Principles of Modern CDMA/MIMO/OFDM Wireless Communications

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Lecture – 43
Capacity of MIMO Wireless Systems

Hello, welcome to another module in this massive open online course on the principles of CDMA MIMO OFDM wireless communication system. In the previous module we had seen how the SVD or the singular value decomposition can be employed to basically convert the MIMO or the multiple input multiple wireless channel into t parallel channels between the transmitter and receiver, we also said that this is basically the example of special multiplexing that is employing this space dimension to multiple streams of information between the transmitter and receiver.

Therefore, this results in the t fold increase in the information rate or the rate of data transmission between the transmitter and receiver in a MIMO wireless system and that is how MIMO leads to higher rates in 3G and 4G and of course, subsequent generations of wireless communication systems.

So, today let us look at the maximum possible data rate or the capacity of a MIMO wireless communication system. So let us look at how to characterize this capacity of a MIMO wireless communication system.

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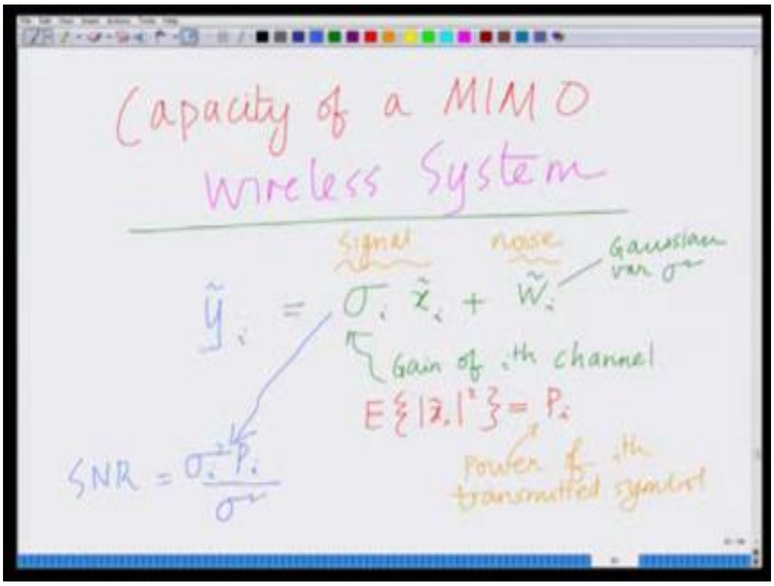

So, what you are going to look at today is the capacity or the Shannon capacity of a MIMO, let us look at the Shannon capacity of a MIMO wireless system. We said in a MIMO wireless system I have

$$\tilde{y}_i = \sigma_i \tilde{x}_i + \tilde{W}_i$$

$$E\{ |\tilde{x}_i|^2 \} = P_i$$

$$SNR = \frac{\sigma_i^2 P_i}{\sigma^2}$$

where $\tilde{W}_i$ is the Gaussian noise of variance σ^2, P_i equals power of the i-th transmitted symbol therefore, now you can see that this is the signal part and this is the noise part and therefore, the signal to noise power ratio or what we call the SNR is equal to $\frac{\sigma_i^2 P_i}{\sigma^2}$.

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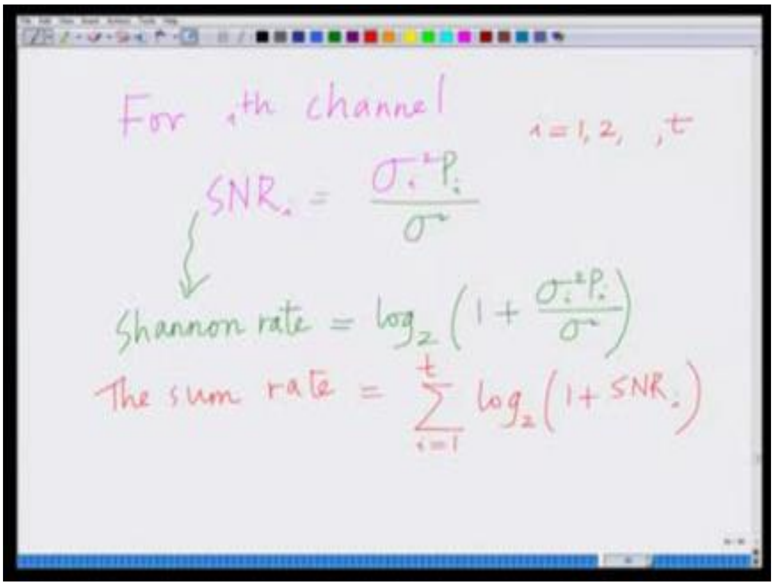

So, therefore, what we have is that for the i-th channel, SNR_i equals,

$$SNR_i = \frac{\sigma_i^2 P_i}{\sigma^2}$$

So, the SNR of the i-th channel, that is we have t parallel channels between the transmitter and the receiver, the SNR of the i-th channel which we are denoting by $SNR_i = \frac{\sigma_i^2 P_i}{\sigma^2}$, Where P_i is the transmit power of the i-th channel and we know from the formula for the Shannon capacity. The Shannon capacity is given as

$$\text{Shannon rate} = \log_2 \left(1 + \frac{\sigma_i^2 P_i}{\sigma^2} \right)$$

Remember we have how many such channels, we have t such channels, we have i equals 1, 2, t we have t such channels therefore; the sum rate equals

$$\text{Sum rate} = \sum_{i=1}^{t} \log_2 \left(1 + \frac{\sigma_i^2 P_i}{\sigma^2} \right)$$

this is the sum rate of the MIMO channels.

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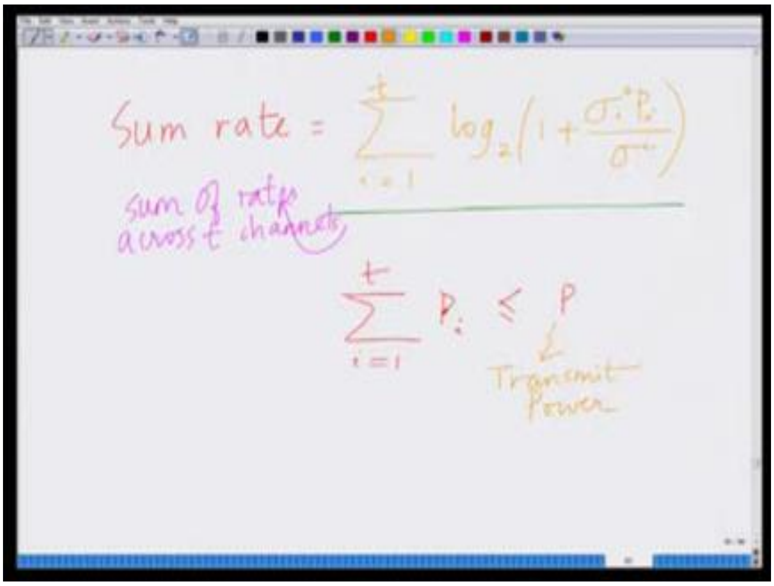

So, what we are saying is the following thing we are saying that the SNR of the i-th channel is $\frac{\sigma_i^2 P_i}{\sigma^2}$. The Shannon rate of this channel is

$$\text{Sum rate} = \sum_{i=1}^{t} \log_2 \left(1 + \frac{\sigma_i^2 P_i}{\sigma^2} \right)$$

This is the sum of the rate is naturally nothing, but sum of the rates across the t parallel channels.

Now, let us also assume that we have a total transmit power of P, which means the total power that is the sum power,

$$\sum_{i=1}^{t} P_i \leq P$$

where P is the transmit, let us assume that we have a total transmit power of P. Which means the transmit powers P_1, P_2 so on up to P_t, these are the transmit powers of the t sub stream or the sub channels. What is the capacity? The capacity is nothing, but the maximum sum rate of the MIMO systems subject to this total power constraint.

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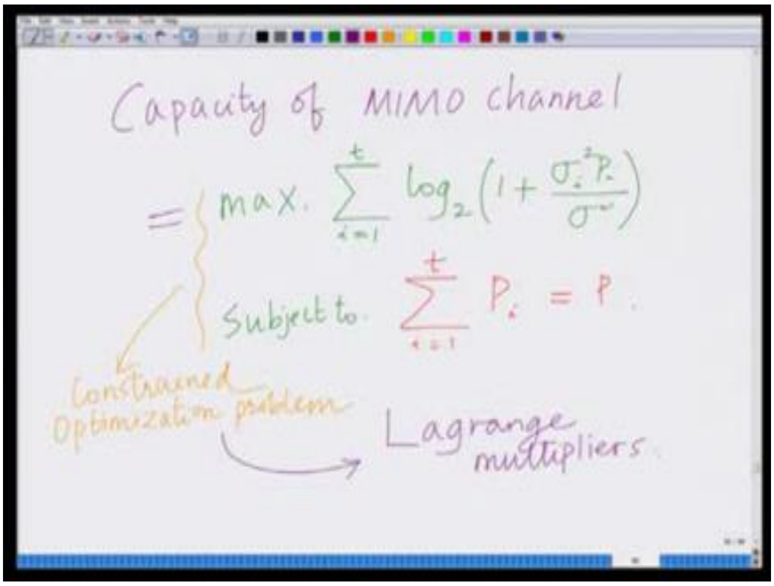

So, therefore, what is the capacity of the MIMO channel? Capacity of the MIMO channel equals

$$= \max \sum_{i=1}^{t} \log_2\left(1 + \frac{\sigma_i^2 P_i}{\sigma^2}\right) \qquad \text{subject to } \sum_{i=1}^{t} P_i \leq P$$

So, what we are saying is very simple, we are saying that the capacity of the MIMO wireless system that is the maximum capacity is the maximum possible sum rate, subject to the constraint that the total transmit power that is $\sum_{i=1}^{t} P_i \leq P$ or we can even say the total power has to be equal to P, that is the total transmit power is equal to P. This is also known as a "Constraint Optimization problem".

Therefore the way we are going to solve this Constraint optimization problem is by using the principle of Lagrange multiplier the way. We do it is, we form the objective function which is

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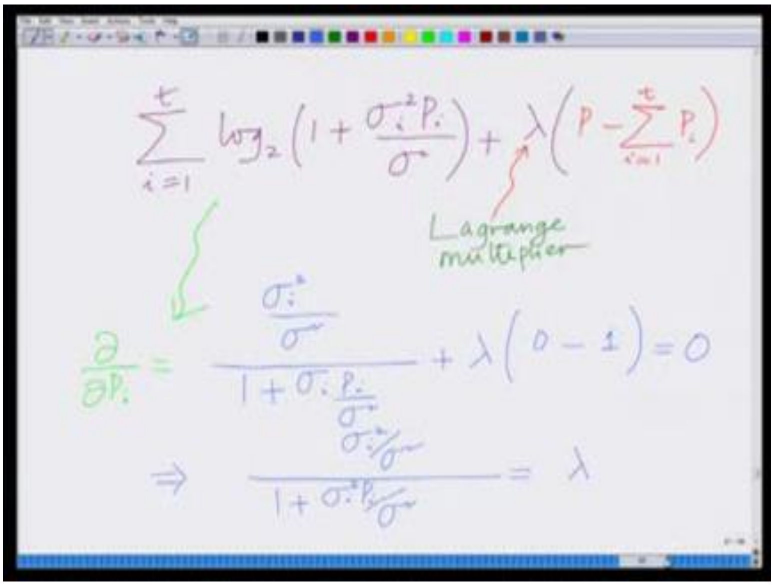

So, this is my objective function

$$\sum_{i=1}^{t} \log_2 \left(1 + \frac{\sigma_i{}^2 P_i}{\sigma^2} \right) + \lambda \left(P - \sum_{i=1}^{t} P_i \right)$$

where lambda, this is termed as the Lagrange multiplier. So, we are forming the Lagrangian for optimization problem, which is basically the original objective function plus lambda times the constraint which is the total power constraint.

Now what we are going to do is, we are going to differentiate this objective function with respect to each of the powers because the optimization variables at the powers that is P_1, P_2 so on up to P_t, we are going to differentiate this with respect to each power in particular with respect to the power P_i and set it equal to 0. So, what we can now do is we can differentiate this with respect to the power. So differentiate this

$$\frac{\partial\left[\sum_{i=1}^{t}\log_2\left(1+\frac{\sigma_i^2 P_i}{\sigma^2}\right)+\lambda\left(P-\sum_{i=1}^{t}P_i\right)\right]}{\partial P_i}=\frac{\frac{\sigma_i^2}{\sigma^2}}{1+\frac{\sigma_i P_i}{\sigma^2}}+\lambda(0-1)=0$$

$$\lambda=\frac{\frac{\sigma_i^2}{\sigma^2}}{1+\frac{\sigma_i P_i}{\sigma^2}}$$

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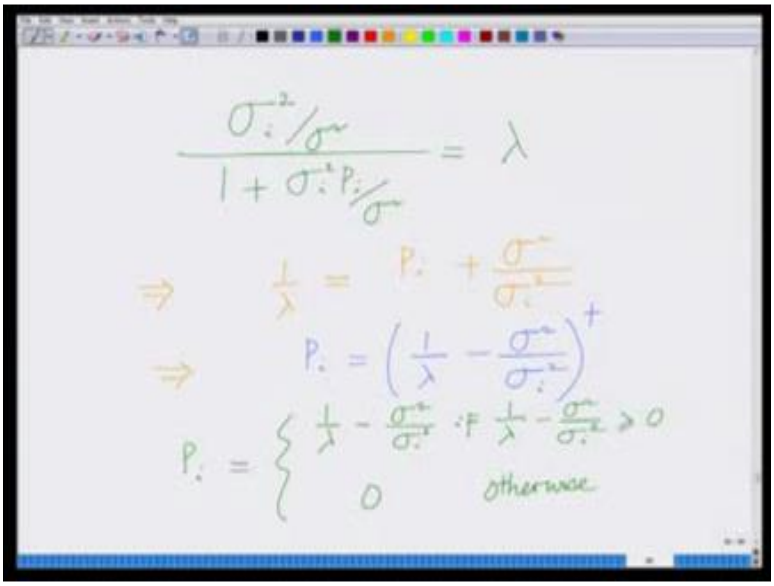

Therefore let me rewrite this again I have

$$\frac{1}{\lambda}=P_i+\frac{\sigma^2}{\sigma_i^2}$$

$$P_i=\frac{1}{\lambda}-\frac{\sigma^2}{\sigma_i^2}\quad \text{if } \frac{1}{\lambda}-\frac{\sigma^2}{\sigma_i^2}\geq 0$$

$$=0 \qquad\qquad \text{otherwise}$$

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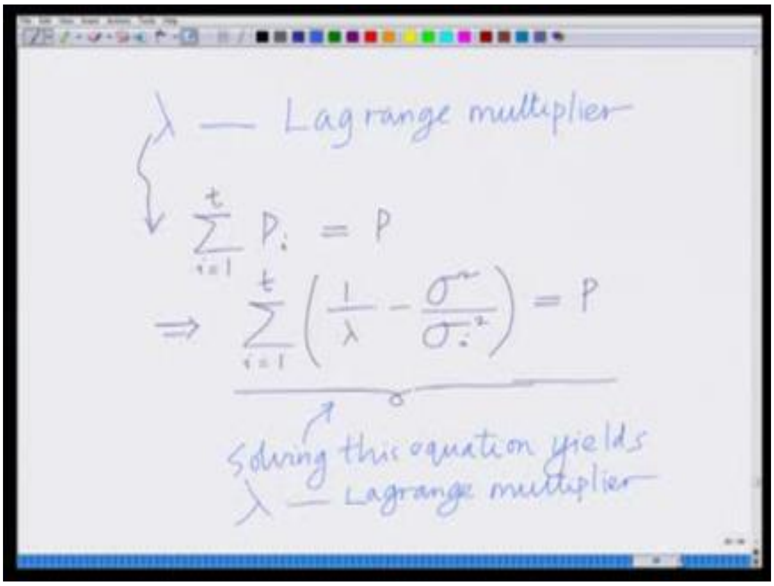

So, what we have therefore, now we still have to find this lambda, which is the Lagrange multiplier this can be found from

$$\sum_{i=1}^{t} P_i = P$$

$$\sum_{i=1}^{t} \left(\frac{1}{\lambda} - \frac{\sigma^2}{{\sigma_i}^2} \right) = P$$

So, solving this equation yields lambda which is the Lagrange multiplier and therefore, once you solve for lambda you can substitute in this back in this expression to get the power P_i of the i-th channel for a to achieve the maximum sum rate which is also the capacity of the MIMO wireless system.

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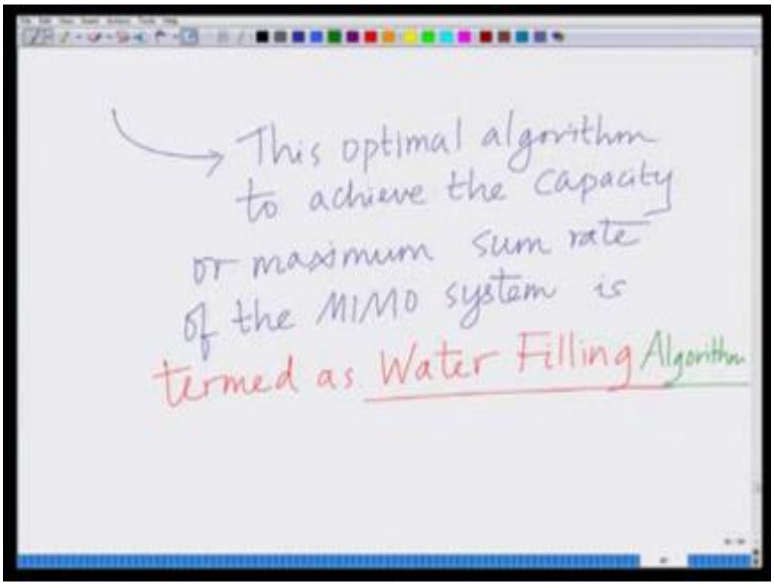

This optimal power allocation is term as the water filling algorithm. This optimal algorithm to achieve the capacity or maximum sum rate of the MIMO system is termed as; this is termed as the Water Filling Algorithm,

So, what we have seen is basically the power allocation P_i, to maximize the sum rate or which achieves the capacity of the wireless communication system is basically P_i equals $\frac{1}{\lambda} - \frac{\sigma^2}{\sigma_i^2}$. And this power allocation algorithm which basically leads to the maximization of sum rate or basically, which achieves the capacity or the maximum rate at which information can be transmitted or which information symbols information bits can be transmitted between the transmitter and receiver in the MIMO wireless communication system. This algorithm of power allocation or this optimal power allocation scheme for the MIMO wireless system is termed as the Water Filling Algorithm and very key scheme or a very key algorithm that helps maximize the data rate, helps increase the rate at which information can be transmitted in MIMO wireless systems.

So, what we have seen in this module and some of the in past module is how the SVD can be employed for a special multiplexing, to convert the MIMO wireless channel to t parallel channels. How the power can be allocated optimally among this t parallel channels to achieve the MIMO capacity or the maximum sum rate for MIMO wireless transmission. So we will conclude our discussion here and look at other aspects of wireless communication systems in the subsequent modules.

Thank you very much.