INTELLIGENT CONTROL OF ROBOTIC SYSTEMS

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Lecture 07: Fuzzy Relations and Rule Base

Good afternoon, everyone, today. We are going to start a lecture on fuzzy relations and rule base. The outline of this lecture will be fuzzy relations with some examples and fuzzy implications relation. Then we will get into the fuzzy rule base, which forms the fundamentals of fuzzy logic, and then we will finally have multiple rules of a fuzzy system, precisely a continuous fuzzy system.

Now, coming to the fuzzy relations, Let us talk about that.

If X & Y : Universal sets,

$R(x,y)$ is given by,

$$R(x,y) = \left\{ \frac{\mu_R(x,y)}{(x,y)} \,\middle|\, (x,y) \in X \times Y \right.$$

Now, coming to the formal definition of fuzzy relations.

A fuzzy relation is a fuzzy set defined in the Cartesian product of crisp sets. Crisp sets say x1, x2, x3 up to xn. Okay, this is a formal definition of a fuzzy relation. It is a fuzzy set defined in the Cartesian product of crisp sets X_1, X_2 up to X_n.

Now, the fuzzy relation

$$R(x_1, x_2, \ldots, x_n) \Rightarrow \left\{ \frac{\mu_R(x_1, x_2, x_3 \ldots x_n)}{(x_1, x_2, x_3 \ldots x_n)} \,\middle|\, (x_1, x_2 \ldots x_n) \in X_1 \times X_2 \ldots X_n \right\}$$

$$\mu_R : X_1 \times X_2 \times X_3 \ldots \times X_n \Rightarrow [0, 1]$$

is the membership function value mu R. Now, let us see the relation inference. Relation inference.

Say fuzzy relation represented by R1, R2, R3, where R1 is relating the Cartesian relation space of two universal sets X and Y, where R2 is a fuzzy relation relating Cartesian space Y and Z, and R3 is for Cartesian space relating X and Z. So, R1 is associated with the

Cartesian space formed by X and Y universal sets, likewise for R2 and R3. Now, if R1 and R2 are given, then R3 can be inferred Using the

Following Relation Or You can say precisely following operation. One is

Fuzzy max. Min. Composition. Operation. Another one is.

Fuzzy. Max product. Composition operation. Max product. Composition.

Operation. Either of this. Either of these two. Operations. R3.

Can be inferred. Given the relation. R1 and R2. Okay. Now let us see.

What is. Fuzzy. Max-min composition, the first one let us take, fuzzy. Max-min composition operation. We can see this with an example.

Let us consider two fuzzy relations. Let us consider two fuzzy relations. R1 and R2 defined on the Cartesian space, you can say Cartesian spaces X cross Y and Y cross Z respectively. That means R1 is defined on the Cartesian space X cross Y, and R2 is defined on the Cartesian space Y cross Z. So now the fuzzy max-min composition of R1 and R2 is a fuzzy set defined on the Cartesian space.

That is the max-min composition of R1 and R2, which is a fuzzy set defined on the Cartesian space X cross Z as R3 equal to R1 composition R2, which is given by mu r3 of x comma z upon x comma z. Okay, so here mu r3 of x comma z is given by a main expression: maximum over y of minimum of mu r1 of x comma y comma mu r2 of y comma z, such that we can say x belongs to capital X, y belongs to capital Y, and z belongs to capital Z. Okay, so this is the formal definition and expression for the fuzzy max-min composition operation. Okay, so that is given by this expression: R3 is obtained as a composition of R1 and R2, which is given by this expression where the numerator mu r3 of x comma z is given by the max of the minimum of these membership functions corresponding to the two spaces. Now, let us see an example to understand this problem: an example to understand this max-min fuzzy composition operation. Say we have R1 equal to between the X and Y sets of the Cartesian space.

So, R1 is defined on the Cartesian space having X and Y as universal sets. They are having the relation matrix given by 0.1, 0.2, 0.4, 0.5, and 0.7, 0.8, let us say. And let us say R2 is obtained by or is defined on the Cartesian space of Y and Z sets. So, Y is in the rho and z for the column of this matrix.

So, they have the elements 0.9, 0.8, 0.7, 0.6. So, then we have R3 obtained with the dimension R3 is meant for x and z. So, 3 cross 2 this is for x and column for z, x cross z. So, let us say that r3 of 1 comma 1, what is that? It is obtained by the max of the minimum of 0.1 comma 0.9, 0.9. This row you take and then this column, so 0.1 and 0.9, the minimum of that. Similarly, the minimum of 0.2 and 0.7, okay, this and this, and 0.2 and 0.7, take the maximum of that. So, the maximum of 0.1 comma 0.2, what is that? It is 0.2. Similarly, we can do for other elements or 1 comma 2 So, in that way, if you do, we will be ending up with this matrix R2, R3 to be 0.2, 0.2, 0.5, and 0.5, and 0.7 and 0.7 by the fuzzy max-min composition operation.

Now, let us talk about the fuzzy max product composition operation. Okay. Let us see that now. Fuzzy max product composition. Composition.

Operation. With the same example, we can take. But before that, let us see. The expression. Or it is defined.

The fuzzy max product composition of R1 and R2. Is a fuzzy set. Defined by $R_3 = R_1 * R_2 = \left\{ \frac{\mu_{R_3}(x,z)}{(x,z)} \right\}$

Where $\mu_{R_3}(x,z) = \max_{y} \left\{ \mu_{R_1}(x,y) * \mu_{R_2}(y,z) \right\}$ that is all.

Okay, this is for the max product composition operation. So now we will see the example given by the previous R1 and R2. Okay, so now again we take an example where R1 is again given by the matrix for X and Y that 0.1, 0.2, 0.4, 0.5, 0.7, 0.8, and R2 is given by Y cross Z. So, it is given by 0.9, 0.8, 0.7, 0.6. So, by fuzzy max product composition operation, we can get R3 coming out to be a 3 cross 2 matrix, 3 rows and 2 columns.

Okay, so now what is R3 of 1, 1? It is given by max over Y of 0.1 into 0.9 comma 0.2 into 0.7. So the maximum of 0.09 comma 0.14, which is 0.14. So this guy is 0.14. The element R3 of 1 comma 1 is 0.14. Likewise, we can get the other elements as 0.12 0.36, 0.32, 0.63, 0.56 for R3 using fuzzy max product composition. Okay.

So now you can cross-check these answers. Okay. Now let us see which involves some deeper understanding that we have two relational matrices. We get the third relation matrix obtained from the practical situations. Okay.

So let us consider this example one. Okay, so let us see this example. Let X be a universal set of well-known objects given by X equal to car, boat, church, bike, tree, and a plateau. Okay, there are six elements in this universal set.

Car, boat, church, bike, tree, and plateau. Again, let Y be a universe of simple geometric shapes that is given by Y equal to square, octagon, triangle, circle, and ellipse. Okay, there are two sets defined now, which are called universal sets. Okay, so now let us define a simple fuzzy set, okay, of objects such as car, square, and edges. Each one is a fuzzy set: a simple set car, another simple set Z square, and the third simple set is edges. Okay, so that is represented as, say, A is a fuzzy set which is meant for car, which is given by membership function value upon element 1.0 upon car

Because the membership function value of car for the fuzzy set car is 1. And 0.4 upon boat. Similarly, 0.1 upon church. And 0.6 upon bike. And 0.1 upon tree.

And 0 upon plateau. Okay, likewise we have the fuzzy set called square, which has the value 1 for square, 0.5 for octagon, 0.4 for triangle, and 0 for circle. And 0.1 for ellipse. So, these are the two sets defined now. Now, let us see the third set as c equal to edge, which is given by 0.6 upon square, 0.9 upon octagon, and 0.4 upon triangle.

And 0 upon. Circle. And 0.2 upon. Ellipse. Okay.

So these are the. Three sets we have. Fuzzy sets we have defined now. Now. Let us have the question.

Q1. Find. A relation. Or. That is between car and square.

Likewise, question 2. Find a relation. Let us have this as yes. Between square and edge. Between the first and second fuzzy set and between the second and third fuzzy set is relation matrix S. And then question 3 is using max-min composition operation to find a relation T between

car and edge. So now T has to be found between car and edge. That's the question. Now, let us see how it is getting done. Okay, so now let's move on to the solution. The solution is the matrix The relation matrix R is the fuzzy set A and B. Okay. So that is R is the relation matrix that is between the fuzzy sets A and B. So now let us see how R can be obtained. R equal to which is given by A matrix that is meant for quasi sets A and B relation.

So, A has what are the names A has? What are the elements? A has car, boat, church, and bike. Tree and plateau, so we can write them in the column: car, boat, church, bike, tree, and plateau. Okay, so this is the thing for set A, and this is for set B, so that is square, octagon, triangle, and Circle with ellipse in the end.

Now, let us form this relation matrix R that is between sets A and B. Now, let us see that the fuzzy set A, what value it was having. Accordingly, we can see which one is the minimum obtained here. Okay, for 1 and 1 here, which is a minimum. So, this is 1. Likewise, the minimum entries between the fuzzy set A for each of the fuzzy sets of set B or the elements or the variables of the fuzzy set B are compared.

Which one is a minimum? We can enter here. The car for the shapes. Car for square is getting 1, and octagon, triangle are getting as per the minimum value. We are getting the first row filled as 1, 0.5, 0.4, 0, and 0.1. And likewise, for boat, for all the shapes, five shapes, we are getting the value, which is nothing but the minimum of these two values.

It is giving us 0.4, 0.4, and 0.4 with 0 and 0.1. Likewise, for church member, we are getting 0.1, 0.1, 0.1, 0, and 0.1. Likewise, for bike, we are getting 0.6 and 0.6. 0.5, 0.4, 0, and 0.1 for the corresponding shapes. Finally, for tree, we are getting 0.1, 0.1, 0, and 0.1.

Likewise, for the plateau, it does not come under any shape. So, we can say 0, 0, 0, 0, and 0 is the minimum value compared to that. So, likewise, we can proceed with computing the matrix S. That is the relation between the fuzzy sets relation matrix between the fuzzy sets B and C. Okay. So, in the same manner, the S matrix is going to be obtained as this one, and we have

The columns represent these shapes: square, octagon, triangle, circle, and ellipse. Likewise, the edge which is for the C subset or C fuzzy set is having the edge which is also based on the shapes: square, octagon, triangle, circle, and ellipse. Now, let us find the relationship between these two fuzzy sets and name it as relation matrix S. So, when we compare that and take the minimum, we get 0.6, 0.9, 0.4, 0, and 0.2 as the first row. And 0.5, 0.5, 0.4, 0, and 0.2 for the second row, which means we are comparing and taking the minimum and placing it here. So, likewise, the third row of this S matrix is coming to be 0.4, 0.4, 0.4, 0, and 0.2.

Likewise, C is having a circle. is having all the elements coming out to be 0, and finally, the ellipse value is getting 0.1, 0.1, 0.1, 0, and 0.1. So now we finish with the examples to get the relation matrices R and S, but we need to get the relation matrix T that is obtained by the fuzzy max-min composition rule between the matrices. R and C. Let us see that now. So, T is obtained by fuzzy composition between the relation matrices R and C. R and S. Okay. So, it is basically to get the relation between fuzzy sets A and C, basically.

And that one, we need to use fuzzy max-min composition. We can use it. We need to use fuzzy max-min composition to obtain this table called the relation matrix between R and S. So, let us have this table form now. R and C. Okay. So, it is basically R and S. Okay. So, R forms the rows, and C forms the columns.

So, car, boat, And church, then bike, tree, plateau. And S has square, octagon, triangle, circle, and ellipse as the members. So, we represent here. Okay.

Now, let us see T11. This one we need to find now. Now, let us see. The T matrix of 1 comma 1 is obtained by the max of the minimum of 1 comma 0.6 and the minimum of 0.5 comma 0.5. I am comparing these values, where I am taking these values from the R and S matrices.

Taking the first row of R and comparing that with the first column of the S matrix. So, I am getting these entries, which is nothing but the maximum of the minimum of 1 comma 0.6 and 0.5, the minimum of 0.5 and 0.5, then the minimum of 0.4 and 0.4, then the minimum of 0 comma 0, and the minimum of 0.1 and 0.1. Okay. So that leads to the maximum of 0.6, 0.5, 0.4, 0, and 0.1.

Which is nothing but the maximum 0.6. So the first entry of this T matrix is going to be 0.6. Likewise, we can fill this table of the T matrix, which is going to be 0.6, 0.9, 0.4, 0, and 0.2. And this first row is obtained by taking the first

A row of the R matrix is compared with each column of the S matrix. Similarly, taking the second row of the R matrix and comparing it with all five columns of the S matrix, we get the second row of the T matrix: 0.4, 0.4, 0.4, 0, and 0.2. Then, the third row comes out to be 0.1, 0.1, 0.1, 0, and 0.1. And the fourth row is 0.6, 0.6, 0.4, 0, and 0.2. Finally, the fifth row is 0.1, 0.1, 0.1, 0, and 0.1, and the last row for the plateau is 0, 0, 0, fully filled with 0 elements. Okay, so like this, we have completed the

relation matrix obtained through fuzzy max-min operation composition. Okay, composition operation. Alright, now let us talk about something called fuzzy linguistic variables. That is very important for constructing the fuzzy rules. Let us see that now.

Okay, so now let us talk about fuzzy linguistic variables. Fuzzy linguistic variables. We know that Algebraic variables Take

Numbers as values. Whereas fuzzy. Fuzzy. Linguistic. Variables.

They take. Words or sentences. As values. Okay. So now.

Let X be a linguistic. Variable. With label. Height. X is a linguistic variable with label height, which means X is a linguistic variable that mentions or represents height as a label.

The fuzzy set height, say, denoted by capital H can be written as H equal to very short, short, medium, tall, very tall. Okay, so this is the fuzzy set called height. represented by H with its fuzzy variables, precisely fuzzy linguistic variables: very short, short, medium height, tall, and very tall. Here, height is the base variable, which is called the universe of discourse. Universe of discourse, and each term or each item in the fuzzy set—very short, short, medium—is a linguistic value of the fuzzy variable X. That means a fuzzy variable is X; it's a linguistic variable which has a value that is very short, short, medium, these things.

So, with this, let us have a discussion about further things on fuzzy rule base and fuzzy implications relations. So, now we will see fuzzy rule base. Having defined the fuzzy linguistic variables, we see fuzzy rule base now. So, Fuzzy rule base, it is also called if-then rule base. Okay, so fuzzy rule base is one way to represent knowledge through natural language.

The generic form of a rule base is of a rule base is: if premise, then conclusion. If you have an introduction, then conclude it. Fuzzy information can be represented in the form of a rule base, which converts a set of rules, which converts or, sorry, which consists of a rule base which is having a set of rules in the conventional premise-conclusion form. Okay.

That means introduction conclusion form. Rule base is nothing but a set of rules, and each rule is having an if-then rule base, which is in the form of conventional premise and conclusion form. Let us talk about a rule. Say rule 1. It is given by if x is A,

Then, Y is B, where A and B are representing fuzzy propositions or fuzzy sets. Okay, or fuzzy sets. Okay, propositions. Fuzzy propositions or fuzzy sets. Precisely, they are nothing but the membership function values pertaining to that rule.

We are going to see that then in the upcoming time. So this is rule 1. So we can say that we now introduce a New introduction or premise. Say A1, and we consider the following rule.

Say rule 2. If x is a1, then y is a2. b1. Now the question I have here is we have two rules, and the first rule is if x is a, then y is b. Likewise, rule 2 is if x is a1, then y is b1. So from the information of rule 1, can we derive the

Consequent, say B1 of rule 2. That's the question. The answer is yes, by composition operation. Okay, that is obtained by composition operation. How it is, say we need b1, which is nothing but a1 composition the relation matrix r. Okay, so I can write a1 here, a1 composition r. Okay, so it is not the superscript; it is the subscript. Okay, so we have coming out this That from the rules, we can get the consequent of the next two rules by composition, fuzzy composition operations using the relation matrices.

Now, let us talk about fuzzy implication relations. So, a fuzzy implication relation for a given rule, if X is AI, then Y is BI, is formally represented by RI of X, Y equal to mu R i of x comma y upon x comma y. Okay, mu R i, you know that it is a membership function value.

So, the fuzzy implication relation is, let us consider a rule which is: if x is A_i, then y is B_i, formally represented by $R_i(x, y) = \left\{ \frac{\mu_{R_i}(x, y)}{(x, y)} \right\}$ So one of the implication Rules. Okay.

$\mu_{R_i}(x, y) \Rightarrow$ implication rules

if $p \rightarrow q$

There are several rules. I am considering here in this lecture. Only one rule. Implication rule, which is. Mamdani. Implication. Rule. So that states that. When. Fuzzy if. Then. Rules are.

True, then using Mamdani implication rule, we can say that P tends to Q implies P and Q is true, that is, taking the AND operations between the membership functions of that rule and. So, by having the membership function values and taking the AND operation of that rule, we can get the fuzzy Mamdani implication rule. We can understand the rule implication, fuzzy implication relations by Mamdani implication rule in such a way that when a fuzzy if-then rule is locally true, then the. It implies that P tends to Q, which means that P and Q are AND operated. Okay.

That means we are going to choose a minimum value of the membership function corresponding to P and the membership function corresponding to Q. Okay. Now, let us talk about fuzzy compositional rules. Okay. Let us talk about fuzzy compositional rules. Rules, so there are different rules, okay? So, to get a matrix B by the composition operation between A and matrix R, okay? So, let us see the max-min fuzzy compositional rule, which is given by the max-min fuzzy composition rule, is given by μB of y.

For this relation, B equals A composition R, for that the fuzzy max-min composition rule is given by μB of y equal to maximum over x belonging to capital X of minimum of μA of x, okay? So, μA of x, μR of x, y. Okay, this is it. Similarly, max-product instead of in max-product, the minimum inside the bracket will be replaced by the product. So, the max-min fuzzy compositional rule is given by μB of y equal to maximum over x belonging to capital X. μA of x multiplied by μR of x, y, okay? Likewise, we can go for max-max fuzzy compositional operation and min-min fuzzy compositional operations also. Let us take an example here that if. X is A, then Y is B, where A is given by, as I said, is given by 0.2 upon 1, 0.5 upon 2, 0.7 upon 3. And let us say B is given by 0.6 upon 5.

0.8 upon 7, 0.4 upon 9. Now, through fuzzy implication, how can we infer B1 for the following rule? So, now we have the following rule. If X is A1, then Y is B1, okay? So, now the question is, how can we get B1 from the rules we have? Okay, definitely, we have to do the composition operation here. A1 for this rule equals 0.5 upon 1, 0.9 upon 2, 0.3 upon 3.

Okay. So, we need to have the relation matrix first of all. Okay. So, that relation matrix is between A and B. The first rule is having the statement: if X is A, then Y is B, where A and B are given. And we have the second rule: if X is A1, then Y is B1.

Now, we need to find or infer B1 from these rules through fuzzy composition operation using Maxmin. Let us say we have A12. Now, the solution is using the Mamdani implication. Relation we have the r matrix coming out to be x. This is for y. The x corresponds to the elements 1, 2, 3, and the y corresponds to elements 5, 7, and 9, as per rule 1. So, we take the minimum of these two elements corresponding to these two fuzzy sets.

So, 0.2, 0.2, 0.2, and 0.5, 0.5, 0.4, and 0.6, 0.7, 0.4. So, here you must know that the elements mu r i of x comma y are obtained by the minimum of mu a i of x comma mu b i of y. That is how we are getting these elements 1, 2, 3, 4, 5, 6, 7, 8, 9. Now, let us see using the max-min compositional relation or operation. We can have b1 equal to a1 composition relation matrix r. That is nothing but the maximum of the minimum of mu a1 of x comma mu r of x comma y. So, b1 is going to have b1 of 1. The first element we are going to find is the maximum of the minimum of 0.5 comma 0.2, the minimum of 0.5 comma 0.5, and the third one, the minimum of 0.5 comma 0.6, which is nothing but the maximum of 0.2, 0.5, and 0.5.

It is 0.5, that is b of 1 is 0.5. Likewise, we can find b of 2, which is going to be by this expression, that is b 1 of 1, b 1 of 2 is nothing but the maximum of the minimum of 0.5, 0.2, and then the minimum of 0.5, 0.5, and then the minimum of 0.5, 0.7. Which turns out to be the maximum of 0.2, 0.5, and 0.5. That is finally 0.5. Likewise, we can do for B1 of 3, that is coming out to be 0.4.

So finally, we can say that B1 is obtained as 0.5 upon 5, 0.5 upon 7, and 0.4 upon 9. That is all. Because B1 is A1 related to R1. Composition relation between A1 and R1. That is coming out to be 0.5 upon 5, 0.5 upon 7, and 0.4 upon 9.

Now let us see multiple rules. The final topic we have come here. We will see in the new slide. Multiple rules with continuous fuzzy sets.

Okay. Now, let us take two rules. Take the following rules. Rule 1, which is given by rule 1, such that if x1 is a11 and x2 is a21. a21, then y is b1. Likewise, rule 2: if x1 is a12 and x2 is a22, then y is b2. You must be noting that in both the rules, I kept a uniformity that

the superscript is for the corresponding rule. So, you also maintain the same thing to avoid confusion in the numerical problems. So here, a11 and a21 are the values of the fuzzy input variables X1 and X2 of rule 1. Rule 1. And B1, B2 are the fuzzy output values of rules 1 and 2, respectively. Okay.

So, this is a continuous fuzzy set. So, this is a continuous fuzzy system with two non-interactive inputs, say x1 and x2, and a single output, which is y. So now, this is the definition about the system with these two rules. Now, finally, let us come to an important part here, which is drawing the membership functions.

Let us see the pictorial representation of rules 1 and 2. This is a very important concept here. This is the membership function for rule 1. Here, we are doing so, say this is the crisp value, crisp input x1, okay? So, which has the membership function value crisp, that is a11. Then, for the second input x2, we have again a triangular membership. For this corresponding crisp value, we have here that is a21, sorry, a21. This is the crisp input X2.

For these corresponding crisp inputs, we get the membership function value as A11 and A21 for inputs X1 and X2 in rule 1, respectively. That leads to an output mu, which is the membership function. Okay, so here comes the a11 and a21. The minimum of this comes here in your output, that is b1, this portion, okay? So, this is now the first rule. Output b1, likewise for rule 2, we have the membership function that is a. This is the crisp input that leads to a12. Is the output the membership function value corresponding to that a x1 input? Now, we have for input x2, for that corresponding crisp input of x2, we have the output that is coming out to be a22, okay?

So, now this leads to the output membership function for Y. So, the minimum of this goes here. So, the minimum cuts here. So, this portion gives you the B2, that is the output 2 for rule 2. Now, the resulting Resulting output membership function is given by the maximum of this and this.

You get the output of rule 1. It is the minimum of these two function values of corresponding inputs x1 and x2. Likewise, for rule 2, the membership function value for rule 2. For X1, it is A12, and for X2, it is A22. The minimum of that forms B2.

That is the output for rule 2. Now, the resulting output membership function is given by the maximum of these outputs for each rule. So, the maximum case here is mu on the Y-axis. The X-axis is Y. So, add this. So, from here, you can see the value obtained from this output membership function y star.

So, in general, the output value is obtained, that is, the crisp output is obtained by a method called the center of gravity approach. By that, you can obtain y star, which is coming out to be the integral of y mu of y dy upon the integral of mu of y dy. So, this is called the center of gravity approach. By that, we can get the final crisp output of this problem.

Okay, so now coming to the conclusion, we have seen today fuzzy relations, fuzzy implication relations, fuzzy rule base, and finally multiple rules for a fuzzy relationship. Continuous system. In the next class, we will be seeing fuzzy logic control of robotic systems. Thank you very much.