

Signal Processing Algorithms & Architecture
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Lec 7: Introduction to Frequency-Domain Signal Processing

Hello, everyone. Welcome to a fresh new lecture on the topic Introduction to Frequency-Domain Signal Processing for the course on Signal processing algorithms and architectures. This is Dr. Anirban Dasgupta and let us get started.

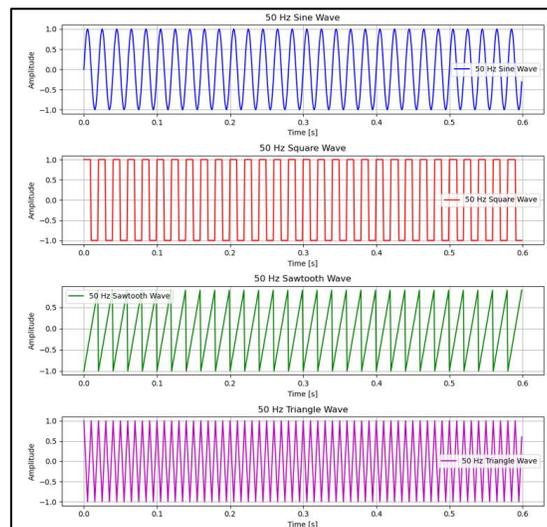
So, while understanding the topic frequency domain. Let us first understand what frequency is? What is frequency? Now, if you have studied in your school day physics, It is something like that the number of cycles per second and in the context of signal processing, I would like to define it as the number of a sinusoidal signal of cycles completes per unit time, usually in 1 second and this is typically measured in the unit hertz. Where in 1 hertz, how many sinusoidal cycles are completed? Now, people often get confused by two terms, one is the frequency content and the fundamental frequency.

And of course, these are different things. So, what is the difference? So, if we see these two signals, now these two signals you I can see that in one second there are 50 complete cycles but the difference is that one is a sine wave, and the other is a square wave. So, both have the same fundamentals frequency which is 50 hertz and which is obtained by

$$f = \frac{1}{T}$$

Time Period

But the frequency content of these signals are different. What does it mean? So, fundamental frequency if you see, it is the lowest positive frequency of the periodic signal and frequency content refers to the distribution of frequencies that make up the signal. For example, how many such sinusoids are there? Different frequencies are present in the signal is referred to as the frequency content of the signal. So, if you see these graphs-



there are a sine wave of 50 hertz exists. A square wave of 50 hertz. It is a sawtooth wave of 50 hertz, and there. It is a triangular wave of 50 hertz and all have the same fundamental frequency that is 50 hertz. But these are made up of different sine waves combination of different sine waves and the frequency of those sines waves that make up this signal is called the frequency content and this introduces us to Fourier analysis.

That's how to know the frequency content of a signal. Like I said, frequency content refers to the different frequency components that constitute the signal. Now, if we perform a Fourier transform analysis on a signal then we can get the fundamental frequency and the harmonics, which are basically multiples of the fundamental frequency and these signals collectively form this complete signal. For example, if we pass white light passing through a prism breaks it into its constituent colors. Frequencies, like if white is passed, we get the VIBGYOR components. Similarly, in Fourier analysis, if we send a signal all the components of the signal which are nothing but sinusoidal waves.

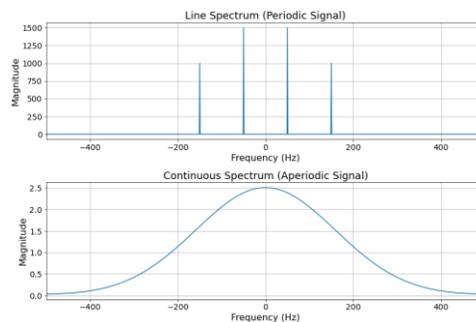
Varying at different frequencies will be split out, and that is the purpose of Fourier analysis. Now, if the signal is periodic in nature, the specific number of sine waves, specific means there can be infinite. Many sine waves also exist, but these sine waves are of discrete frequencies. Like, say if one frequency is f_0 the other frequency will be $2f_0$, and so on. That is multiples of the fundamental frequency f_0 . And if it is an aperiodic signal, then we will have all possible frequencies range from, say, $-\infty$ to $+\infty$ or 0 to ∞ , whatever you say. So, we will all discuss these categories. So, the question is: what is Fourier analysis? So, this is just a mathematical technique. To decompose such signals into the frequency. components of which that signal is made up of.

And why is it used? So, this is very helpful to analyze and process signal specifically in the frequency domain, like for an example in the time domain we have. I learned some smoothing techniques but technically, I say I have a signal which has some frequency content like when I am speaking, my audio will have a specific frequency content, whereas there is some background noise that may be of some different frequency range. So, by observing their Fourier analysis, then we can actually determine that which portion is my signal and which is not? The portion is noise, and we can easily filter them out. So Fourier analysis as I said that it is the principle which states that any periodic signal can be represented as a sum of sines and cosines cosines that are at different frequencies and there are two types of Fourier analysis. Depending on whether the signal is periodic or aperiodic, and if it is a periodic signal. We have Fourier series, and if it is a aperiodic signal it is a Fourier transform.

And to visualize this, the spectra are called a line spectrum and a continuous spectrum for your periodic and aperiodic signals respectively. What is a spectrum? Spectra is just a fancy name for the frequency domain representation. So, we have a line spectrum. Line spectra means, suppose I have a signal says

$$\sin(2\pi 10t) \text{ and } \sin(2\pi 20t).$$

So, in the spectra, we can see that these two components are present, and this is periodic signal the sum of two signs, and if you do, the In Fourier analysis, we see that the 10 Hertz And the 20 hertz components will be present. If I give some magnitude to this, like this is 4 times this, and this is 5 times this, so 10 hertz. will have a magnitude of 5 to 20 hertz Will have a magnitude of 4. Of course, it will be split into positive and negative frequencies, which I will discuss. So, line spectra basically are that. So, if you see this in the frequency domain, you see one at 10 and the other at 20. And similarly, there will be something in the negative frequency also at minus 10 and minus 20.

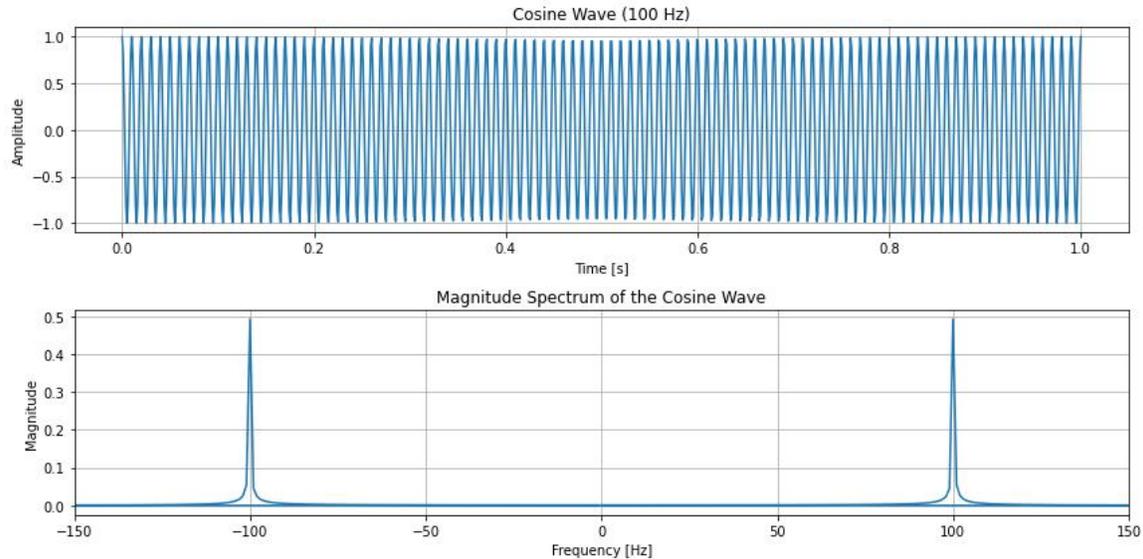


So, these are all the lines and the other thing, which is the Fourier transform, is that for aperiodic signals, your spectra will be continuous. Now there is a very nice mathematical explanation for why it becomes continuous. So, if you look closely, what is this spacing? Say that if I have your periodic signal, there is an f_0 , which is $1/T$; there can also be some 0 frequency. This is said twice f_0 , this is said thrice f_0 . So, if you see this spacing, what is this spacing? This spacing is f_0 , and what is f_0 ? f_0 is the inverse of your T ; T is my time period.

So now, if we see as T reduces, what happens? This spacing will increase, and vice versa; if T increases, then my spacing will reduce. So what happens in this case when T tends to infinity? Then the spacing will tend to 0. This f_0 spacing, what does it mean? This means that my time period of the signal is tending to infinity; that is my signal it will repeat. After infinite time, which means it will never repeat. Practically, and when it will never repeat that means it is an aperiodic signal. And similarly, the spacing tends to 0 that means you no longer get a line spectrum. Now the spectra will be continuous like this. So that is the continuous spectrum for aperiodic signals. And hence, you see that in the periodic signals.

Where you have Fourier series, you get the summation. Term whereas in the synthesis equation. Fourier transforms give you the integration term. Because this is no more a discrete amount of frequency components are continuous in nature.

So, this is what I was talking about. So, if you see clearly that in line spectra or spectra of a periodic signal, say we have two waves somewhere near, say, 50 and 150, so we see the magnitude spectra, and there are components at 50, -50, 150 and -150. But in a continuous spectrum for an aperiodic signal, We see that the frequency values are defined for all. Values of the frequency axis, maybe if it is a band limited signal you will have within certain range.



Otherwise, it may not be band-limited. Now, coming to something more interesting. concepts like zero frequency. So, what is 0 frequency? And people say that there is 0 frequency. Is the DC component of a signal, and what does it mean? It means the average value of the signal. Now typically when we represent a signal, say it is a sine or cosine; whatever it is, So, if you see the average, it is typically 0. And hence, we do not have any. DC component. But if we bias the axis such that we add something to this signal then the signal shifts vertically which I have already discussed In the time domain, that amplitude shifts. So that is introducing this DC component or DC frequency of this signal. So, if you have a DC frequency, that means that the signal has a bias and it does not having a zero mean 0 frequency because the definition of frequency states that number of oscillations in a second or unit of time. So, 0 frequency will never oscillate, even if it is.

Something like this means it is. Having a tendency to oscillate. But if it is totally straight with respect to the x-axis or the time axis, so that means It is a 0 frequency; there is no change. in the signal that is that component. An infinite frequency is the exact opposite. that means that in a unit of time there will be infinite such oscillations which means the signal will not proceed and this happens when there is a sharp discontinuity or an edge in the signal in the time domain and here you clearly see that there are infinitely many values at this specific time instance t not. So, if there is an infinite frequency present, In a signal, it typically means that there is. is a discontinuity at the signal and this is technically a very idealized case. In real cases, you do not encounter such sharp transition

in signals. But theoretically, this is still useful for analyzing many signals. So pictorially, this is our 0 frequency. If you see that the amplitude has shifted to plus one level, and this is my zero frequency component, this 1 value of this amplitude. So, this is a sine wave plus one. While infinite frequency occurs in a square wave, a perfectly square wave where at this instance, at this instance wherever there is an edge, there is a sharp. Sharp means very idealistically sharp change, where this value of time will have multiple values of amplitude which means the signal is not proceeding. Further, it is oscillating at the same point. Now, some more interesting concepts like negative frequencies and complex frequencies. So, you can see that in the spectra, there are both positive and negative frequencies. And specifically, this is happening in real time. signals where it has a positive component and a negative component.

So negative frequencies can be attributed. To the counterclockwise rotations of the signal. Now, if you recall simple harmonic motion, so a signal is found when there is rotation. And if you see, if you have a point, then if you draw. A right-angled triangle, so you can see that this the height of this is basically telling you about the motion of the sign. So, naturally, if you see Euler's expression, so here, you also get that the sign can be expressed as

$$\sin(x) = \frac{(e^{ix} - e^{-ix})}{2i}$$

So, these two are complex components of the sign. Similarly, cos will also have

$$\cos(x) = \frac{(e^{ix} + e^{-ix})}{2}$$

So, these are comprising of the two frequencies, positive and negative. And negative frequency is basically representing the time reversal of the signal that means if I have a signal like this, so, that means there is a positive flow of the wave, there is a negative flow of the wave. So, both the positive and the negative. frequencies are present in such a signal. What about complex frequencies? Now, complex frequencies are basically frequencies that have a real part, which has an imaginary part. Typically, you see this in your Laplace where there is a σ and a $j\omega$ term. Where your $j\omega$ is used for your Fourier analysis. What is your real frequency? Oscillations and sigma are a damping factor. Why is this damping factor used? Like we know, some Fourier transforms do not exist because of the convergence issue. So, by multiplying by

$$e^{-j\sigma t} \text{ NOT } e^{j\sigma t}$$

Rather than σt , you actually do it with an exponentially decaying function. So, you force a signal that is not absolute summable or integrable to be absolute summable or integrable, and that creates a plane, this S plane instead of an axis.

So this brings the concept of complex frequencies. So that is the complex frequencies. involve both oscillations and decay or growth. So, if you see this beautiful animation, so this is the formation of your cosine and sine. So, if you see the height and width of the graph, So this is actually giving you the instantaneous.

Amplitude as this is rotating, and this is a unit circle. So, you see that sine squared θ and cosine squared θ , their sum is 1 based on the geometry. So, this is rotating in a counterclockwise direction. Similarly, if I make it rotate in the clockwise direction, Then you get the negative frequency field, which is the opposite of the positive. So, this is an illustration of this cosine wave. A frequency of 100 hertz has a positive frequency component. and a negative frequency component. Now, ideally, you should have a single component.

In the spectra, but I plotted this in Python. Where I used a window of my sine wave. Naturally, you cannot plot the total sine wave because it is of infinite duration. So, because of this windowing effect. You get only two values or two magnitudes which are spread and which are not exactly at the desired frequency point. But there is a spread; this is a windowing effect. So, this is again, if you recall, the properties of your convolution and multiplication in the time and frequency domain.

So when you are windowing this signal you are multiplying by a rectangular function. Multiplying a rectangular function in the time domain means the convolution of the Fourier transform of the rectangular function in the frequency domain. So, because of that, you get a spread. This is windowing. So, next is the concept of phase. The question it answers is, can do two different signals have the same spectra? And if so, how will you distinguish that? They are different if they have the same spectra? Like I say, I have two sine waves the only thing is that this is sine 50Hertz T plus 45 degrees phase. And if I see the magnitude spectra again, this should be ideal peak, but because of this windowing. You get a spread around the peak. So, these two magnitudes spectra are exactly the same.

So, by looking at the spectra, I will ensure that they are the same signals. So, how do we distinguish? Using the phase spectra. So that is another takeaway that When I say "spectra," there are two components because of this Fourier analysis will give you a complex number. So, if I have a complex number, something. Like if I say

z is a complex number x plus jy ,

$$z = x + jy$$

So I can always plot the magnitude and phase. So, the magnitude spectrum shows how. Each frequency component is present. So, that is the magnitude of the frequency components. So, this is how you get the magnitude. Spectra just by taking the magnitude. So if this z is your complex number, then your Magnitude is obtained by x squared plus y . Square, and your phase is obtained by \tan inverse of y by x , which is the phase spectrum.

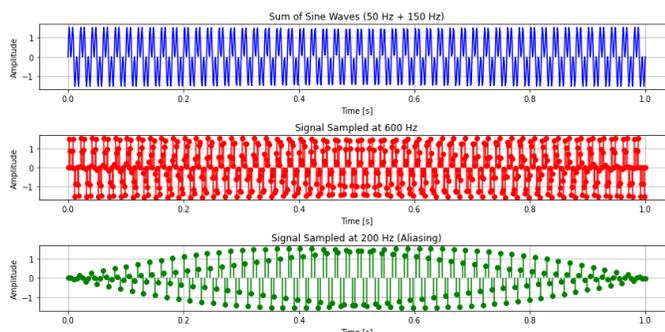
$$M(f) = |X(f)|$$

$$\theta(f) = \arg(X(f))$$

So, by the phase shift, you can actually distinguish that these two signals have the same magnitude spectra but different phase spectra. So, they have a lag with each other and this. Is the argument function, which is? Nothing but the tangent inverse. The next concept I want to discuss is the sampling frequency.

So, the sampling frequency refers to the number. Samples taken per second are also measured in Hertz. So, what is typically the sampling frequency? So, as I said, when I want to convert an analog signal to digital. What I do is pick some samples at typically, a uniform time gap T_s . So, this is my sampling frequency. That is how many samples of the signal I am collecting in one second and this is very important to understand. The conversion of analog signals to digital. or continuous to discrete. And the important theorem here is Nyquist's Shannon's sampling theorem, which states that you should sample at least twice the highest frequency component or bandwidth of the signal to be accurate. reconstruct a signal from its sample and this is done to avoid something called aliasing. So, this minimum sampling rate, which is at least twice the maximum frequency Content is called your Nyquist rate. For example, if you have a signal that is made up of frequencies up to 500 Hertz, then you should sample at least at 1 kilohertz to accurately reconstruct the signal.

But this is just the theoretical sampling rate. Practically, you should sample at least more. than four to five times this highest frequency component. The next concept is aliasing, which I just discussed. So, what is aliasing? So, if you see this diagram, so this is a signal which has two sine waves.



One is 50 Hz, and the other is 150 Hz. Now, what is the highest frequency component? This is 150. So, my Nyquist rate should be 300. So, if I sample at 600, see, I am still able to get the structure of the waveform.

So, this pattern is somewhat captured. Although if I sample even further, like 1000 hertz. It will be even better or even say, 2000 hertz will be very nice. But 600 is sufficient to capture the structure. If I take samples at 200 hertz, which is you. See less than my Nyquist rate of 300, then. I do not at all get the signal structure; I get a signal that is very different from my actual signal. So, this is called aliasing and this is aliasing in the time domain, that if, say, this is a wave, so if I take just two. I just took these samples, so I miss this curve.

If I take four samples, I am very much capturing the curve. The more samples I take, the better the representation. of my time domain original signal in the continuous domain is represented. Now, what is aliasing in the frequency domain? So, the same thing can be seen in. The frequency spectra, like if this is the original.

Frequency domain representation of the signal. So, if I sample at 600, I clearly retain. Both the components of the analog domain. But if I do sampling at 200, then basically this version is aliased into this. So, these two replicas are indistinguishable.

Like, say this low-frequency version and the high. The frequency version has the exact sample. Points, and you cannot distinguish between these two signals. So, this is called the folding frequency. So, the folding frequency is half the sampling rate.

So, this is where the frequency is, basically. Folded, like you see here, this is the folding frequency. So, if my sampling rate is, say, 200, my folding The frequency will be half of this, which is 100. So, this is the maximum frequency that. Can be accurately represented in the sample signal and any frequency that is above the folding. Frequency will be aliased into one of. The frequencies below the folding frequency. So, this is what happens. So, if this is your bandwidth of the original signal, then if it is like oversampled, oversampled means greater than Nyquist rate, then We see there is a clear distinction. clear separation between the spectra. However, if it is less than the Nyquist rate, these two samples overlap. And this brings us to the concept. of normalized frequency.

So, what is normalized frequency? This is the frequency with respect to. to or relative to our sampling rate. So, if f is the frequency of the signal and f_s is my sampling rate. So, normalized frequency, which is a dimensionless quantity because We are taking a ratio of 2 hertz. So, this is normalized frequency. It is in the range of minus 0.5 to plus 0.5. So, what happens is that, say, the maximum I can represent is $0.5 f_s$.

$$\omega = 2\pi f_{norm} = 2\pi \frac{f}{F_s}$$

This is my Nyquist theorem. So, say anything if I have a frequency. which is, say, 0.6 fs, so it will be aliased to some other value, it will be folded back in this range. Like if you have a signal with a frequency of 200 hertz.

And we are sampling at 1,000 hertz, so my normalized frequency will be 200 by 1000, which is 0.2. And this is the meaning that this conveys. The frequency is 20 percent of the sampling rate.

So you can go at a maximum of 50. Percent of the sampling rate. The next concept is digital frequency. So, this normalized frequency is digital. Frequency technically, but we also represent digital frequency or digital angular. Frequency in radian per sample. And this is two times π into a normalized frequency. Since the normalized frequency can vary, from minus half to plus half, this digital Angular frequency will vary from minus π to π . Now technically when you study this DTFT That this will be periodic with this as the fundamental period. But for all analysis purpose This range is used from minus pi to plus pi. So, this is what I have typically explained. So, like, say if you have a frequency that is 100 hertz. And if your sampling frequency is 1 k, then you can.

Easily find your digital angular frequency using this. Formula, which is 0.2 radians per sample. So, what are the applications for frequency analysis? So, in signal processing, this is used for. designing filters and noise reduction.

Noise reduction; I give an example of filter design. Typically, it is done in the Z domain, Z-transform. In communication, we used "in." Modulation and spectral analysis. And even in image processing, DCT is used.

For compression like JPEG compression, you I have used the file jp.jpg or .jpg, which uses this JPEG compression. And then in feature extraction, This Fourier analysis is also useful. To know whether the features are.

High-frequency features or low. Frequency smooth features. So, these, in a nutshell, are the construction of this module. that we will discuss Fourier analysis next, Then frequency estimation and power. Spectral density estimation: then we I will study the Z-transform and finally, coherence analysis.

Thank you, everyone. Have a nice day.