Signal Processing Algorithms & Architecture

Dr. Anirban Dasgupta
Department of Electronics & Electrical Engineering
Indian Institute of Technology Guwahati

Lec 11: Z-transform

Hello everyone, welcome to a fresh new lecture on the topic of Z-Transform, and this is part of the course on Signal Processing Algorithms and Architectures. I am Dr. Anirban Dasgupta and let us get started. So, what is this Z-transform? We already have a transformation in the frequency domain, which is the Fourier transform. So, the Z-transform is another transform that converts my discrete-time signal x[n] into a complex frequency domain representation and this is kind of analogous to the Laplace transform for continuous time signals.

So, in Laplace, we extended the concept of the continuous time Fourier transform from a single j ω axis to the s plane. Similarly, here we extend the concept of the Fourier transform or the discrete-time Fourier transform to the whole z-plane. So, why do we need the z-transform? So, this is very useful for analyzing linear time-invariant systems. Specifically with respect to stability and other properties and it is also useful in solving difference equations, just as Laplace was used to solve differential equations. This is useful in solving difference equations. What is a difference equation? Suppose I say

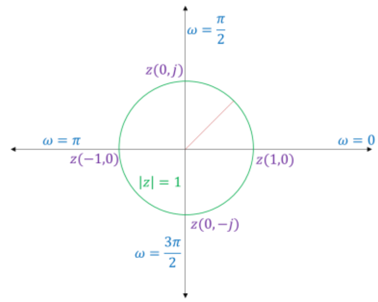
$$y[n] = x[n] + 2x[n-1]$$

and there can also be terms of y[n-1] as well. So, this kind of equation is called a difference equation, and this is the discrete-time counterpart of differential equations.

So, for a discrete-time sequence or signal x[n], we will define the Z-transform X(Z) as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

So, here, what is Z? So z is a complex variable, and usually complex variables or complex numbers are represented as a real axis like x and something like j y axis as the y axis. But here we are more interested in the polar representation than in this cartesian representation and the polar representation is in the r ω form, where r is the magnitude and ω is the phase. So, whenever you go radially outward from the origin, your r will increase, and that is the representation of the magnitude of the component, while the phase or ω will vary from 0 to 2π via a rotation.



Okay, so now how do we interpret this? This is what I was talking about. So, we have this z-plane, and these are the values of ω where this angle is my 0 degrees; the right of the y-axis is my π/2, the negative x-axis is my π, and the negative y-axis is ω equals to 3π/2. So, this is again just an extension of the DTFT across the plane. So DTFT is typically the evaluation at the unit circle if you see this. So, this green line represents the unit circle denoted by $|z| = 1$, and this means I have a number that lies on the unit circle, which is the evaluation of the DTFT.

So, you can see and compare these two formulas-

$$X(z) = z \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

So, X(z) and X(ω), which is the DTFT, you see they are very similar and if you compare these two, you will clearly see that the DTFT is an evaluation of the z-transform where z equals e to the power of j ω, or the unit circle. So why is this needed? Now, as we know, the Fourier transform is not defined for all signals, specifically if that Fourier summation is not finite. So, in this case, the Z transform is defined, and this is defined for almost all signals, but not over the whole Z plane; rather, it is in a specific region, and this region is called the region of convergence.

So clearly when $|z| = 1$, this is the unit circle in the z-plane and the z-transform will reduce to the DTFT. So, what is the region of convergence? The region of convergence is the set of values in the z-plane where the z-transform exists, or in other words, where the formula for summation converges and gives a finite value. Now, clearly, the ROC will never contain any poles. You know what poles are, so if you represent the system as a transfer function ($\frac{Y(z)}{X(z)}$), where (x(z) = 0), this will make the transfer function infinite; these are the poles, the locations of those z the ROC cannot contain poles because the ROC will be those points where the z -transform has finite values.

So, ROC is crucial in determining the properties of the system or the signal which is being analyzed. For example, if we have a causal signal, causality is mainly described in terms of systems, but causal signals are also defined for signals that are right-sided. Like for

$$x[n] = 0 \text{ where } n \leq 0,$$

This kind of signal is called a causal signal. So, in such signals, the ROC will extend outward. Like if I draw a circle, the ROC will be somewhere outside that circle, and this is defined by the outermost pole. For anti-causal signals, which are left-sided, the ROC is inside the circle, and if the signal is two-sided, that means there are positive indices as well as negative indices in such cases, the ROC is typically an annular region, which is the region between two disks. So, we will understand each of these meanings but let us try to see this pictorially and we will start with finite duration signals. So, if it is a causal signal, causal means the values are 0 for n <0. So, here we see that the entire z-plane is the ROC except for the point z = 0.

Why is that? I will explain. For anti-causal signals where the signal values are 0 for positive values of n, the entire z-plane is the ROC except when z = ∞ and if it is a two-sided finite signal with values for both positive and negative indices of n, then the ROC will again be the entire z-plane except at z = 0 and z = ∞. Now, what about infinite-duration signals? So infinite duration signals will also have a very similar pattern, but here instead of just the origin, we will have a disc, and this infinite and finite will play a major role in filter design and implementation, where we will discuss FIR and IIR filters. But as of now, if we see this, if I have an infinite-duration causal signal or right-sided signal, then the ROC will be some outer circular region beyond a circle.

Like |z| is greater than r_2 . And if it is an anti-causal signal, the ROC will be contained within some circle like this, mod $|z| < r_1$ and finally, if it is a two-sided signal, having values for both positive and negative indices, it is kind of an annular ring like this. So, let us try to break this down and clearly understand why this is happening in such a manner. So, we will start with an example of a finite signal that is causal.

So, this is our signal where this 3 is my 0 index, or I would say that x[0] is 3, x[1] is 6, x[2] is 8, and so on. So, how do I calculate the Z-transform? I will use the formula. But the formula says from $-\infty$ to $+\infty$. But I know that the sequence will not exist below - 1 or below 0, I would say. Also, this infinity can be written because after index 3, everything will be again 0, as this is a finite length sequence.

So if I plug in the values, the first value I put is n equals 0, and I get x[0], which is 3, and z^0. Then x[1], which is 6, then z^1. Then x[2], which is 8, then z^{-2}. And then $x[3]z^{-3}$, and the rest are zeros.I can expand this in this manner: 3 plus 6 z inverse plus 8 z to the power of minus 2 and 7 z to the power of minus 3.

$$x[n] = \{\underbrace{3}_{\uparrow}, 6, 8, 7\}$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$X(z) = 3z^0 + 6z^{-1} + 8z^{-2} + 7z^{-3} = 3 + 6z^{-1} + 8z^{-2} + 7z^{-3}$$

The ROC cannot contain $|z| = 0$

Now here we clearly see that this is a polynomial in Z inverse. So, if I get a value of 0 for Z, then that Z inverse term becomes infinite or undefined, and the transform will not exist. So, the ROC cannot contain Z=0, which defines my first concept. Similarly, we can define it for the finite anti-causal signals.

And this is the example. So, here my 0 index is 6, and these are all negative indices like n equals minus 1, minus 2, and minus 3. I will quickly plug in the formula and my values again; although it is from minus 3 to 0, I can say minus infinity to 0 just to show that it is an anti-causal signal. I will plug in the values and let us see what I get: I get this expression. You can verify this. So there are all positive powers of z.

$$x[n] = \{3, 5, 2, \underbrace{6}_{\uparrow}\}$$

$$X(z) = \sum_{n=-\infty}^{0} a^n z^{-n}$$

$$X(z) = 6z^0 + 2z^1 + 5z^2 + 3z^3 = 6 + 2z + 5z^2 + 3z^3$$

The ROC cannot contain $|z| = \infty$

Here, if I put z equal to 0, it is not an issue, but z equal to infinity will be an issue, and hence z cannot take the value of infinity, either positive or negative, and this is clear. And I think, by intuition, the other thing will also be clear, but still, let us take an example. So, this is my index 0; this is index 1, 2, 3, and these are indices minus 1, minus 2, minus 3. So again, if I put on the formula, and again since this is a two-sided signal, I am putting from negative infinity to positive infinity, I plug in the values because outside this range everything is 0, so this is what I get. And clearly we see that at z equal to 0, this goes to positive infinity and at z equals infinity, either positive or negative, this term will go to infinity. So the z cannot contain infinity or 0 for the z-transform to exist, which means the ROC is the entire z plane apart from these two values. So these are examples of FIR systems or finite impulse response systems. Now infinite causal signal, these are examples of filters which are of infinite length. So here there will be a denominator part, and there will be some poles.

$x[n] = \{3,5,2,\underbrace{9}_{\uparrow},6,8,\ 7\}$

$X(z) = \sum_{n=-\infty}^{\infty} a^n z^{-n}$

$X(z) = 3z^3 + 5z^2 + 2z^1 + 9 + 6z^{-1} + 8z^{-2} + 7z^{-3}$

The ROC cannot contain $|z| = \infty$ or 0

So, let us take an example. So x[n] is, so here I am talking about signals, but when I talk about poles, these are with respect to systems. But I am not specifying a signal or a system here; in general, I say I have a sequence represented as X of n, and I want to find the Z transform. Now, of course, for signals, Fourier is a better method, while for systems, the Z transform is a better method. But in this case, just for the introduction to the Z transform, let us assume that this is a function; this can be a signal or a system, and this is defined by A to the power of n U of n. So the Z-transform will be, if I plug in the formula, and again I have said that there is $a^n\ u[n]$.

$x[n] = a^n u[n]$, where $u[n]$ is the unit step function

$X(z) = -\frac{1}{1-az^{-1}} = \frac{z}{z-a}$ ROC; $|z| < a$

What is u[n]? u[n], as we discussed in the first lecture, if you remember, this is 1 for n greater than or equal to 0, and this is 0 otherwise. which makes it a right-sided or causal signal. And now we see that this is a geometric series because if I plug in the value of n, this is a power of 0, z to the power of minus 0, then n equals 1, a to the power of 1, z to the power of minus 1, and n equals z to the power of minus 2. So this is a sequence that is of infinite length, an infinite GP series like this. And we know that if I have an infinite GP series of the form r equals r to the power n, then the output is a by 1 minus r.

So a is the first term, basically, which is r to the power 0, so it makes it 1 by 1 minus r in this case. So, what do we get? So here this will converge to this provided that mod r is less than 1 because even if mod r equals 1, I have infinite terms which are 1, so that is infinity. And if that is greater than 1, it will diverge and explode to infinity. The only way is if mod r is less than 1; then the sequence will converge, and you will get a finite value. So here my r is a times z to the power of minus 1.

So if this is negative, then only the sequence converges. So, if you just put it and simplify properly, mod z is greater than mod a, and this is nothing but the region outside the circle with a radius of a, and this is the z-transform z by z minus a. Now I will put a box around this formula because we will reuse it in my next slide. And I will take another example, which is an anti-causal signal, and it is also infinite here. So minus a power n, u of minus n minus 1.

$x[n] = -a^n u[-n-1]$, where $u[n]$ is the unit step function

$X(z) = -\frac{a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a}$ ROC; $|z| < a$

So if I do the math, I plug in, so this is the minus, this is the transform, and again here what I do is I just make a small change: I put a new variable m, which is minus n. So here what happens is my summation of m now becomes 1 to infinity, and this pretty much looks like the previous case where I have this r; r here is a inverse. So, this r is the r for the GP series, r^n, which is, right? This is n so now, how will I solve this? I will put the formula, and I will see that this will exist only if the value of a inverse g is less than 1. And if I rearrange, we will see that mod z is less than mod a, which is the region inside the circle having a radius of a.

My drawing may not be too scale or bad here, but I hope you have understood the concept. So now if I just plug in the formula, I get this value a/(1-r), and if I rearrange things I get z/(z-a), but hang on, in the previous case I also got the same Z transform, and here I also got the same Z transform. So, which means Fourier, where two signals, two sinusoids having the same frequency but different phases, the magnitude spectra were the same. Here, we also see that two different signals have the same Z-transform. So, how do we distinguish between them? So, we distinguish them only by the ROC.

This is the same signal, not the same signal, but the same Z-transform; however, the ROCs are different. So, the Z transform does not just mean this expression for X(Z). The Z transform is the collective concept of this expression X(Z) and the ROC. This complete thing is called my Z-transform. Let us take another example in which we will calculate the transfer function.

$$y[n] - 0.5y[n-1] = x[n]$$

$$Y(z) - 0.5z^{-1}\,Y(z) = X(z)$$

$$Y(z)(1 - 0.5z^{-1}) = X(z)$$

$$Y(z) = \frac{X(z)}{(1 - 0.5z^{-1})}$$

$$\frac{Y(z)}{X(z)} = \frac{1}{(1 - 0.5z^{-1})}$$

Now, the transfer function is a very trivial thing, and in this case, we have a discrete-time system. So here we call it a pulse transfer function h of z, which is nothing but the ratio of Y(z) to X(z). So, this is a difference equation, and this is an IIR system. Why is that? I will explain later while talking about filter design and implementation. So, if you see, if I take the Z-transform of both sides, y[n] gives me Y(Z); this is a constant, and the

Z-transform of $y(n - 1)$ will be $Z^{-1}\, Y(Z)$.

Why is it so? This is a property. So, I am introducing a delay. So, a delay in the time domain will be a Z inverse multiplication in the frequency domain. And x[n] will give me X(Z). And if I just arrange them, what do I basically get? So, I understand this expression.

So Y(z), in terms of x(z), is this thing and again, if I bring down X(z) to the left side of the denominator, I get Y(z) over X(z), which is my transfer function. So, if a system is defined in the time domain by this difference equation, in the frequency domain or the z domain, the transfer function can be described by this. Let us take another example where the output is an advanced version of my input x of n plus 3. So I have to find y of z in terms of x and z. So, if I take the z-transform of both sides again, what happens is I get Y(z), and I will not use the relation or any property but rather let us use the formula.

So here I will let n+3 be a dummy variable m. Then my n will become m-3, and I will get this. I bring out z to the power of 3 outside. So, what I get is z cubed, and then what I do is, if I take out the terms z to the power of 0 and minus 1, this should be, I think, negative 3, n is m minus 3. So what happens is, if I take it out, I get $x[0]z^0$, which is 1, and $x[1]z^{-1}$

$y[n] = x[n + 3]u[n]$, where $u[n]$ is the unit step function,

$Y(z) = z^3X(z) - z^3x[0] - zx[1]$

So, this is the expression that I get. Now, unless this x[0] and x[1] are zeros, you cannot find this transfer function. Now take this as an exercise and think about whether this system is a linear time-invariant system or not, and under what circumstances it will become a linear time-invariant system, or if it is always a linear time-invariant system. So now we have understood the forward Z-transform, let us think about the inverse transform. Like I have the X(Z), I want to get my time domain signal back. So, this is given by the contour integral based on the cauchy integral theorem, which is

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz$$

But in the forward x[n], there was a discrete thing, so we did summation. So, this is the equation, and here the contour will encircle the origin and lie within the ROC. So basically, this is the evaluation of x in the ROC of z. So, I will end this topic with some properties, and the first property is the linearity property, which I have defined for most of the transforms. So, if you see that

$$x_1(n) \longrightarrow X_1(z)$$

$$x_2(n) \longrightarrow X_2(z)$$

Then if I multiply and do a linear combination of these two, I get the same linear combination in the z-transform domain as well.

$$ax_1[n] + bx_2[n] \to aX_1(z) + bX_2(z)$$

Similarly, time shifting means that

$x[n] \to X(z)$, then

$$x[n-k] \to z^{-k} X(z)$$

if I right shift or delay the signal by some amount k samples, it is multiplied by a factor of z^{-k} , just like $e^{-j\omega}$ was multiplied in the case of Fourier ω k. Similarly, advancing the signal or left shifting will result in multiplication by z raised to the power of k.

$$x[n+k] \to z^k X(z)$$

So, convolution and convolution multiplication are very well-known properties for most of the transforms.

$$x_1[n] * x_2[n] \to X_1(z) X_2(z)$$

So, the only thing is that you have to remember when to use circular convolution and linear convolution. So here, if we do linear convolution in the time domain, it is equivalent to multiplication in the Z domain.

Two more interesting properties of the Z transform are the initial value theorem, which states that

$$x[0] = \lim_{z \to \infty} X(z)$$

if I know my X(Z) and I am just interested in the initial value of the signal, which is small x(0). I do not need to do the entire inverse z-transform; I just need to find the limit X(Z) as z tends to ∞. Similarly, if I want to find the final value, this is the final value theorem that

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1)X(z)$$

The limit of x[n] tends to ∞ is the limit of z as it tends to 1, z-1 capital X(Z). So, this is the definition of the final value theorem for a stable system. So, I will conclude with some common Z-transform pairs, such as

Discrete-Time Sequence, $x[n]$	**Z-Transform,** $X(z)$	**ROC**
$\delta[n]$	1	All z
$u[n]$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-k]$	z^{-k}	If $k > 0$, all z except at $z = 0$ If $k < 0$, all z except at $z = \infty$
$\frac{1}{n}; n > 0$	$-ln\,(1 - z^{-1})$	$\|z\| > 1$

the discrete time sequence $\delta(\,n)$, which is always 1 in all the transforms we have covered.

So here we have 1, and the ROC will be all Z, and probably this is the only sequence where you have all the ROCs in the whole Z plane where the transform is defined. What about u[n]? This is a very well-known case, which we saw as an extended version. So, it is $\frac{z}{z-1}$ or $\frac{1}{1-z^{-1}}$. The ROC is greater than z and another signal, $-u(-n-1)$, which has the same Z-transform, at least the X(Z), but the ROC will be different.

Then delayed impulse δ(n- k). So this will give you Z^{-k}, and if k is positive, then it will be the entire z-plane except z = 0, and if k is negative, it will be the entire z-plane except z = ∞, and that is very easily seen from the transform. Another is 1/n, where n is positive. So here we get

$$-ln\,(1-z^{-1}),$$

and you should take this as an exercise to find out using the formula. And here the ROC will be |z| $>$ 1 because this is a right-sided infinite sequence, not, yeah, it is kind of infinite.

So here it will be |z| $>$ 1.

So we will end up here. Thank you so much. Have a nice day.