Mathematical Methods and Techniques in Signal Processing - I
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Lecture – 09
Linear independence and spanning set

So, we will now look into the linear combination of vectors, this is very important.

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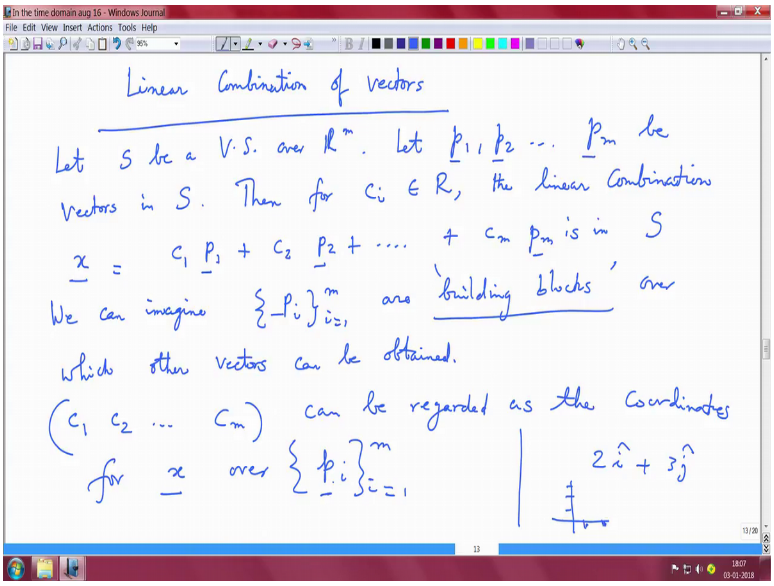

Let S be a vector space over R say, some R n. Let p 1 p 2 so on p m be vectors in S. Then for some scalar belonging to R the linear combination x which is c 1 p 1, c 2 p 2 some c m p m is in S. So, we can imagine that this set of vectors ps i are basically building blocks over which other vectors can be obtained. So, we can sort of imagine them to be as building blocks over which other vectors can be obtained and basically your c 1, c 2, so on till c m can be regarded as the coordinate, as coordinates for x over the set.

No different from our notion of for example, if you think about in 2 dimensional space. If you think about it as 2 i plus 3 j right, you think about it as 2 i plus 3 j; that means, in I direction you are going 2 steps and in the j direction you are going 3 steps is basically a position vector 2 i plus 3 j right. So, your i and j you can think about them as p 1 and p 2 and your 2 and 3 are basically coordinates clear.

So, do we think about this in terms of vectors? Of course, we have to always have this in our mind that somehow our objective should be to take a signal and get this in the form of a coordinate representation that is our objective and this is the aim of this module. Well, we have to give an underscore for all these vectors following our convention and often I miss this.

Now, let us start with some definitions here.

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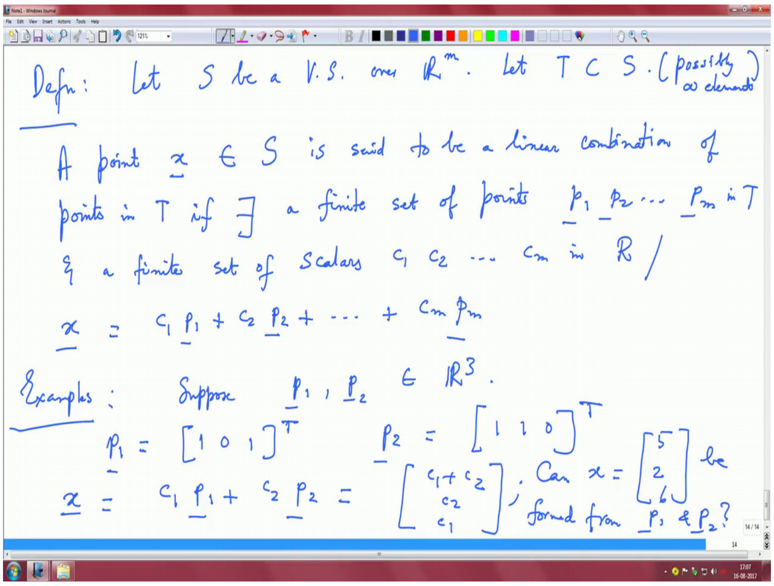

Let S be a vector space over R some m, let T be part of this space and you can imagine T can have infinite elements right, T can have possibly infinite elements. A point x belonging to S is said to be a linear combination of points in T if there exists a finite set of points p 1, p 2, so on till p m in T and a finite set of scalars c 1, c 2, so on till c m in R such that this vector x can be written in this form.

So, we will see some examples. Suppose I choose p 1 and p 2 in R 3. Suppose I say p 1 is 1 0 1 transpose and p 2 is basically 1 1 0 transpose and I form x to be c 1 p 1 plus c 2 p 2 right, I take x to be a linear combination of p 1 and p 2, c 1 and c 2 are basically scalars over the real right. And if I just did this I land up with basically, I take c 1 times p 1 plus c 2 times p 2. So, the first coordinate is what? c 1 plus c 2 next coordinate is c 2, next coordinate is c 1.

Now, can you formulate can x say suppose it is given as 5 2 and 6 we formed from p 1 and p 2? Obviously, not right you want c 1 plus c 2 to be 5, c 2 is 2 and c 1 is 6, it is not right. You cannot think about this as basically a linear combination. So, therefore, you really cannot think about having any vector using a linear combination of these vectors is very very very important to think about. Because in signals also we can think about analogously; can we represent this signal belonging to the space as some linear combinations of some signals in the set. I have not gotten into the basis and dimension yet, but here is a feel and intuition as you start scribbling through your notes to think through if such a thing is possible or not and that is the whole idea about signal representation right. I mean we will get to that point, but I think you should get a feel for this intuition.

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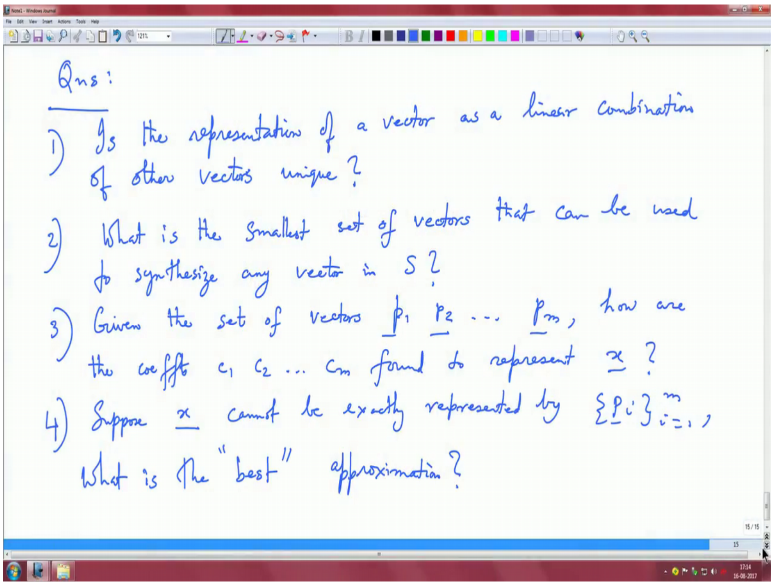

Now, at this stage since most of you are graduate students you have to go a little more deeper asking more basic questions. Now, if you are in high school you just imagine thinking about yeah I my teacher taught me coordinate geometry so therefore, I am writing all these in a Cartesian plane and I trust what these results are true. But again if you were to at this stage posed this question what is the rational behind construction of this coordinate axis in this particular way, can I do differently, is my representation unique. So, these are very very basic questions. If you ask these questions you will really go back to your high school and revisit that what you did was indeed right and there is a theory behind what you are doing right.

So, let us ask this set of questions, is the representation of a vector as a linear combination of other vectors unique? What is the smallest set of vectors that can be used to synthesize any vector in S we want the smallest set of vectors, the collection of vectors should be as the cardinality of that set should be very small, what is the smaller set of vectors that can be used to synthesize any vector in S. Given the set of vectors p 1 p 2 so on till p m how are the coefficients c 1, c 2, so on till c m found to represent x right. This is a basic question if you were to think about it. If you know the answer probably you will rattle it off, but if did not know the answer I give you this vector x and I give you this set of vectors p 1 p 2 p 3 so on till p m, how can I get this c 1 c 2 from this right, techniques for doing that. Suppose x cannot be exactly represented by p i, what is the best approximation?

These are basic questions I want to represent let us say my set of polynomials is one T and T square this is this, this set let us say the polynomials up to degree 2 is one T and T square they form you know this these are the maximum polynomials up to degree 2 right if you use monomials one T T square is what I have. But I want some signal T power fifth root of T, T power 1 by 5 I want that to be represented using one T and T square. Some linear combination of this how can I do that, some curve I want approximate with something else, what is the mode, how can I do that, what is the best approximation, in what sense. So, these are basic questions which one has to ask right that automatically leads us into this representation in geometric sense and the field of geometry of these vectors and is what we have to get. So, we will try to address some of these questions.

But before we get into the details we should. Now, define what is linear independence is very where we just started with linear combination of some vectors and we said that you know it is sometimes not possible to express any vector as a linear combination we may be restricted. But if I wanted express it as a linear combination of these vectors when can I do so, these are questions which we have to ask that gives us an idea into an important concept called linear independence. So, let us see what this is.

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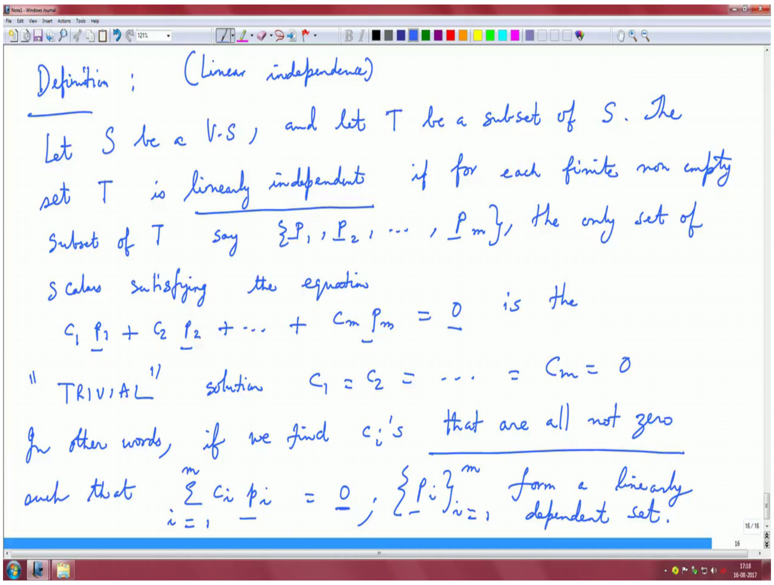

Let S be a vector space and let T be a subset of S. The set T is linearly independent if for each finite nonempty subset of T say p 1, p 2, so on till piece of m, if for each finite nonempty subset of T say p 1, p 2, so on till p m the only set of scalars satisfying the equation c 1 p 1 plus c 2 p 2 plus 1 and so forth till c m p m this is equal to 0 is the trivial solution. That means, if you find c i's that are all not 0 such that sigma c i p i is 0 then this set p i, i equals 1 to m they form a linearly dependent set.

In other words if we find c is that are all not 0 I mean it is a very important step some can be 0, but some cannot, some can even if I give you one element which is non zero then it breaks. That are all not 0 very important such that sigma i equals 1 to m, cs of i, ps of i if this is equal to 0, sometimes with a slight abuse of notation you may omit this bar for 0 and you have to imagine that it is a 0 vector. So, there is a difference between a 0 scalar and a 0 vector, here it is a 0 vector. So, sometimes I may miss this underscore, but you may have to just register in your mind that it is a vector. Then this piece of i, i equals 1 to m they form a linearly dependent set. So, we will just take some examples and just cross check this.

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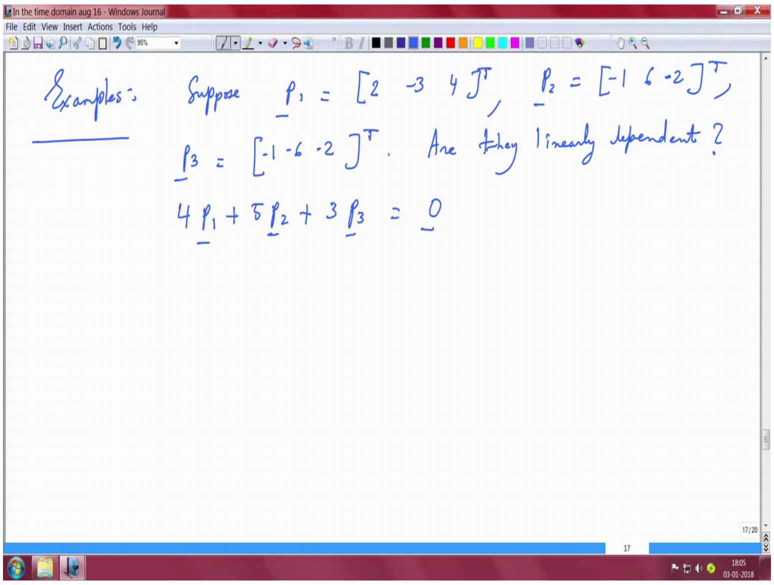

Suppose I give you p 1 equals 2 minus 3 4 transpose p 2 is minus 1; minus 1 6 2 transpose and p 3 is so 1 6 2 transpose are they linearly dependent. So, I will also give you the answer if you did 4 p 1 plus 5 p 2 plus 3 p 3 you get this none vector. Just cross check and then conclude that they are basically linear linearly dependent.

Well, I think which is 4 p 1 5 p 2 and 3 p 3 this might not be the case they may not be linearly dependent. So, let us just fix this up. So, I think a negative sign here on the third coordinate of p 2 and p 3 completely negated would make this work. I mean you can just quickly verify this is 4 times 2 which is right this is 5 times minus 1, which is minus 5, 8 minus 5 which is 3 and 3 minus 3 is 0 and you can verify this for the rest of the coordinates that if we change p 2 and p 3 in this form and this satisfies is equation and they are linearly dependent.

So, we will see a few more definitions.

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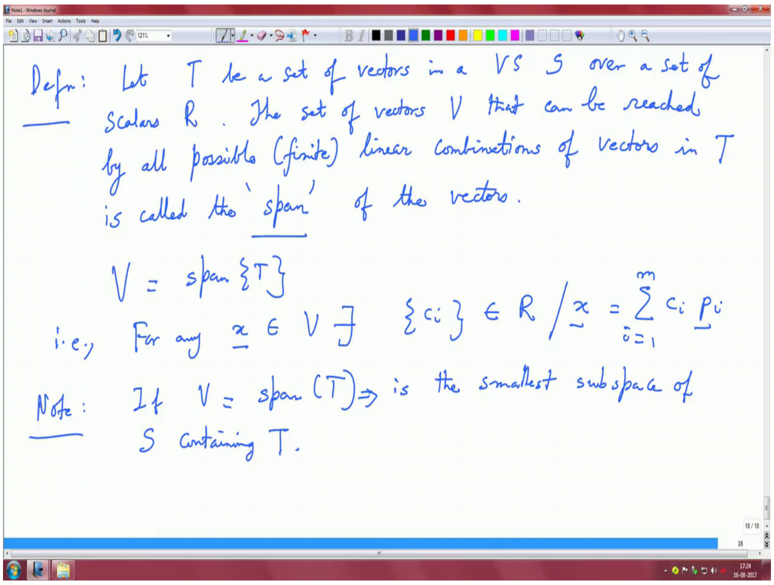

Let T be a set of vectors in a vector space S over a set of scalars R. The set of vectors V that can be reached by all possible it is a finite here, by all possible finite linear combinations of vectors in T is called the span of the vectors. So, let T be a set of vectors in a vector spaces over a set of scalars R. The set of vectors V that can be reached by all possible linear combinations of vectors in T is called the span of the vectors. So, V is basically the span of T, which means for any x belonging to V there exists some c is belonging to R such that x is written down in this form. Now, we have a small note here if V equals span of T. So, this is basically the smallest subspace of S containing T. Now, we have some definitions on spanning set.

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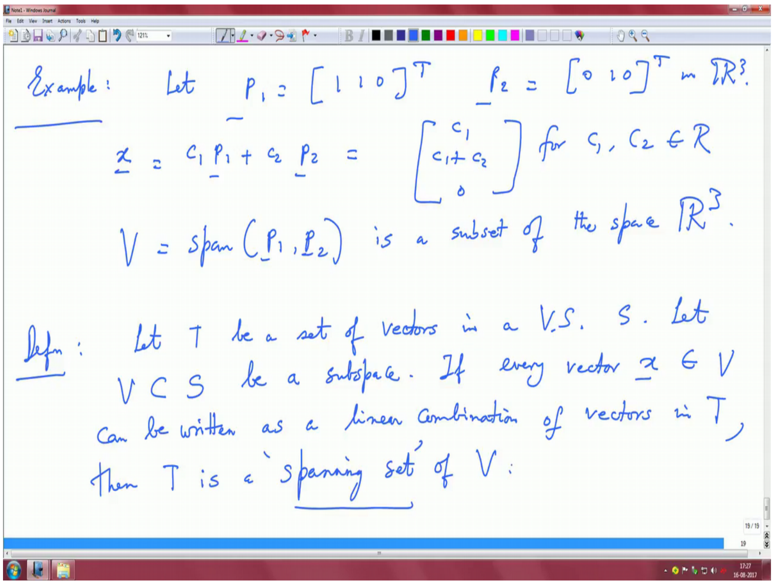

So, but before we get into the definition let us take an example. Let p 1 be this vector 1 1 0 transpose and p 2 is this vector 0 1 0 transpose in R cube. So, let us say x is c 1 p 1 plus c 2 p 2 which is basically c 1 c 1 plus c 2 and the last element is basically 0. So, V is basically the span of these 2 vectors p 1 p 2 and this is a subset of this space R 3.

Now, just imagine what is happening here right. I mean we take a linear combination of p 1 and p 2, the third coordinate is basically a null and all we have is basically a plane a 2 dimensional plane right. I have c 1, I have c 1 plus c 2 these 2 combinations will basically define this plane in 2 dimensions, so that is what this is. And it is basically the plane where 1 1 0 and 0 1 0 lie right. So, this is basically the idea.

So, we have one more definition here. Let T be a set of vectors in a vector spaces, let V be a subset of S and I just let us denote this and we call this as a subspace actually to be very precise. If every vector x belonging to V can be written as a linear combination of vectors in T, then T is a spanning set of the space V. This is an important definition as well. But we will end this with an example.

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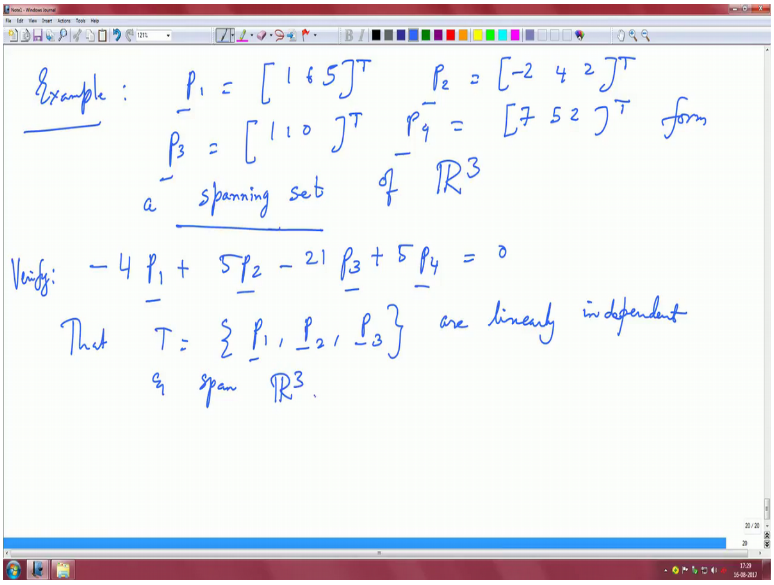

Suppose I give you vectors p 1 say 1 6 5 transpose, some p 2 is minus 2 4 2 transpose, p 3 is 1 1 0 transpose, p 4 is 7 5 2 transpose. They form a spanning set of R 3 and you can think if you just verify minus 4 p 1 plus 5 p 2 minus 21 p 3 plus 5 p 4 is 0 and the set p 1 p 2 p 3 or linearly independent and span R 3 right. So, though we have these set of vectors p 1, p 2, p 3 and p 4 they are linearly related as per this equation minus 4 p 1 plus phi p 2 minus 21 p 3 plus 5 p 4 is 0 and the set T which is p 1 p 2 p 3 they are linearly dependent and they span this 3 dimensional space. So, this is basically a spanning set.

Now, I think the important notion that we have is linear dependence, linear independence is an important concept and linear independence provides us with what we need a unique representation as a linear combination of vectors. That means, given some vector I can express this vector as a linear combination of some vectors. And the question is how to get vectors that are linearly independent and spanned the space right because I want this to span the space and I want these vectors to be linearly independent. So, this is the next notion that leads us automatically into the notion of basis of this. And with this notion of the definition of basis which is basically a set of vectors that are linearly independent and that can span the space now I can construct the space I want, it can be a vector space it possibly can be a signal space. If it is vector it is basis vectors, in signals this is a base for this signal set right and we will get to this in the next lecture.