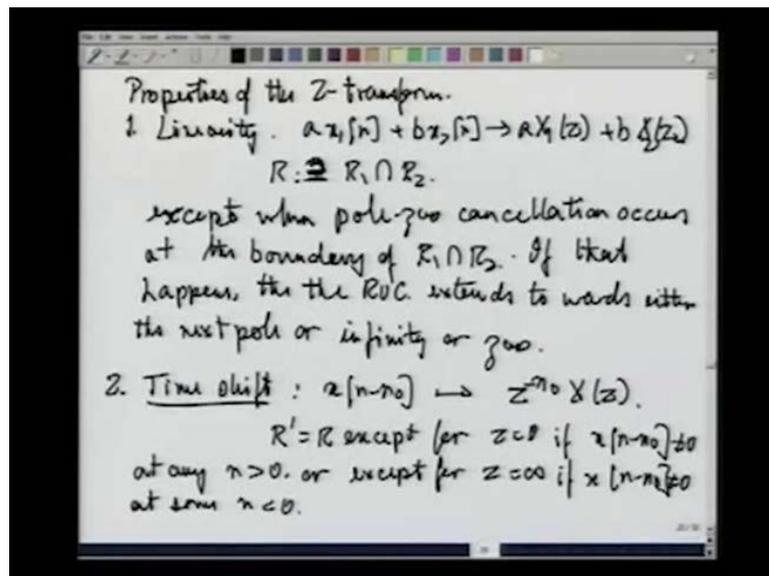


Signals and Systems
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Lecture - 45
Properties of Z Transform

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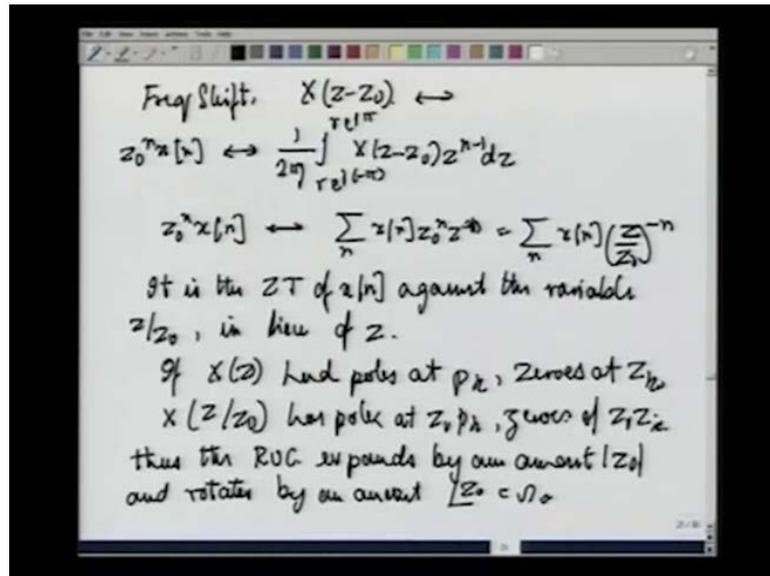


Now, let us move on to properties of the Z transform. Now, most of the properties resemble their counterparts for the Laplace transform. So, we can be very brief and quick with this discussion. First property is linearity. So, we have that $a x_1[n]$ plus $b x_2[n]$ transform to $a X_1(z)$ plus $b X_2(z)$. Now, the ROC R of the whole combination is normally equal to $R_1 \cap R_2$ except when pole zero cancellation occurs at the boundary of $R_1 \cap R_2$. When that happens, well if that happens then the ROC extends towards either the next pole or infinity or 0.

So, more appropriately this should be this or the overall R of the combination being the median conversation of the combination is a super set in the most general case of the individual ROC's R_1 of $x_1[n]$ and R_2 of $x_2[n]$, so much for our discussion of linearity. Next, time shift; the most straight forward manner you get that $x[n-n_0]$ transforms to $z^{-n_0} X(z)$. Now, what about the ROC? R' is the ROC of $x[n-n_0]$.

How does it relate to the ROC of $X(z)$? It is practically the same except for the usual caveat, except for z equal to 0 which has to be excluded; if $X(z)$ has points on the right side. That is if $X(z)$ is not equal to 0 at any n greater than 0 or except for z equal to infinity if $X(z)$ is non zero at some n less than 0.

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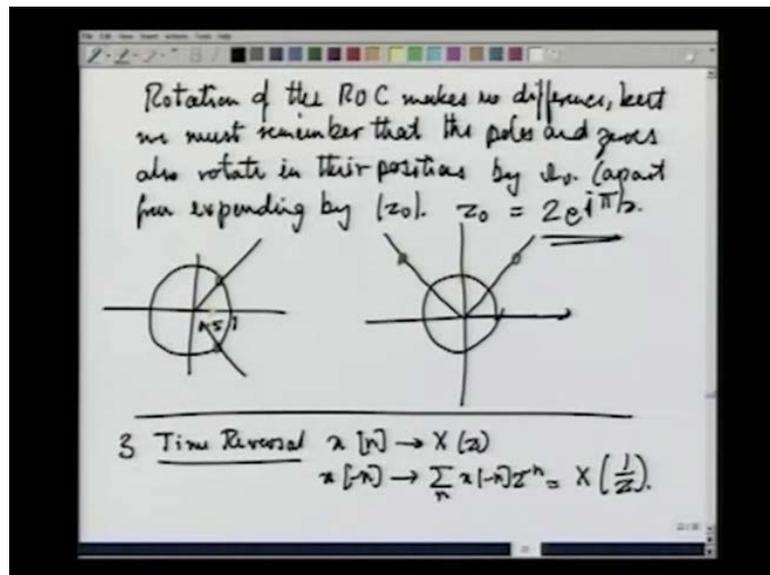
Frequency shift, this is interesting, and it effects the poles and zeroes in a very complicated manner. So, let us look at it. What we are saying is suppose we have $X(z)$ minus z naught, what happens to $X(z)$ minus z naught? What is the inverse transform of $X(z)$ minus z naught? Now, if $X(z)$ minus z naught is given to us then we can apply the inverse Z transform formula, which says $\frac{1}{2\pi j} \int_{\Gamma} X(z-z_0) z^{n-1} dz$ to get what will turn out to be z naught to the power n times $x[n]$. So, this is the sequence that we will get. Now, let us understand z naught to the power n times $x[n]$ by taking its Z transform once again.

We have to really get the same thing, but we will get it in a different appearance. z naught to the power n times $x[n]$ will transform to summation over n times z naught to the power n times $X(z)$ inverse which is equal to summation to the power, summation overall n times z naught by z or rather z by z naught to the power minus n , fine. So, essentially it is the Z transform of $x[n]$ against the variable z by z naught in lieu of, in lieu of z . That means wherever z had a pole or a 0, now z by z naught will have that pole or 0. That means to

say that all poles and zeroes which are at points Z_k and p_k will now shift to points $Z_{\text{naught}} p_k$ and $Z_{\text{naught}} Z_k$.

If X of Z had poles at p_k zeroes at Z_k then $X Z$ by Z_{naught} has poles at $Z_{\text{naught}} p_k$ zeroes at $Z_{\text{naught}} Z_k$. Thus, the ROC expands by an amount $\text{mod } Z_{\text{naught}}$ for example, if Z_{naught} were equal to 2 all the pole magnitudes would multiply by 2. All the 0 magnitudes would multiply by 2, as well as the ROC expands by an amount $\text{mod } Z_{\text{naught}}$ equal to 2 and rotates by an amount angle of Z_{naught} equal to ω_{naught} . Rotation the ROC however has no impact because they are all circularly symmetric. So, anything which circularly symmetric about the origin can rotate freely without bothering us, but the poles and the zeroes will also rotate and that is important to understand.

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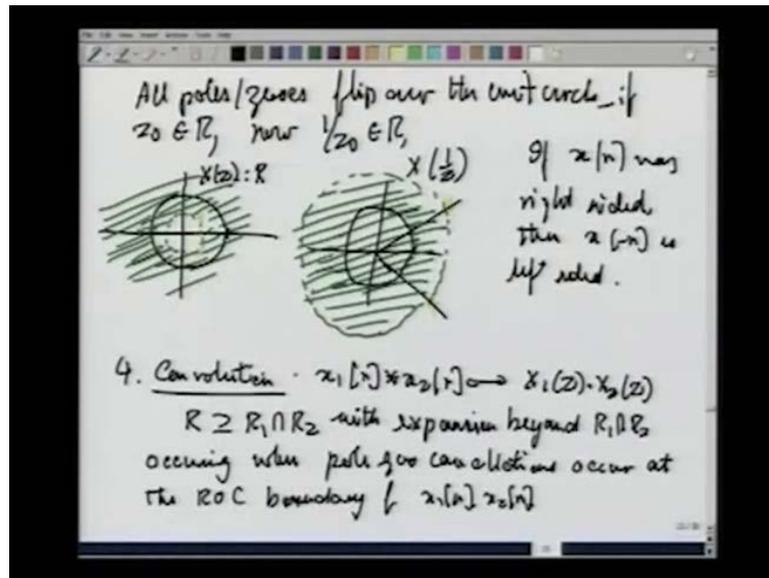
Rotation of the ROC makes no difference, no difference, but we must remember that the poles and zeroes also rotate in their positions by ω_{naught} apart from expanding by $\text{mod } Z_{\text{naught}}$. So, let us take an example suppose we say Z_{naught} equals $2e^{j\pi/2}$. Let us say this is Z_{naught} to $e^{j\pi/2}$. Now, let us take a pole zero plot, that is the unit circle and let us put a few ((Refer Time: 12:38)) over here. Let us have a pole at 0.5 and let us have pair of zeroes at say here and here. So, there are two zeroes in green and there is a pole in yellow. The pole in yellow is at 0.5, 0.5, the zeroes are at 45 degrees and minus 45 degrees.

Remember, that poles and zeroes must all come in conjugate pairs if they are not real. If they do not then the coefficient of the difference equation from which the X/Z has been elicited will not be real. It is the same argument as before, there is nothing new. Now, we have this and we have Z naught given by this. Now, $\text{mod } Z$ naught equals to... So, everything gets farther away from the present position, everything blows up by a factor of 2. So, if this was the unit circle as we have assumed it is, fine; then we will see what happens to the new. The unit circle is still there of course, the unit circle is not part of the function.

Then you have the 0.5 the polar 0.5 shifting by expanding by two's of 0.5 becomes 1. Also, there is a rotation by π by 2. So, the pole now lands up here. Magnitude has doubled rotation of 90 degrees. Now, these two zeroes which were there on the unit circle at minus 45 and plus 45 degrees, ω equal to minus 45, ω is equal to plus 45. The one at ω equal to minus 45 and of magnitude 1 will become magnitude 2 and shift to plus 45. So, you will have now the ((Refer Time: 14:57)) placed over here and over here. The entire thing rotated by 90 degrees. The relative positions remain the same, the relative angular positions of the poles and zeroes are maintained, but they have expanded and the whole system has rotated.

So, this is how it relates to frequency shifting. Next property, time reversal; if $x[n]$ transforms to X/Z then $x[-n]$ will transform to summation over all n of $x[-n]Z$ to the minus n which will turn out to be equal to X of $1/Z$. So, what is the ROC, if this was with an ROC of R what will be the ROC of this? Let us first try to understand where the poles and zeroes will go. Now, if you had a 0 or a pole some singularity at say Z^k . Now, $1/Z$ earlier it was at Z equal to Z^k . Now, it is at $1/Z$ equal to Z^k that is at Z equal to $1/Z^k$. So, what happens in the pole zero plot of $x[-n]$ is that everything gets flipped over the unit circle.

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All poles zeroes flip over the unit circle. Now, apart from their flipping over the unit circle the entire ROC also flips over the unit circle that is to say if Z naught was in R , now 1 by Z naught is in R . So, let us make up an example again. Unit circle and let us say there are poles at this place and this place that is plus or minus 45 degrees with a magnitude of 4, with a magnitude of, with an angle of the magnitude of 0.5. So, now what will happen if we take X , this is for X sorry, this is for $X Z$. So, what will it be for X minus n ? It will be X 1 by Z .

So, I will just plot that, X 1 by Z unit circle again. Now, these two get thrown over to the other side of the ROC. Suppose, this had an outward ROC, let us say that where R is given by this, let us go back to the first one and complete the story. So, it is an outward ROC starting at 0.5 because you have complex pair of poles at 0.5 magnitude and 45 and minus 45 degrees. Now, these two will go to these new places same direction of course, as before; but now the poles are at the reciprocal of the present magnitude distance. So, they are somewhere over here sorry and here, this is where the poles now are located and also the ROC gets inverted. It was outside of 0.5 earlier.

Now, it will be inside of 2. So, you will have to make an angular boundary over here like this and fill it up inside. This is what you get an inward ROC, but it extends all over inside the unit circle and everywhere. This is the ROC of the new function. Now, it is evident that if X n was right sided then X of minus n is left sided and thus worn out by

this. Inward ROC left sided, outward ROC right sided and that is what is happening. Fine this is for time reversal. Next, let us go to the next item. Convolution, this is pretty straight forward. All you have to say is that $x_1[n]$ convolved with $x_2[n]$ will map to $X_1(z)$ times $X_2(z)$.

The proof is practically the same as we showed. Furthermore, R dash is a super set of R_1 intersection R_2 just like in the case of linearity with expansion beyond R_1 intersection R_2 occurring when pole zero cancellations occur at the ROC boundary of $x_1[n]$ and $x_2[n]$. If there is no pole zero cancellations, then it will be the intersection. Even if there are pole zero cancellations, but they are inside and not at the boundaries it will be the same. Expansion occurs only if any outer pole or inner pole, any boundary pole gets dislodged by cancellation with a 0.

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5. Differentiation in the z-domain:

$$X(z) = \sum_n x[n] z^{-n}$$

$$\frac{dX(z)}{dz} = \sum_n x[n] z^{-n-1} (-n)$$

$$z \frac{dX(z)}{dz} \leftrightarrow -n x[n].$$

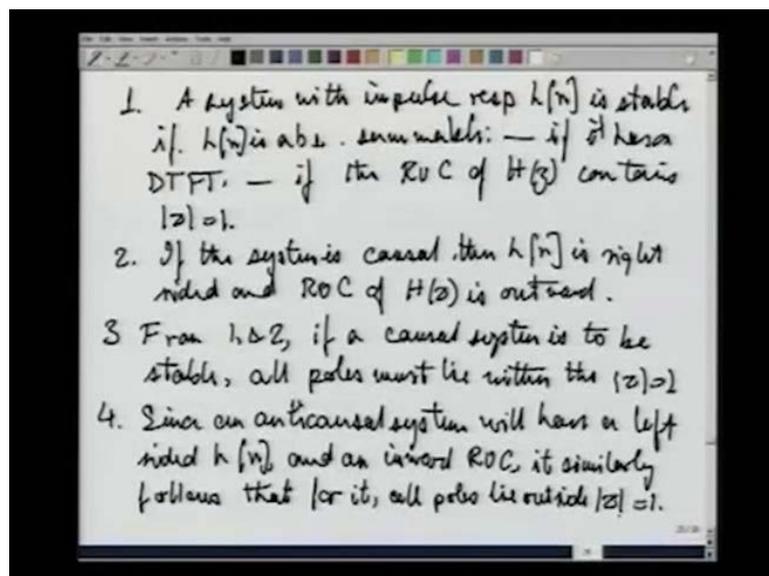
End of the properties of the ZT.

Next, Z domain differentiation, differentiation in the Z domain. Now, what we are saying is if we differentiate $X(z)$ with respect to z what happens? This is easily found by using the normal Z transform expression and carrying out the differentiation. We have $X(z)$ equal to summation over n of $x[n] z^{-n}$. So, if you differentiate it with respect to z $\frac{dX(z)}{dz}$ with respect to z equals summation over n . The only thing that weights differentiation here is z to the minus n , so it becomes $x[n] z^{-n-1}$ multiplied by minus n .

That is what you get. So, carrying out appropriate manipulations you get that Z times dX Z by dZ is the transform of n minus $n \times n$; this is the expression you get for the transform. Now, if $x[n]$ has a certain ROC n times $x[n]$ will have the same ROC because multiplying by a linearly increasing sequence will not affect the ROC since the modifying function used in the Z transform is exponential. So, it will make more difference to, modification by a linear function the ROC does not change. So, this brings us to the end of the properties.

Finally, along the lines of our discussion regarding the properties of systems described by Laplace transforms whether they were causal, whether they were stable etcetera. We can make similar remarks about systems LTI systems, about their stability and causality on the basis of the poles and zero locations of the Z transform, of their Z transform. In order to do this we will talk of, we will say, we will just note the following arguments.

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First a system with impulse response $h[n]$ is stable if $h[n]$ is absolutely summable. That is if it has a DTFT that is to say if the ROC of $H(z)$ contains the unit circle that is what we know. Second, if the system is causal then $h[n]$ is right sided and ROC of $H(z)$ is outward. Taking one and two together, so one and two we find that if a causal system is to be stable all poles must lie within the unit circle.

Since, an anti causal system will have a left sided $h[n]$ and an inward ROC. It similarly follows that for it, for it all poles lie outside the unit circle. So, that neatly winds up the

kind of conclusions one can come to about stability, causality etcetera of discrete time systems on the basis of the poles and zeros of the Z transform of the systems impulse response.