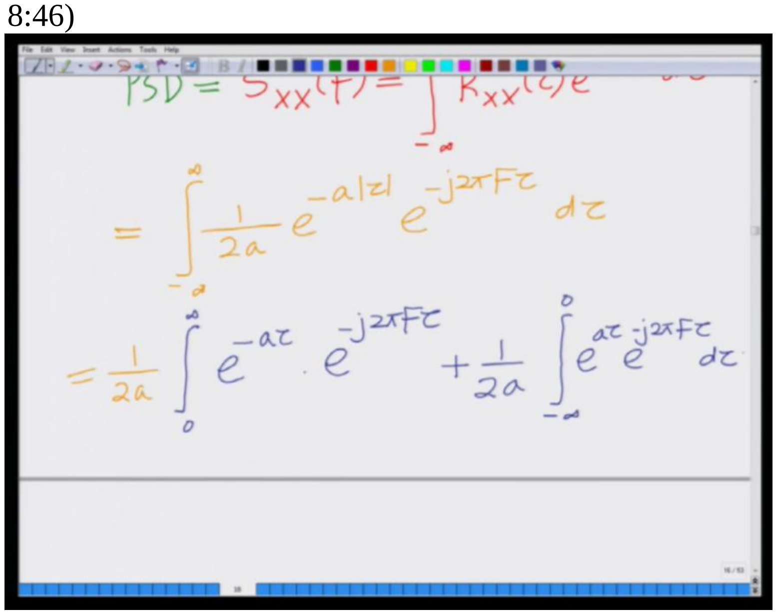

Now since the magnitude is basically I can split this just because as I'm looking at the modulus of τ, I can basically split this into 2 integrals. This is basically –

$$= \frac{1}{2a}\int_0^{\infty} e^{-a\tau}\, e^{-j2\pi f\tau} d\tau \; + \; \frac{1}{2a}\int_{-\infty}^{0} e^{a\tau}\, e^{-j2\pi f\tau} d\tau$$

Okay?

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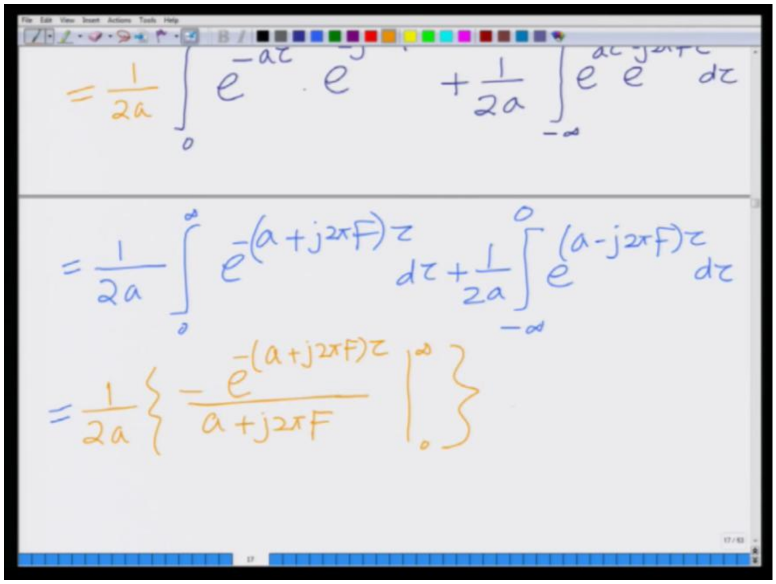

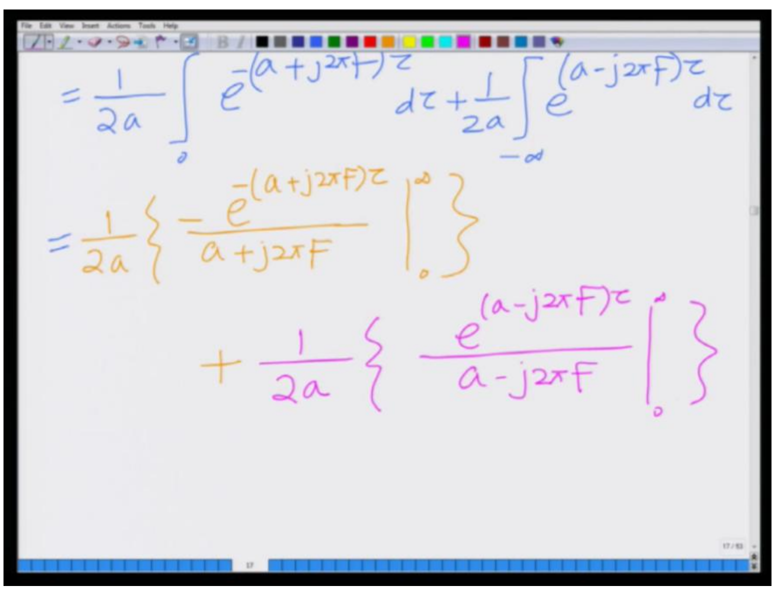

So I have split this into 2 integrals and now I can write as –

$$= \frac{1}{2a}\int_0^{\infty} e^{-(a+j2\pi f\tau)} d\tau \; + \; \frac{1}{2a}\int_{-\infty}^{0} e^{(a-j2\pi f)\tau} d\tau$$

$$= \frac{1}{2a}\left\{\frac{-e^{-(a+j2\pi f)\tau}}{a+j2\pi f}\Big|_0^{\infty}\right\} + \frac{1}{2a}\left\{\frac{e^{(a-j2\pi f)\tau}}{a-j2\pi f}\Big|_0^{\infty}\right\}$$

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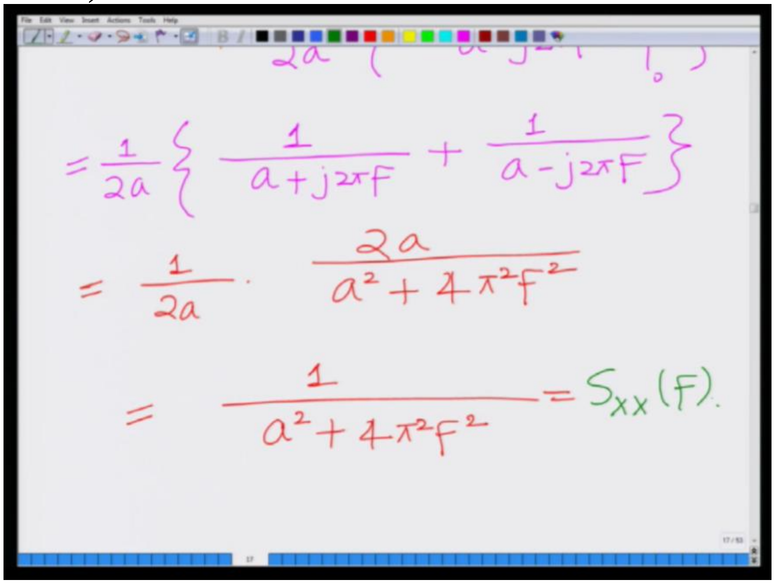

And now therefore you can see that this is basically –

$$= \frac{1}{2a}\left\{\frac{1}{a+j2\pi f} + \frac{1}{a-j2\pi f}\right\}$$

$$= \frac{1}{2a} \cdot \frac{2a}{a^2+4\pi^2 f^2}$$

$$= \frac{1}{a^2+4\pi^2 f^2}$$

$$= S_{XX}(f)$$

and this is therefore your power spectral density. We have derived the power spectral density for this random message signal $X(t)$ with the autocorrelation $\frac{1}{2a}e^{-a|t|}$.

We have shown that the power spectral density, which is basically the Fourier transform of the autocorrelation function that is given as –

$$S_{XX}(f) = \frac{1}{a^2+4\pi^2 f^2}$$

So this is the power spectral density of the message signal under consideration.

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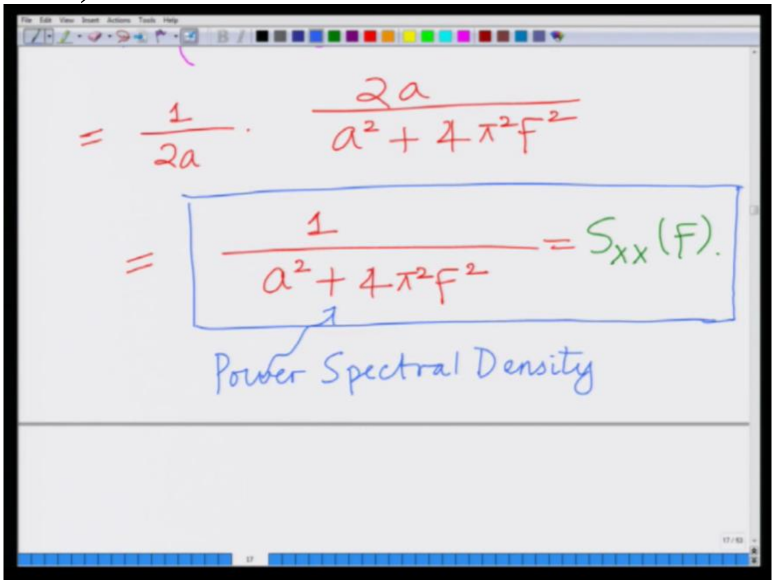

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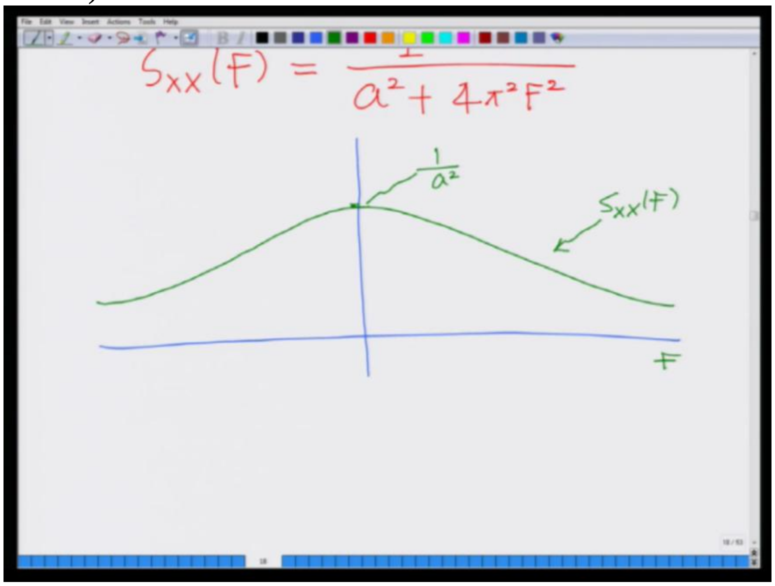

If I plot this power spectral density at f = 0, it is –

$$S_{XX}(0) = \frac{1}{a^2}$$

This is the power spectral density $S_{XX}(f)$ of the wireless signal under consideration which means if you look at the frequency band, the power spectral density is nonzero over the entire frequency band.

Basically it decays towards f = 0. At f = ∞ and f = -∞, it becomes 0 but it is otherwise basically present over the entire frequency band which means if I have to transmit all the power in this signal, I have to utilise the entire frequency band which is not possible because the bandwidth of frequency transmission in a communication system or a wireless communication system is limited. Right? So we cannot have an infinite frequency bandwidth, a frequency band of infinite bandwidth available for transmission of this signal.

Therefore this leads to the next question that is, we would like to find out the bandwidth that is required for transmission of 90% of the signal energy. So the next question is very interesting.

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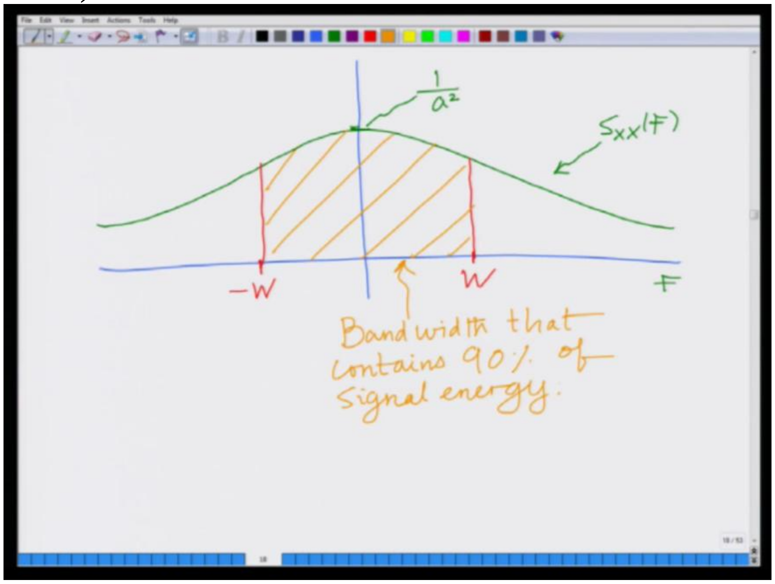

What is the bandwidth? So let us say I consider a bandwidth [-W, W]. So what is the band [-W, W] ~~that consider that contains so what is the band or bandwidth that contains 90% of~~ that contains 90% of the signal energy and that is very simple.

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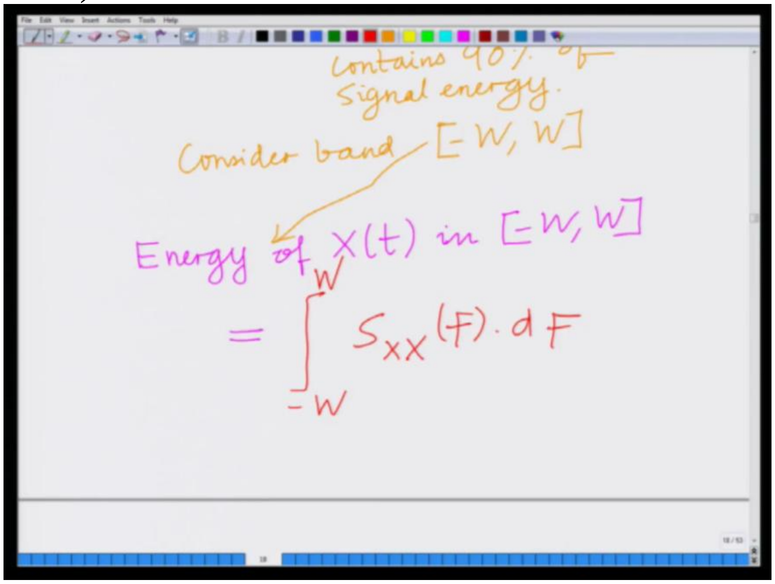

We said the energy of this signal in this band that is –

$$\text{Energy of X(t) in [-W, W]} = \int_{-W}^{W} S_{XX}(f)df$$

This energy is basically the integral of the power spectral density that is the area under power spectral density in [-W, W].

And this has to be 90% of the total energy. Remember, we found earlier that the total energy is 1/2A. Therefore this energy in the band [-W, W] has to be 90% of 1/2A.

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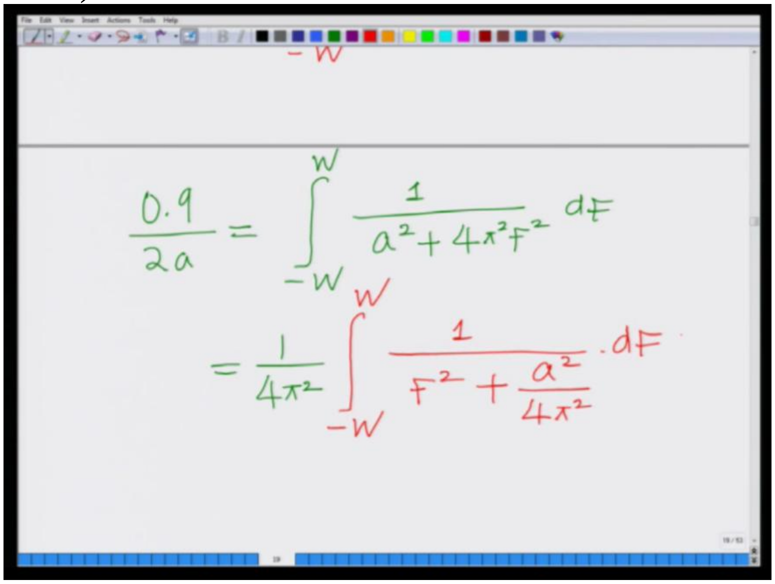

So this energy has to be equal to 90% or 90 over 100 times one over 2A and therefore that gives me basically that –

$$\frac{0.9}{2a} = \int_{-W}^{W} \frac{1}{a^2 + 4\pi^2 f^2} df$$

$$= \frac{1}{4\pi^2} \int_{-W}^{W} \frac{1}{f^2 + \frac{a^2}{4\pi^2}} df$$

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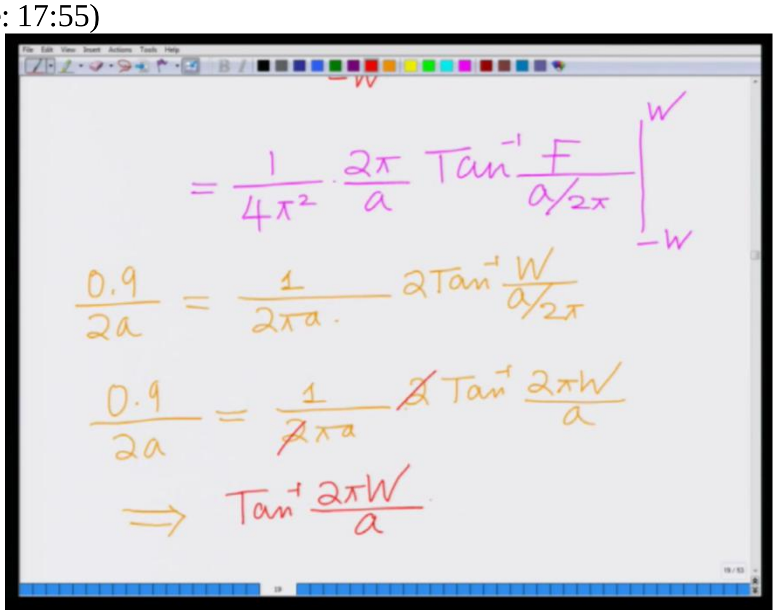

$$= \frac{1}{4\pi^2} . \frac{2\pi}{a}\ tan^{-1}\frac{f}{\frac{a}{2\pi}} \Big|_{-W}^{W}$$

$$\frac{0.9}{2\cancel{a}} = \frac{1}{\cancel{2}\pi\cancel{a}} . \cancel{2} tan^{-1}\frac{W}{\frac{a}{2\pi}}$$

$$\frac{0.9}{2} = \frac{1}{\pi} . tan^{-1}\frac{2\pi W}{a}$$

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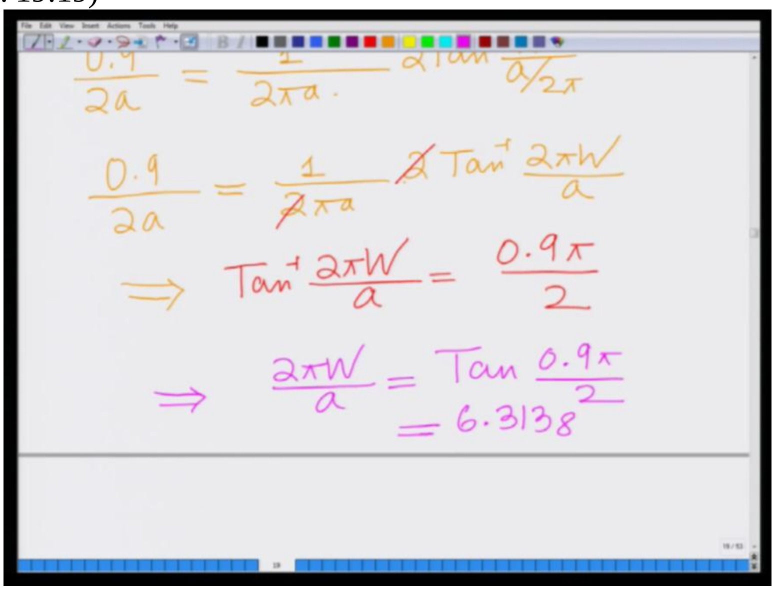

$$tan^{-1}\frac{2\pi W}{a} = \frac{0.9}{2\pi}$$

⇨ $\frac{2\pi W}{a} = \tan\frac{0.9}{2\pi}$

$= 6.3138$

⇨ $W = \frac{a}{2\pi}\ x\ 6.3138 = 1.005a$

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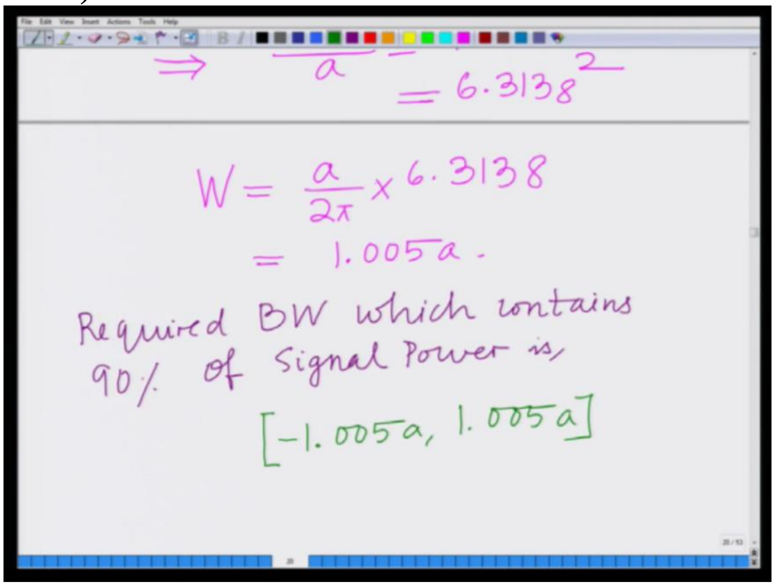

Therefore the required bandwidth which contains 90% signal power ~~bandwidth required bandwidth which contains 90% of the signal power~~ is [-W, W] that is [-1.005a, 1.005a].

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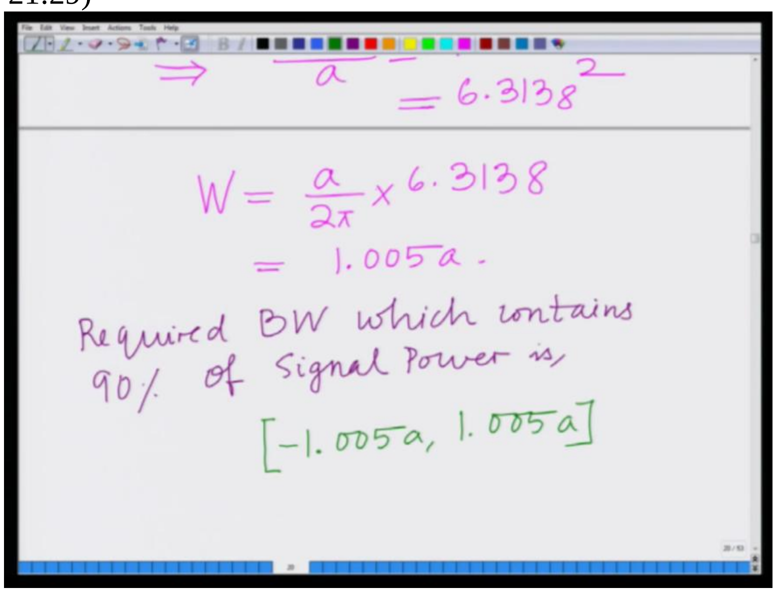

So required bandwidth which contains 90% so now what have we done? We have evaluated the power over this bandwidth, [-W, W] and we said, it has to contain 90% of the signal power and from that equation, we have derived that ~~this capital~~ W = 1.005a.

Therefore the required bandwidth for transmission, ~~for transmission~~ of this wireless signal which contains ~~90% of the power,~~ 90% of the signal power is -1.005a that is -W to +1.005a that is +W.

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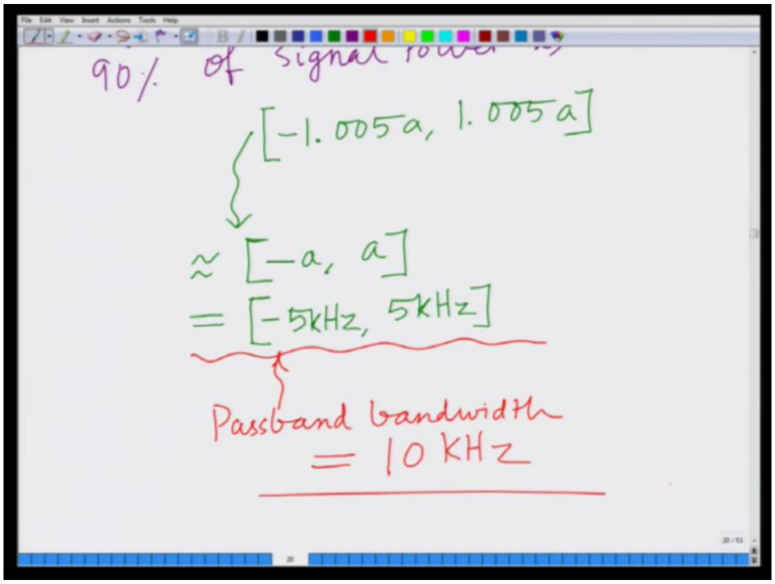

And 1.005 ~ 1. So this I can say this –

$$[-1.005a, 1.005a] \approx [-a, a]$$

$$= [-5\text{ kHz}, 5\text{ kHz}]$$

So one can say the total bandwidth i.e. the pass band signal bandwidth after modulation by the carrier ~~the passband bandwidth required~~ is 2a which is 10 kilohertz.

And this is a very interesting problem. What we are seeing is basically although the power spectral density is spread over there infinite frequency band that is from [-∞,∞], it has an infinite spread or infinite support. However most of the signal energy is contained in a bandwidth from [-5 kHz, 5 kHz]. How much is the band? How much is the signal energy that is 90% of that signal energy? That is the dominant fraction is contained in this band that is from [-a, a] which is [-5 kHz, 5 kHz].

Therefore, the spectral bandwidth, the band that is required to transmit a majority or this 90% of this signal energy is basically as we have seen here, this is 10 kilohertz.

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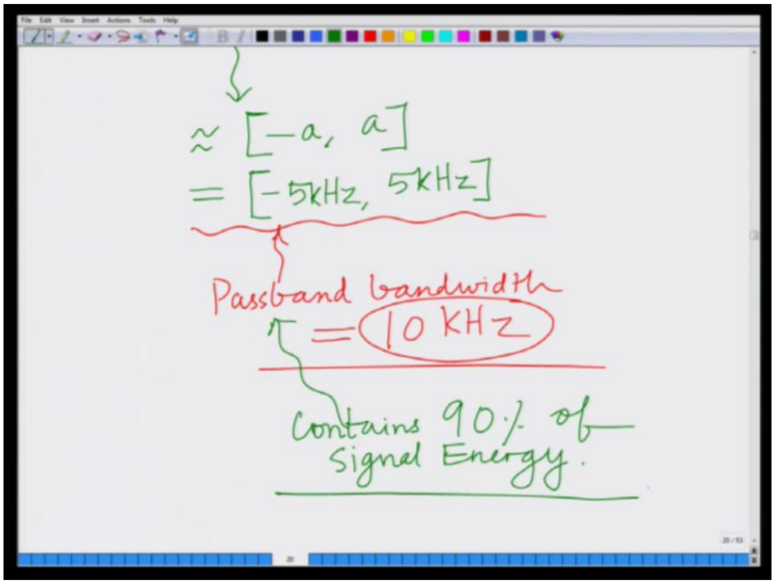

Although this contains, So this basically contains 90% of this basically contains 90% of the signal energy which is a very interesting result because now we have derived what is the bandwidth required for the transmission of a wireless signal such that even though it requires ideally it requires an infinite bandwidth, practically for 90% of the signal energy because it only requires a bandwidth of about 10 kilohertz, a pass band bandwidth of 10 kilohertz.

So this is an interesting example of a power spectral density application of power spectral density in the context of communication and wireless communication systems. That is given the autocorrelation function, we have derived what is the power of the message signal. the average power yes? And we have derived what is the power spectral density. That is $\frac{1}{a^2+4\pi^2 f^2}$ which is the Fourier transform of the autocorrelation function.

Also interestingly, we have derived the band that is required for transmission of 90% of the signal energy which we have said is basically a pass band bandwidth of 10 kilohertz. So this example illustrates application of this, important quantity, very important quantity of power spectral density in the context of wide sense stationary signals which arise very frequently in the analysis of communication systems and specially wireless communication systems. So we will stop this module here. Thank you very much.