Neural Networks for Signal Processing-I
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Lecture – 42
Hilbert Space

In this module, we will begin by exploring the fundamentals of Hilbert space and then progress to the concept of Reproducing Kernel Hilbert Space (RKHS). Following that, we will delve into the proof of the representer theorem, which has significant implications.

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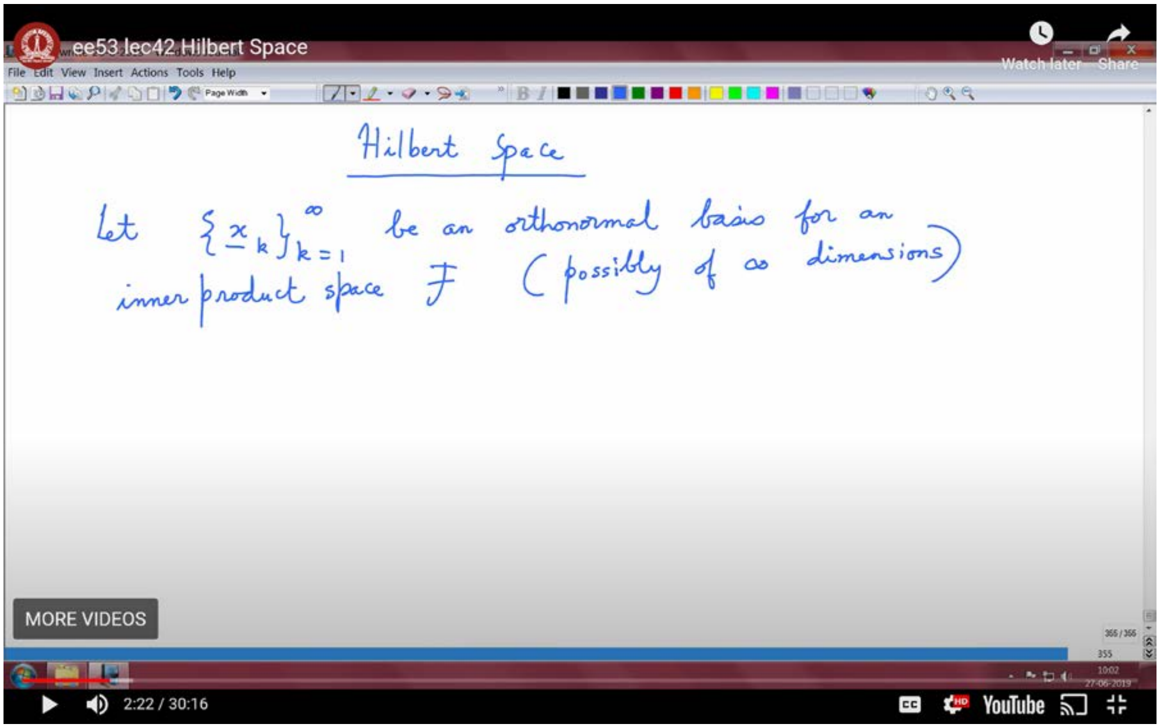

Our journey starts with the concept of an orthonormal basis, and we will build upon this by incorporating the completeness property, leading us to the formal definition of a Hilbert space. We begin with an orthonormal basis, define it with the inner product, and ensure that it is complete, culminating in our understanding of Hilbert space. From there, we will extend this understanding to define the Reproducing Kernel Hilbert Space based on kernels.

Let's denote $\{x_k\}_{k=1}^{\infty}$ as an orthonormal basis for an inner product space F, which may potentially be of infinite dimensions. An inner product space is defined on a vector space where an inner product is established. We start with a vector space that has an orthonormal basis, and by endowing this space with an inner product, it becomes an inner product space.

We are familiar with inner products from previous studies. Formally, for two vectors x_j and x_k, the inner product is denoted as $x_j^T x_k$. Since $\{x_k\}_{k=1}^{\infty}$ forms an orthonormal basis, this inner product equals 1 when j = k and 0 otherwise, ensuring the orthonormality condition where the norm is 1.

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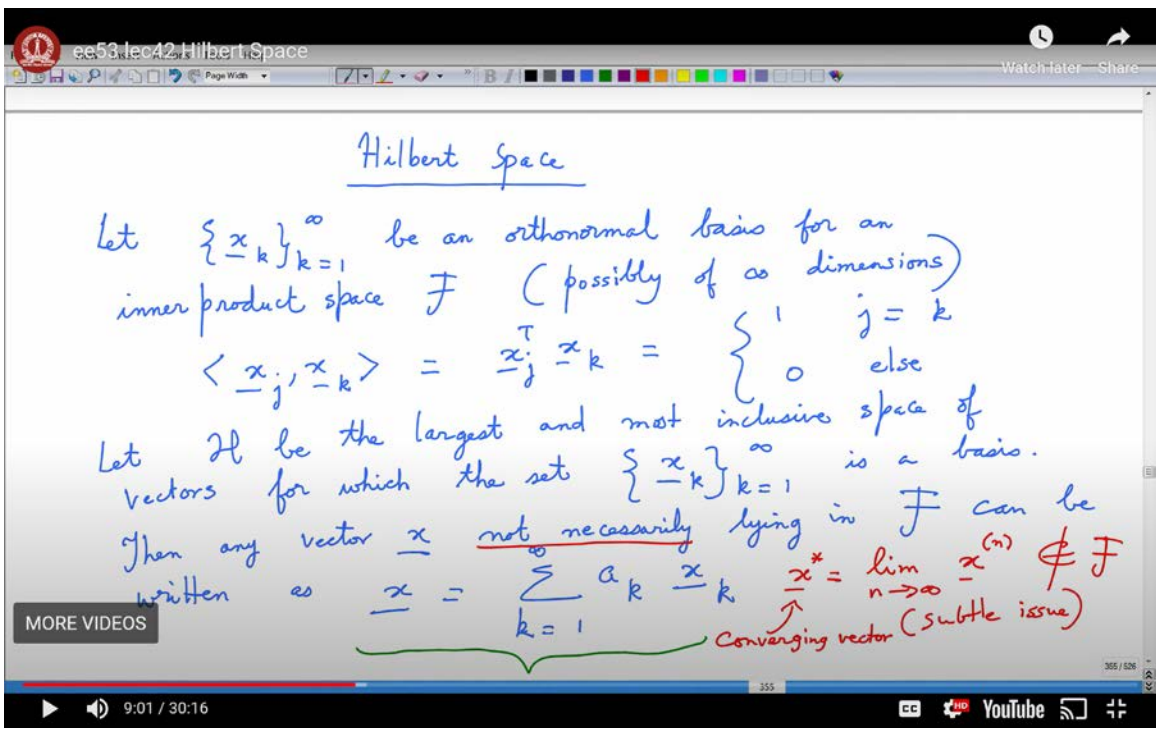

Let H be the largest and most inclusive space for which the set $\{x_k\}_{k=1}^{\infty}$ serves as a basis. For vectors not necessarily lying within the space $\mathcal{F}$, where $\mathcal{F}$ is an inner product space endowed with this orthonormal basis, the space H provides a broader context. Any vector x in this larger space H (which may not be part of $\mathcal{F}$) can be expressed as a linear combination of the orthonormal basis vectors:

$$x = \sum_{k=1}^{\infty} a_k x_k$$

This is crucial because a vector x might not belong to $\mathcal{F}$ even if it is the limit of a sequence of vectors within $\mathcal{F}$. The limit of such a sequence, x_n as n approaches infinity, need not be part of $\mathcal{F}$. This highlights the need for a more inclusive space, H, which encompasses $\mathcal{F}$ and allows us to handle cases where the limiting vector does not reside in $\mathcal{F}$. This subtle issue will be discussed in more detail as we proceed through this module.

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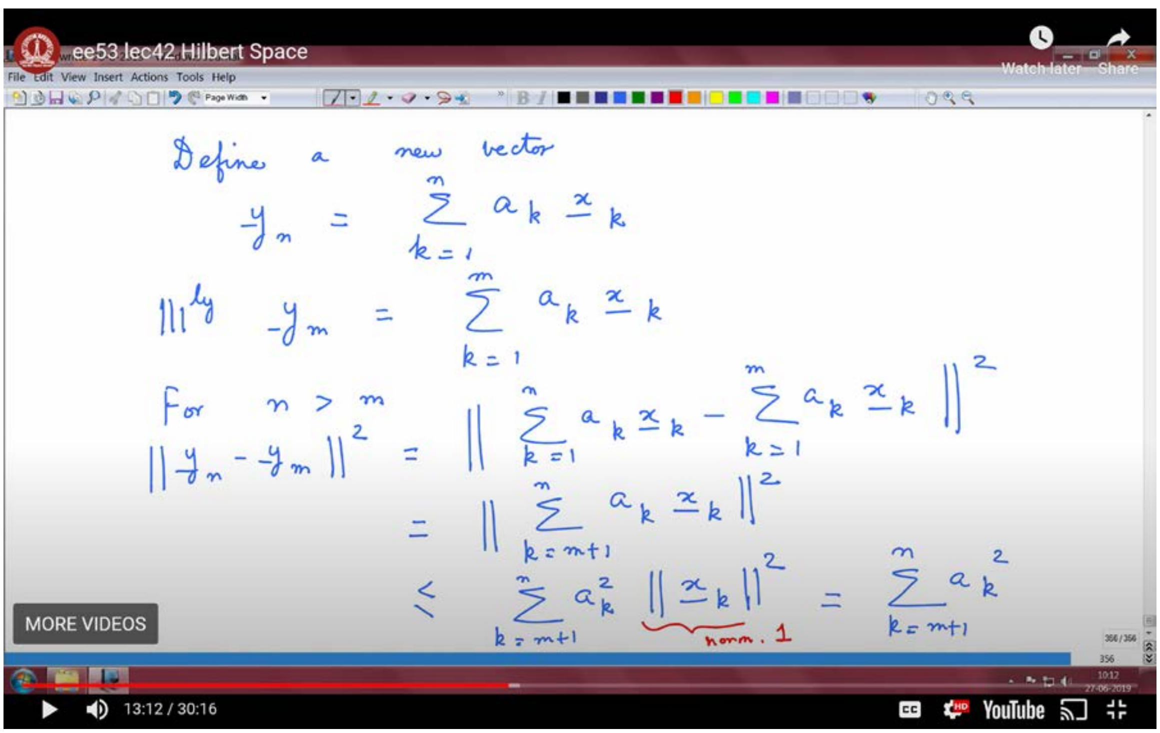

The key detail here is that we are dealing with a converging vector. Specifically, the limit of a sequence of vectors $\{x_n\}$ is a converging vector x^*. However, this converging vector x^* does not necessarily belong to the original inner product space $\mathcal{F}$. To address this issue and to include x^* within a suitable framework, we consider a larger, more inclusive space H. This space H accommodates this subtle condition by incorporating the completeness property, which allows us to account for vectors that are limits of sequences in $\mathcal{F}$. We will explore these details more thoroughly later.

A minor point of confusion might arise from the notation used. The equation suggests that any vector can be expressed as a linear combination of the basis vectors $\{x_k\}$. When using x_n, it denotes a sequence of such vectors. If this notation is unclear, it might help to denote the sequence with a superscript n to indicate that x_n represents a sequence of vectors, and x^* is the limit of this sequence as n approaches infinity. This limit vector x^* may not belong to the inner product space $\mathcal{F}$.

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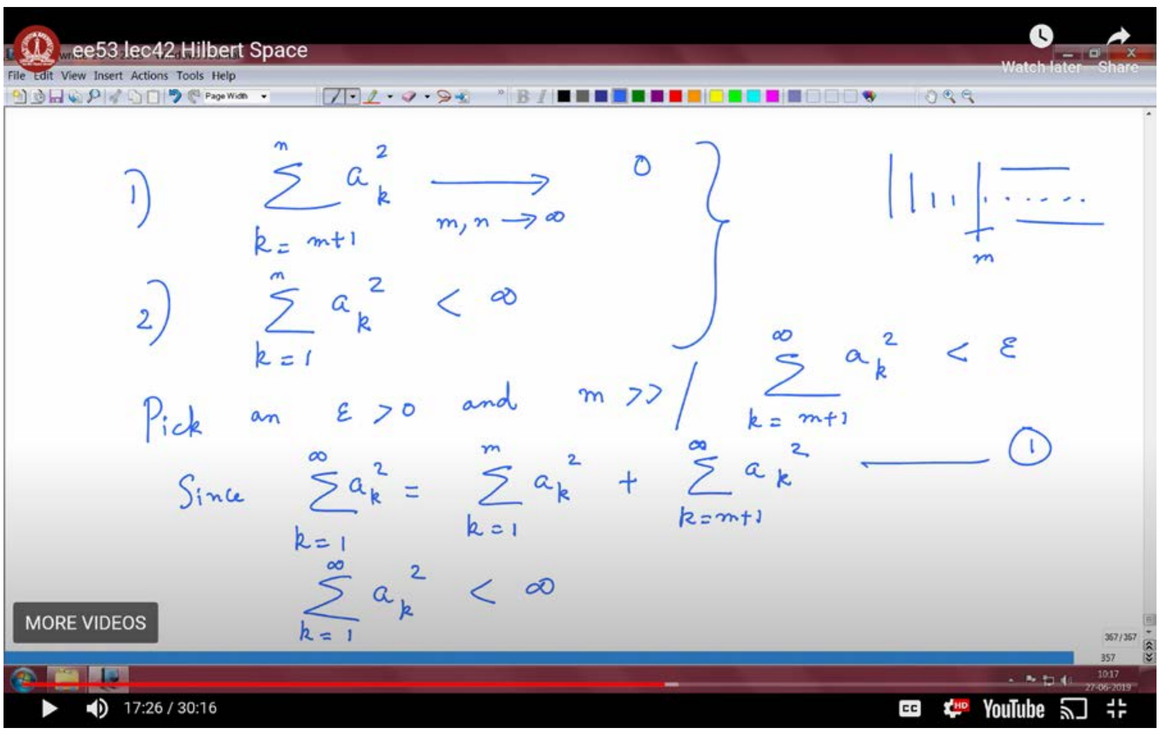

Now, let's define a new vector y_n as:

$$y_n = \sum_{k=1}^{n} a_k x_k$$

This represents a linear combination of n vectors. Similarly, define y_m as:

$$y_m = \sum_{k=1}^{m} a_k x_k$$

where m < n. To measure how y_n differs from y_m, we compute the squared norm of the difference:

$$|y_n - y_m|^2 = |\sum_{k=1}^{n} a_k x_k - \sum_{k=1}^{m} a_k x_k|^2$$

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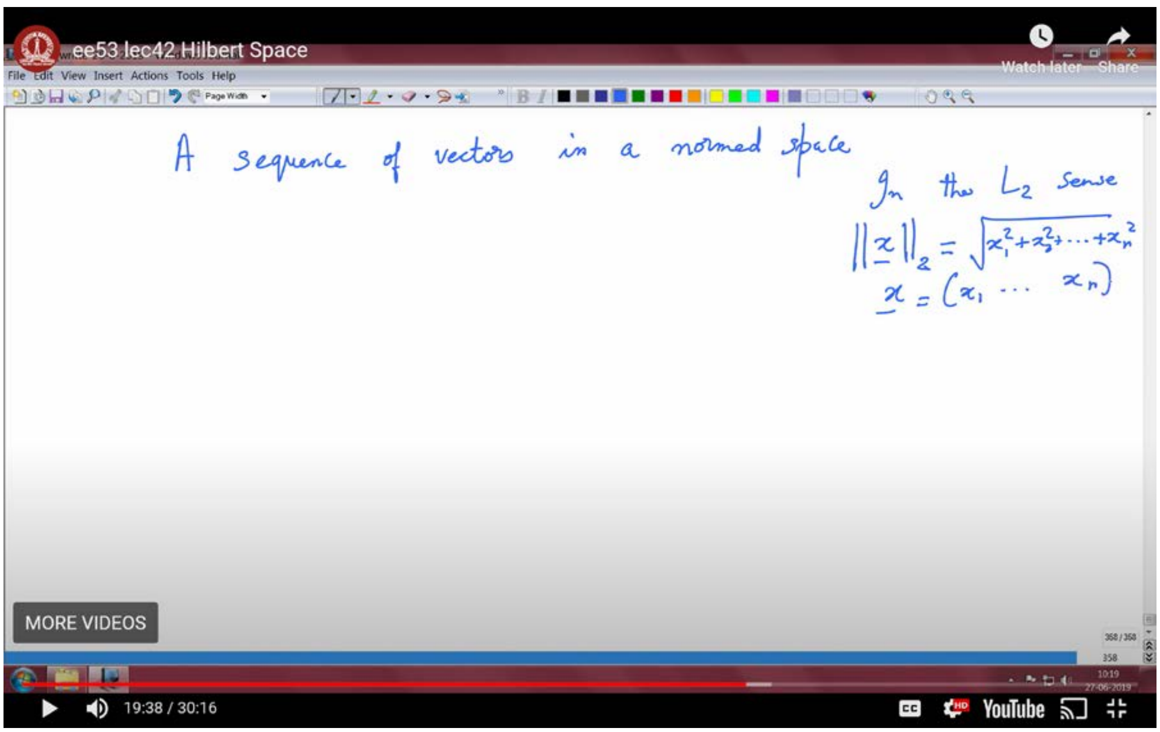

This simplifies to:

$$|y_n - y_m|^2 = |\sum_{k=m+1}^{n} a_k x_k|^2$$

Using the inequality property that the norm of a sum is less than or equal to the sum of the norms, we get:

$$|y_n - y_m|^2 \leq \sum_{k=m+1}^{n} a_k^2 |x_k|^2$$

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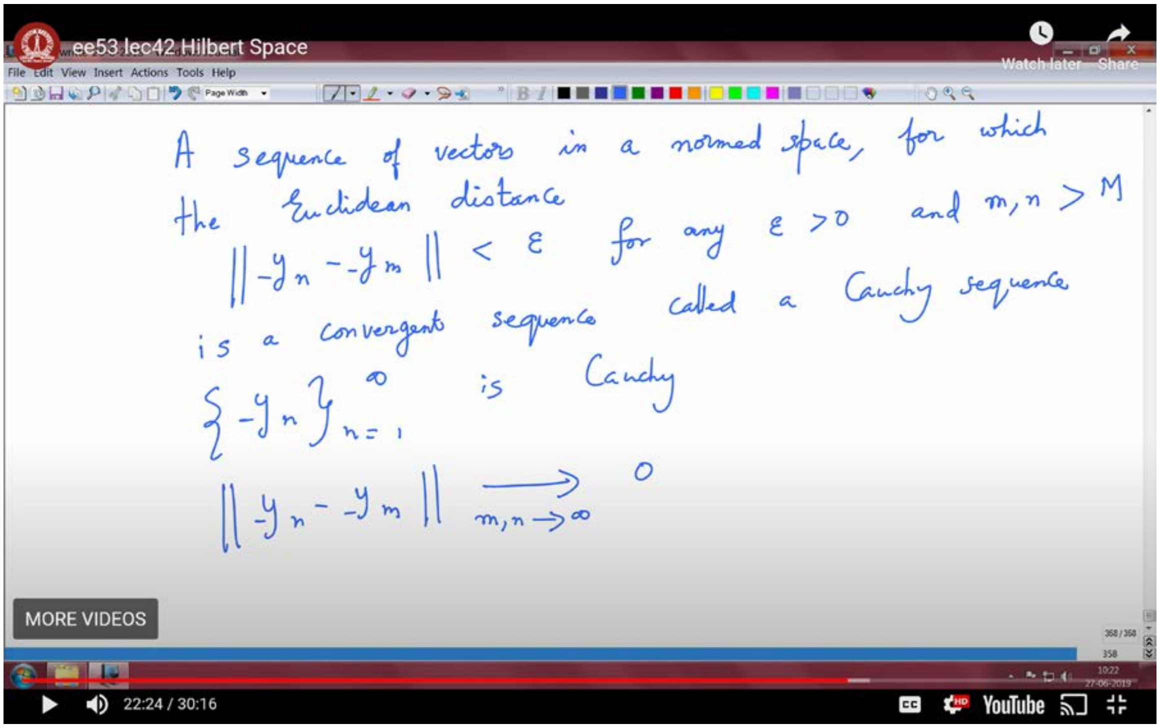

Given that the basis vectors are orthonormal (i.e., $|x_k|^2 = 1$), this reduces to:

$$|y_n - y_m|^2 \leq \sum_{k=m+1}^{n} a_k^2$$

This result follows straightforwardly from the orthonormality of the basis vectors.

This essentially boils down to the summation from k = m + 1 to n of a_k^2, which is less than some tolerance ϵ when m and n tend to infinity. The intriguing aspect here is that, in the limit, this partial sum should approach 0. To clarify, let's list the key properties:

1. The sum $\sum_{k=m+1}^{n} a_k^2$ approaches 0 as m and n tend to infinity.

2. The sum $\sum_{k=1}^{n} a_k^2$ is finite; that is, it converges to a finite value.

The goal here is to introduce the concept of a Cauchy sequence, which is why we are examining these summation properties. We start by choosing an $\epsilon > 0$ and setting m to be sufficiently large so that:

$$\sum_{k=m+1}^{\infty} a_k^2 < \epsilon,$$

Given that the total sum $\sum_{k=1}^{\infty} a_k^2$ is finite, it can be split into two parts:

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{m} a_k^2 + \sum_{k=m+1}^{n} a_k^2$$

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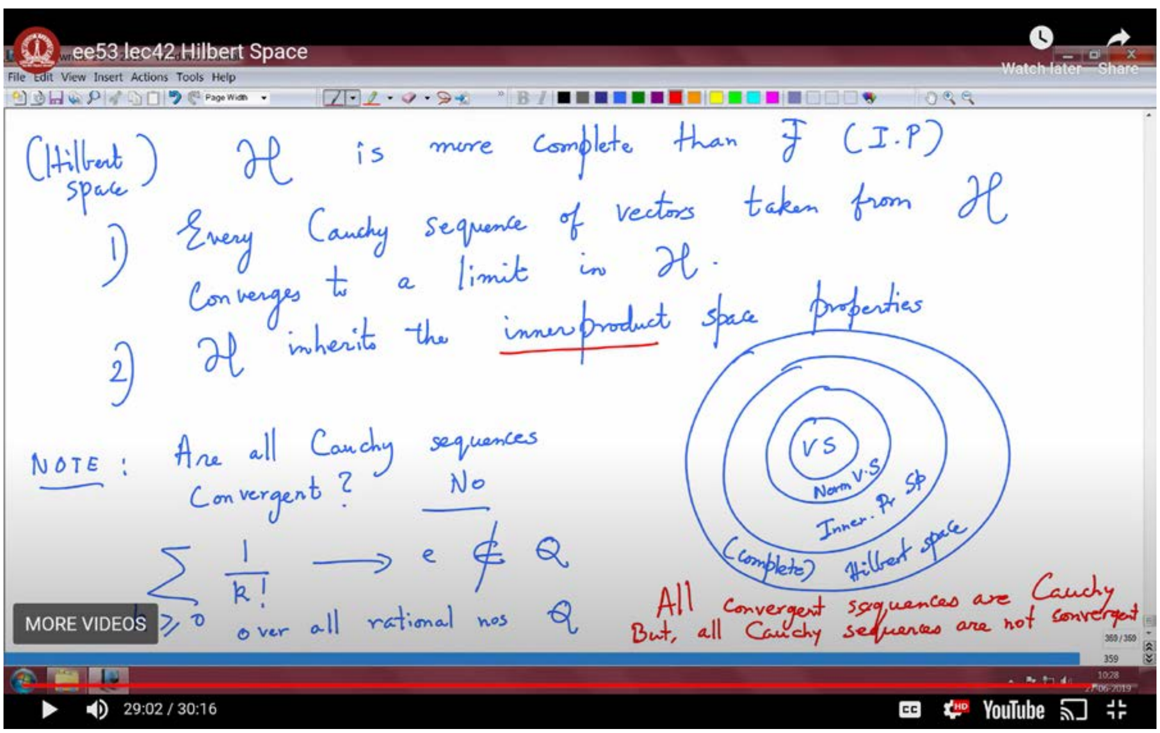

Since $\sum_{k=m+1}^{\infty} a_k^2 < \epsilon$, this implies that:

- The first part of the sum, $\sum_{k=1}^{m} a_k^2$, is finite.
- The second part, $\sum_{k=m+1}^{\infty} a_k^2$, is bounded by ϵ.

Therefore, the entire series $\sum_{k=1}^{\infty} a_k^2$ is bounded and finite.

With this intuition from the series summation, we can frame it in the context of sequences of vectors in a normed space. Here, the norm of the difference between vectors in the sequence is within ϵ, which leads us to define a Cauchy sequence in terms of its convergence properties.

A sequence of vectors in a normed space, where a normed space is a vector space endowed with a norm, is considered. A norm is a non-negative quantity that measures the length or size of a vector. Different norms exist, such as the L_1 norm, L_2 norm, L_∞ norm (or sup norm), etc. Each norm provides a different way to measure the distance between vectors. A vector's norm is 0 only if the vector itself is zero; otherwise, the norm is positive.

These details about norms and Cauchy sequences were covered in the earlier course I offered through NPTEL, titled *Mathematical Methods and Techniques in Signal Processing*.

For instance, in the L_2 norm, the norm of a vector x is expressed as:

$$|x|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

where $x_1, x_2, x_3, \dots, x_n$ are the coordinates of the vector $\boldsymbol{x}$, which can be written as $(x_1, x_2, \dots, x_n)$.

When we refer to a normed space, we mean a vector space that is equipped with a norm. To keep things tidy, let’s clear out some of these details.

In a normed space, a sequence of vectors $\{y_n\}$ is considered a Cauchy sequence if the Euclidean distance norm $|y_n - y_m|$ is less than any positive ϵ for sufficiently large integers m and n (greater than some threshold M). In other words, a sequence $\{y_n\}$ is Cauchy if:

$$|y_n - y_m| \to 0 \text{ as } m, n \to \infty.$$

This is analogous to how a sequence of numbers converges: just as a sequence of scalars can converge, so can a sequence of vectors. The convergence in this context is defined by the norm, which in the L_2 case is the Euclidean distance. The subtlety here is that the limit of a converging sequence of vectors might not belong to the original inner product space, which is why we consider extending to a larger space.

We introduce the concept of a Hilbert space $\mathcal{H}$ to accommodate such sequences. Specifically, $\mathcal{H}$ is a Hilbert space, or a complete inner product space, that includes the original inner product space $\mathcal{F}$. The Hilbert space $\mathcal{H}$ is more comprehensive than $\mathcal{F}$ for two main reasons:

1. Every Cauchy sequence of vectors in $\mathcal{H}$ converges to a limit within $\mathcal{H}$.

2. $\mathcal{H}$ retains the properties of an inner product space.

To summarize the hierarchy: starting with a vector space, we endow it with a norm to obtain a normed vector space. Adding an inner product to this space yields an inner product space. For a sequence of vectors in this space to converge, it must be a Cauchy sequence, and if it converges, its limit should belong to the same space. This is the essence of a Hilbert space, which is complete by definition.

A subtle but important question arises: are all Cauchy sequences convergent? This will be addressed in further detail.

The answer is no. For instance, consider the series $\sum_{k=0}^{\infty} \frac{1}{k!}$, where k is a non-negative integer. If we examine this series within the space of all rational numbers, denoted as Q, we know that the sum of this series approaches e as k tends to infinity. Here, e is a transcendental number, which does not belong to the space of rational numbers.

This example illustrates that while a sequence may be Cauchy , meaning that the norm of the difference between its elements can be made arbitrarily small as m and n approach infinity, the limit of a Cauchy sequence might not necessarily be within the original space. Specifically, all Cauchy sequences do not have to be convergent within the same space, though every convergent sequence is indeed Cauchy.

The distinction is crucial: while every convergent sequence is Cauchy, not all Cauchy sequences are convergent. This highlights the importance of defining completeness within a given space. Completeness is a property of a space that ensures every Cauchy sequence converges to a limit within that space.

Understanding this distinction helps clarify how we progress from a general vector space to a Hilbert space. A Hilbert space, being more comprehensive, ensures completeness for its inner product space. This broader space allows for the manipulation of vectors or functions within a complete framework.

This becomes particularly relevant when considering the expansion of functions. By using an orthonormal basis for such expansions, we must carefully examine the convergence of these sequences and ensure that the resulting sequence remains within the space. These subtle issues of convergence will be addressed in this module to provide a clear understanding of how we apply these concepts.

With this foundational understanding, we will proceed further in the module to explore these details more deeply.