Signal Processing Algorithms & Architecture
Dr. Anirban Dasgupta
Department of Electronics & Electrical Engineering
Indian Institute of Technology Guwahati
Lec 8: Fourier Analysis

Hello, everyone. Welcome to a fresh, new lecture on the topic of Fourier analysis for the course on signal processing algorithms and architectures. This is Dr. Anirban Dasgupta and let us get started.

So, as I said in the previous lecture, these components of frequency domain analysis are like we have Fourier analysis of frequency estimation, PSD estimation, Z-transform and coherence analysis. So here we will start with Fourier analysis.

So, Fourier analysis is just an intro which I gave in the previous video is used to represent the signal as a sum of sinusoids. Like sines and cosines, which are with different frequencies, amplitudes, and phases. So, the goal is to understand. the frequency content of the signal.

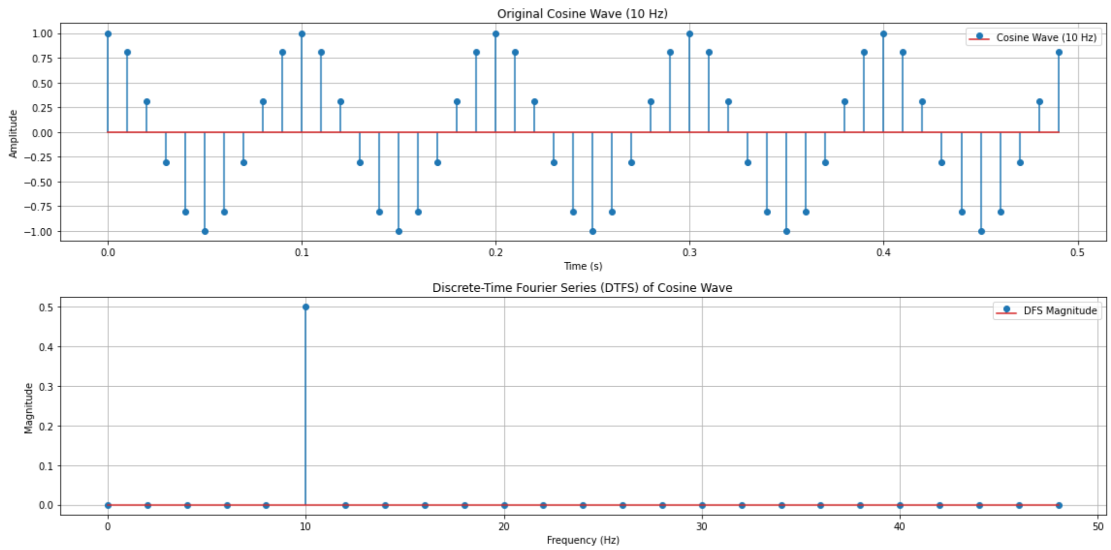
So, for discrete-time signals because here we are mainly focusing on discrete-time signals. So, there are two common types of Fourier analysis. One is the discrete-time Fourier series or DTFS and the discrete-time Fourier transform (DTFT). So, what is the discrete-time Fourier series? So, this decomposes a periodic discrete time signal into a sum of sinusoids with discrete frequencies. So, this is the formula. So, here is the synthesis equation-

$$X[K] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi kn}{N}}$$

Here we see that this $x[n]$, which is a periodic signal having a period N. It is the sum of these N components and how do I get this value, c_k? So $c_k$is obtained by this formula-

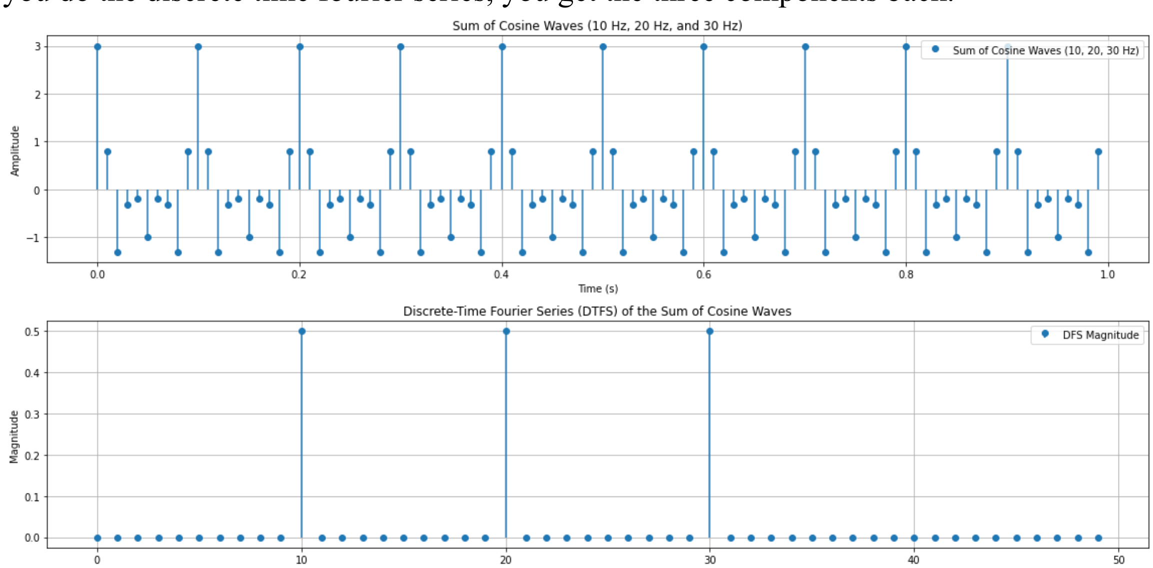
$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

Now this complex exponential is introduced earlier in the time domain topic. So, if you see this signal, this is a cosine wave, and this cosine wave is discrete cosine wave, which is periodic. Although you are seeing part of this cosine wave, this is varying from $-\infty$ to ∞. So if i decompose it into the fourier series we will see that this has only one component. And this component, of course, there is a component at 10 and minus 10 because of this.



It is a 10 hertz signal, and this is mapped back to the original continuous domain spectrum because Here, this is frequency in hertz, just to demonstrate how I can represent this cosine wave of 10 hertz has just one value. Now, one takeaway in this graph is that, So, in the time domain, you have to use ∞ number of points or minimum you have to use points that contribute to one period. But in the frequency domain, you just need that one frequency point, which is 10. which is describing the signal and maybe one more point which defines the amplitude. Now the amplitude will typically be 0.5. which is half of this because you have one component of 10 and one component of -10.

Suppose we have the sum of 3 sines and 3 cosines. So one is 10, 20, and 30. So here, if you do the discrete time fourier series, you get the three components back.

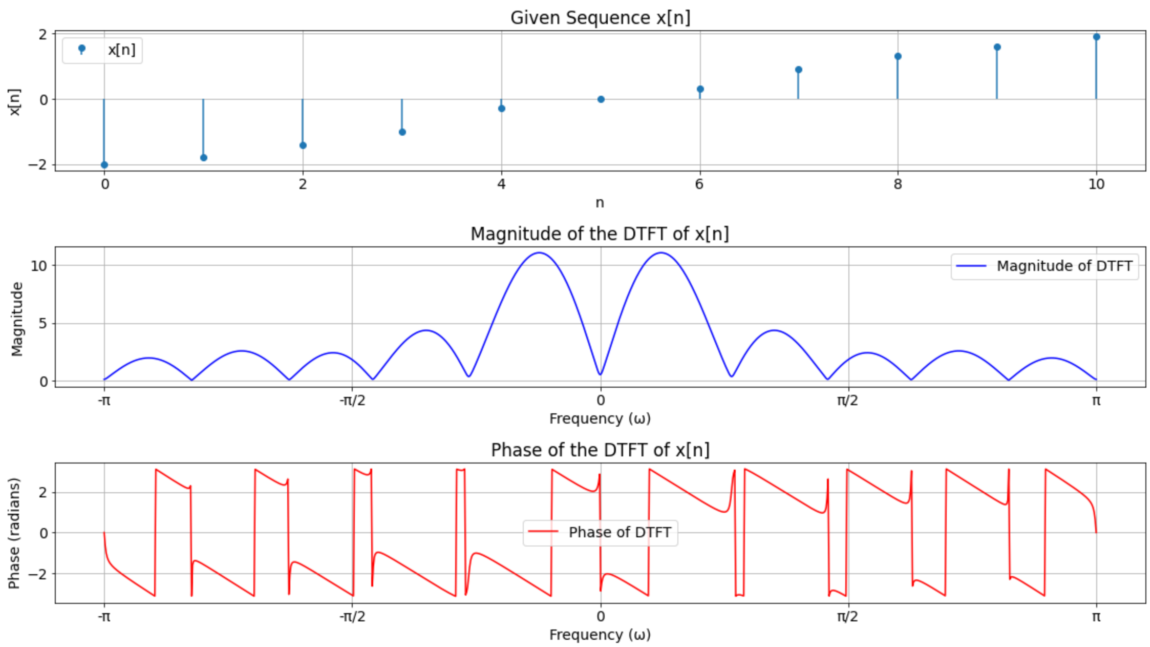


Again, I am plotting it back to the original analog frequency domain. So most discrete-time signals are aperiodic in nature. Even if they are like periodic, we will not analyze periodic functions are important for the obvious reason.

Periodic signals vary from - ∞. to + ∞ and - ∞ is much much much before we were born and + ∞ is if you do not know when will it end? So naturally we will restrict practical signals which are periodic in nature and of finite length. So, in this case, we have our discrete time fourier transform, which is given by this-

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

This gives you the spectrum and naturally, since this is aperiodic in nature, you get a continuous spectrum and the second thing is that this spectrum has we learned in the previous video that has a range from - π to + π, something like this, because any other frequencies will be folded back in this region because of aliasing. So, this is one example. So, here I have a signal; this is some random values that I have selected to show you.



So, this signal is changing from some - values to + values in the time domain and this is typically the DTFT. So, if you see the zero frequency, it is not exactly zero; there is Some small value, which is the average or the sum of these components. So again, as I said, that spectra is not just magnitude. Spectra are complex-valued vectors.

You have a magnitude spectra and you also have a phase spectra. Like the forward DTFT, we also have the inverse DTFT. The inverse DTFT is used to recover the signal in the discrete time domain from the continuous spectra $X(\omega)$ by this formula-

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\, d\omega$$

since here ω is continuous in nature. We perform an integration to obtain the IDFT.

Remember that in the forward transform since $x[n]$is discrete, we do the summation. In the inverse, we do the integration and then again, $X(\omega)$is periodic. As I said, we are just considering here and the fundamental period is - π to π, but this will be basically repeating itself. Now the next question is for everyone. What signals do we get from the DTFT? So, if the signals are not converging, or, in other words, if they are not absolutely summable, So it is difficult to obtain the DTFT. So, for signals that are absolute summable, that is where the DTFT will converge; we can get the signal spectra, okay? So let us understand some properties of the discrete-time Fourier transform. So, the first property is linearity.

$$DTFT\{\, a_1\, x_1[n] \,+\, a_2\, x_2[n]\} \,= a_1\, X_1(\omega) \,+ a_2\, X_2(\omega)$$

So the question is, is the DTFT linear? Yes, it is a linear operation and we can prove it very easily using the superposition theorem.

Like if $x_1[n]$ gives capital $X_1(\omega)$. and small $x_2[n]$ gives capital $X_2(\omega)$, then a linear combination like $a_1\, x_1[n] + a_2\, x_2[n]$ and if I do the DTFT of this. A linear combination, we can easily see that it will be

a_1 multiplied by DTFT of x_1 which is $X_1(\omega)$ plus a_2 multiplied by $X_2(\omega)$. Here, a_1 and a_2 are any constants.

What about time-shifting?

$$DTFT\{x[n-k]\} \,= e^{-j\omega k}\, X(\omega)$$

So, suppose I have a signal in the discrete time domain and I shifted by some value of k samples. So, this is equivalent to multiplying this term $e^{-j\omega k}$ $w$ith the original spectra of X[ω]. So this is the meaning of shifting a signaling in time will result in multiplication its DTFT by a complex exponential factor. And what is this factor doing? It is not changing its amplitude. Because it has a unit amplitude. So it will only change the phase spectrum, and this is why phase spectra are also very important in many signal processing applications. What about frequency shifting? What is frequency shifting? So here we get a change in the spectrum.

$$DTFT\{x[n]e^{j\omega_0 n}\} = X(\omega - \omega_0)$$

In the frequency domain, and this happens when we multiply a complex number exponential in the time domain. The next is scaling.

$$DTFT\{x[an]\} = \frac{1}{|a|} X(\frac{\omega}{a})$$

Now this is slightly different from the others continuous domain, as I also explained this is not just time scaling. Time scaling, but this is also changing the sampling rate. So, will we or we might also encounter multi-rate signal processing, where so, here this is if a value that is greater than 1 means we are doing a down sampling operation. We are picking, say, every alternate sample. For every sample, after skipping some samples. So, this is something called a down-sampler. So, this downsampling will create a spectrum which is related to the original spectra by this Equation that-

$$\frac{1}{|a|} X(\frac{\omega}{a})$$

And then the very common property that it is a convolution property where if you convolve two signals and take their spectra, It is equivalent to multiplying them. Individual spectra in the frequency domain. And what about? Multiplication in the time domain? So, if you multiply two signals in time domain, it is convolution. In the frequency domain. And then there is another fundamental theorem.

$$Y(\omega) = X_1(\omega) X_2(\omega)$$

Which is coming from the concept of energy conservation which Is the energy in time?

$$\sum_{-\infty}^{\infty} |\ x[n]\ |^2 = \frac{1}{2\pi} \int_{\pi}^{-\pi} |\ X(\omega)\ |^2\, d\omega$$

Domain and in the frequency domain will be the same. So, how do you find the energy in the time domain of a signal. So, this is your takeaway. The magnitude and square the signal values and then sum over n. So, this is your energy in the time domain and this will be the same as the energy in the frequency domain that is obtained by this formula,

$$Y(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega')X_2(\omega - \omega')\ d\omega'$$

Because this is the content of this span of your frequency spectra and again, this is basically from - π to π. The second thing is this ω, as I said is a continuous variable. So, you get this integration.

So next is the symmetry property,

$$X^*(\omega) = X(-\omega)$$

which is also very important, specifically when you will learn about the FFT algorithms. So, if you have a real valued signal, So the property says that the DTFT of $X(\omega)$. I will have this conjugate symmetry, which means $X(-\omega)$ will be the the complex conjugate of this and further, if your signal is real and even, then your spectra will also be real and even.

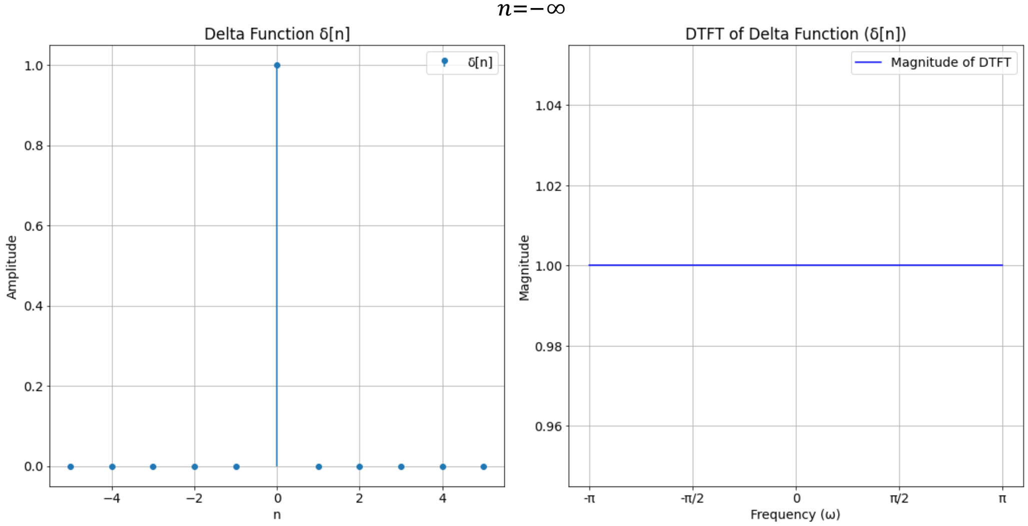
$$x[n] = x[-n]$$

Similarly if your signal is a real and odd function like this, then your spectra will be completely imaginary and odd.

$$x[n] = -x[-n],$$

And good examples are your signs and cosines, which are odd and even, respectively. And these are hard to compute discrete cosine transform and the discrete sine transforms. And now let us try to compute the DTFT of some standard functions and we start with the delta function.

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = 1$$



So if you write the formula, which is this, you see that this will sum to 1, which is very similar to the continuous domain. Now, how do we get this? So, if you see clearly, you keep putting the values, so here only for. n = 0, your delta function will be defined as 1, and this Is also 1, so 1 times 1 is 1.
So this is the plot, and you may be wondering why I have. Do not put these edges at $-\pi$ and π because there would not be be edges, and if, as I said, that this. DTFT is periodic, and it varies to ∞.

So, this is just the fundamental period. When you have the values defined. To be 1 from $-\pi$ to $+\pi$. Then what about a sinusoidal signal? Now, I already said that a sinusoidal signal. Like a cosine wave, it is periodic and will be extending from $-\infty$ to $+\infty$ So, you typically would like to perform the DTFS, the discrete-time Fourier series. But even if you perform the Fourier transform. Transform by putting it in the formula, you will possibly not be able to come to this value because of the absolute summable problem, because this cosine and sine are not absolutely summable when considered $-\infty$ to $+\infty$, So, what should we do in this case? So, there is a special way by which you can find, but from the concepts you already know that sines and cosines will have two frequencies at -ω_0 and $+\omega_0$ and if it is a single complex exponential, at ω_0, this will have only One frequency component, ω, is not. So, this is a pure wave.

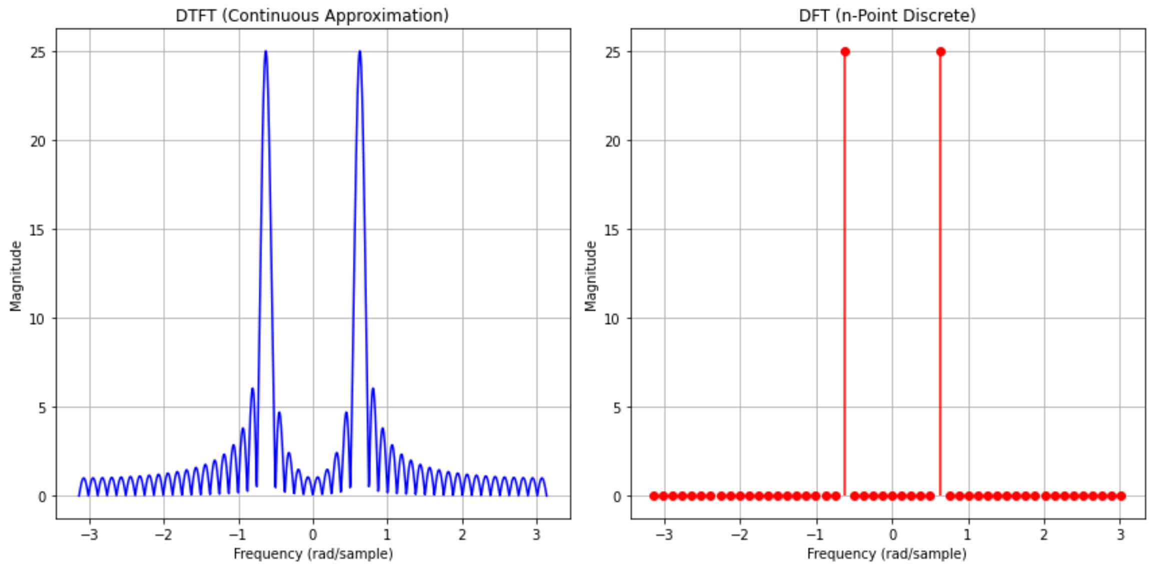
$$X(\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

There is only one frequency component and this again comes from the property of shifting, frequency shifting. So, the next kind of transformation Is your discrete Fourier transform? So, as I said, this omega axis is continuous. From $-\pi$ to $+\pi$, how do you? Write an algorithm to implement. This or how did I implement this? So, typically, I have to use some samples of the ω, say I take. Some 1,000 samples of the ω from $-\pi$ to $+\pi$ and then I can plot based on the function. But really, when you want to put it on a digital. Hardware is digital signal processing is mainly computing on digital signals processors or microcontrollers.

So, you need some specific points. So that means the spectrum. DTFT needs to be sampled. So, if I sample the spectra using n numbers of points, this is nothing but this is called my n-point DFT Discrete Fourier transform.

And this solves the practical implementation problem of the DTFT or in others. I represent this DTFT by n points. So, this is solved again. This leakage problem, See, like, when I have a windowed sample.



So my two frequencies components say this are the two frequency components of say - 0.5 and + 0.5; these are appearing at two peaks, but there are A spread and there are all side lobes but if I take just two samples, although. These are not exactly zeros, so these are. I have done some kind of post-processing. To make them zero, but these two are prominent. Peaks are now very easily visible. What is called my N-point DFT? So this is frequency-domain sampling.

I am sampling the frequency spectrum. So how do I define mine? Discrete Fourier Transform? So this is discrete. Version of my DTFT where I take N samples as I do said, so this $\frac{2\pi}{N}$ And what is the spacing? Between two consecutive points? It is $\frac{2\pi}{N}$ and this is the Formula for my N-point DFT.

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-\frac{j2\pi kn}{N}}$$

So let us find some properties of this DFT. The first property is periodicity,

$$X[k] = X[k+N],$$

which is very important, like my DTFT that, So DTFT was periodic. With a period of 2π. Here, my DFT is periodic with a period. of N which means that my signal sample x[n], which has value 0 to n -1, its DFT will be periodic with n and also now when We see a periodical discrete spectra, lines Spectra, this means in the time domain also my signal should be periodic. So how is this ambiguity? Should it be resolved? So, it is assumed that in the time domain.

Also, that x[n], which is finite duration has a periodic repetition. So, that signal is repeated ∞ To make it periodic or you do a spectral analysis. Analysis of the periodic repetition of my x[n], and then you get this line spectrum. which is my X[K], which is my DFT linearity is just like your DTFT. So I am not going to explain it again.

$$DFT\{ax_1[n] + bx_2\ [n]\} = a \cdot DFT\{x_1[n]\} + b \cdot DFT\{x_2[n]\}$$

Now, symmetry is also very important.

$$X^*[k] = X[N-k]$$

Here, but what is important is to look. It is that it's conjugate symmetric for this.

So, for a real-valued signal x[n], The DFT coefficient X[k] will exhibit conjugate symmetry and this is very important, specifically for the development of my FFT algorithms and these properties are very similar.

To the DTFT, so I will just quickly go through it. Like when I have a shift in the time domain. It is multiplied by this phase factor. So here you see that this omega is typical. replaced 2πk with n and the rest is kind of the same. So next is frequency shifting, and here the The main thing you have to understand is that Since both these x, which is the DFT, and the small x, which is the time, they are considered to be periodic extensions. So, we have this formula modulo. So here, this is not just linear. Shift, but it is a modulo n shift. Modulo n shift means that if the signal It is periodic like this, so after this.

$$DFT(x[n-n_0]) = X[k] \cdot e^{-\frac{j2\pi k n_0}{N}}$$

$$DFT(x[n] \cdot e^{\frac{j2\pi k n_0}{N}}) = X[(k-k_0) \bmod\ N]$$

I get this value which. is same as the first value. So, this is like a circular shift. So, the shifting is happening in a circular manner. The same with convolution.

$$DFT(x[n] \circledast h[n]) = DFT(x[n]) \cdot DFT(h[n])$$

So here, if you see that. If I, so in your DTFT, So your spectra multiplication is. equivalent to convolution, which is linear convolution, but in your DFT It is the circular convolution. which is denoted by this symbol. So, what is circular convolution? So in linear convolution, you have this signal and the kernel and you keep shifting this.

So, in a circular, you basically have circular kernels, and you keep on shifting this shift multiplier. Multiplier but perform circular shifts and you will also come. to know about the hardware There are circular registers or barrel shift registers used for this purpose.

Then multiplication in time will give.

$$DFT(x[n] \cdot h[n]) = \frac{1}{N} DFT(X[k] \circledast H[k])$$

Convolution in frequency, but again and this is circular convolution. And then Parseval's theorem, which is Just the energy conservation, which It is given by this relation-

$$\sum_{n=0}^{N-1} |\, x[n] \,|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |\, X[k] \,|^2$$

Is it a reversal with respect to time or with respect to the periodicity because this signal is periodic. In nature, something like this. So, if you see this, it is like a circle rotating. So, the circular. Rotation or circular time I would say reversal will give DFT which is the conjugate? of the original DFT. Like

if x[n] gives me X[k], then if I do a circular time reversal, then it It will be the complex conjugate of this. So, what is this? Circular convolution? Why is there so much hype about this? So, this is a type of convolution, which is applied to periodic sequences. Like I said, there is a circle. which is rotating. So, there are some. But in circular convolution, Both should be of same length because you are doing things in a circular manner. So, say there are four values of the signal, So one signal you. keep fixed; the other signal you rotate and then multiply add, rotate, multiply, add, rotate. Multiply and add like this. So, this is given by this formula. This module operation is just. depicting the same. And this is because say if you have a signal 1, 2, 3, 4, since this is a periodic extension, so After 4, 1 will come before. 1, 4 will come like this. So you take any window, so when. You are shifting, see? It is like a circular rotation.

$$y[n] = \sum_{m=0}^{3} x[m] \cdot h[(n-m) \bmod 4]$$

So let us take an example to make this clear. So, we have two sequences.
$x[n] = [1,2,3,4]$ and $h[n] = [4,3,2,1]$, both of length 4
Both are of length 4. This is very important for both. They have to be of the same length. So, this is the definition. Which I said. And now using the You can formula. Easily calculate. And I will not waste. Take time on this and you can verify. This is the result. And there is one more. Matrix multiplication method. Which we can discuss quickly. So, you have two matrices. So this is 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4. 1, 2, 3, 4, and this is probably 4, 3, 2, 1, 4, 3, 2, 1, and if you calculate. If you try calculating this, you will get.

This is the same result. So, this is one. easier way to do the calculation by hand. Let us try it out quickly. So, 4 plus 1 times 4 is 4 plus 4. 3 is 12 plus 3 times 3. Is 9 plus 2, sorry, 3 times 2 is 6. And 2 into 1 is 2. So, if I do this, this is 10, and this is. 14, so 24; this is 24. This is matched. Second, 2 times 4 is 8. 1 into 3 is 3, 4 into 2 is 8. 8 and 3 into 1 is 3. And if I Do, so this is 6, and this. How much is 6 plus 16? 22. And if you check, this is the result.

So, thank you very much.

We will meet again.

Hope you enjoyed this lecture.

Have a nice day.