

INTELLIGENT CONTROL OF ROBOTIC SYSTEMS

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Lecture 06: Introduction to Fuzzy Logic

Good afternoon, today we are going to see an introduction to fuzzy logic. The outline of today's lecture will be as follows. First, we have the introduction to fuzzy logic, and then we are going to see about fuzzy sets, which form the basis of fuzzy logic. Then we will be seeing the building blocks of fuzzy logic, which are membership functions, and then we are going to see crisp relations so that in the forthcoming classes we will be having studies about fuzzy relations and inference from fuzzy relations so that we can form fuzzy rules, which are the building blocks of fuzzy logic control. Let's see the introduction part of fuzzy logic. So, the term fuzzy means hazy, which means unclear.

And fuzziness happens if the data or the information is not clearly defined. It is not clearly defined. So, the opposite of fuzzy is you can say crisp, and fuzzy logic is designed from the concept of fuzzy set theory, which is nothing but an extension of classical set theory.

In fuzzy set theory, we have the degree of membership. In fuzzy set theory, we have something called the degree of membership or something called the membership grade. Sometimes it is called the membership function value for each element that is present in a fuzzy set. So, the value of a membership grade is

Will be lying in the interval of 0 to 1. So accordingly, a membership function is related to a fuzzy set, is related to a fuzzy set, say for example A. In such a way that each element x of A is assigned to a membership value of 0 to 1 interval. So it is characterized, the membership function is characterized by an expression. I can say the membership function grade or membership function value is characterized by an expression, which indicates that X belongs to fuzzy set A, X is a member of fuzzy set A.

And the membership value of X is given by $\mu_A: x \rightarrow [0, 1], x \in X$

So, when I say X is the universal set, what is A? A is nothing but a subset of X, where X basically represents an object or an attribute. Thus, a fuzzy set can be expressed or defined as, can $A = \{(x, \mu_A(x)) : x \in X\}$, be defined as a set of ordered pairs, can be defined as a set of

ordered pairs, So X here is the universal set or universe of discourse. And μ_A of X is nothing but the grade of the membership function.

The grade of membership of object X present in fuzzy set A, such that μ_A of X will lie in the interval between 0 to 1. Okay. Now, let me also express or give the definition of a fuzzy subset. Fuzzy subset. Let us say fuzzy subset F. A fuzzy subset F of a fuzzy set A can be defined as a set of ordered pairs $F = \{(x, \mu_F(x)) : x \in A\}$ where the first element is an element of the set A, and the second element is a value in the interval 0 to 1. Now, let us see an example. To understand more about the fuzzy set. Now consider. A universal set. Let us consider.

A universal. Set. Say H, representing the. Heights. Of.

Say. Humanoid. Robots. Okay, let short, medium, and large, or let us say not large, it is tall. Okay, so let us take short, medium, and tall. Be the subsets of the universal set H. Okay, so in that case, these subsets can be defined by the user or the designer in the classical approach. Note down, it is in the classical approach. These subsets, short, medium, and tall, can be defined as, say, Okay, that is $\text{Short} = \{\text{height} \in H : 1 \text{ ft} \leq \text{height} \leq 3 \text{ ft}\}$ defining short, which is a subset.

The medium subset is defined as $\text{Medium} = \{\text{height} \in H : 3 \text{ ft} \leq \text{height} \leq 5 \text{ ft}\}$

Likewise, tall is equal to $\text{Tall} = \{\text{height} \in H : 5 \text{ ft} \leq \text{height} \leq 7 \text{ ft}\}$

So, here it must be noted that 2.9 feet, here it can be noted that you can see the range for short is 1 foot to 3 feet. For medium height, it is 3 feet to 5 feet.

For taller height, it is 5 feet to 7 feet. In that situation, we can observe that 2.9 feet belongs to the short subset. Whereas 3.1 feet belongs to the medium subset, which implies that the boundaries of the classical set are rigid. They are strict.

Okay. That is what our observation is. Okay. In contrast, what we are going to see is, in contrast, the fuzzy set boundaries, the boundaries pertaining to the fuzzy sets, are soft.

They are not so strict. They are not so rigid. Okay, so one approach to define the fuzzy subsets for the universal set H is shown below. That means the So, one approach here is shown below through the schematic of fuzzy membership.

Height is basically the member which has the elements that are nothing but short, medium, and tall. So, height is a fuzzy variable here which is going to be short, medium, and tall.

So, the height is defined here in this case in feet, whereas this is the membership function of the universal set H, okay. So, the x-axis is the fuzzy variable, not the fuzzy variable, it is the variable height, and The y-axis is the membership function.

So, we can see that it is basically a fuzzy variable, only height. So, we can see that this schematic we can see here. Okay. So, This is value 1.

This is 0.5 membership function values. And the first triangular membership function is short. And the second membership function is medium. And the third triangular membership function is tall. So we can observe one thing here.

Say the height of 3 feet is a member of it. It is observed here from this membership function schematic of the fuzzy variable feet, which involves short, medium, and tall subsets. We can say that the height of 3 feet is a member of both the fuzzy sets, which are nothing but short and medium. You can say that short and medium involve the value of 3 feet, okay. That corresponds to the 3 feet. One corresponds to the membership function value, which is μ_h of medium is equal to μ_h of short, which is equal to, from the schematic, what is the value? You are saying it is 5? Sorry, it is 0.5, which is the membership function value. You must note one thing: the membership value is going to have an interval of 0 to 1. The maximum value is 1, which means that the the significance of that value, the relation of that element in that particular subset or set of a Z is prominent. It is more.

If it is 0, then it is not a member at all of that set. Okay. So, we will continue this topic. Now, let X_1, X_2, \dots, X_n be the elements of set X. Then the fuzzy set A, which is a strict subset of the universal set X, can be represented by one of the

Following are the standard forms, standard forms which is $A = \left\{ \frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, \frac{\mu_A(x_3)}{x_3}, \dots, \frac{\mu_A(x_n)}{x_n} \right\}$

Where it is to be noted that $\frac{\mu_A(x_i)}{x_i} \neq \text{division}$ is not a divisor; it is not something having a division operation. Rather, it is a tuple, okay? It is a tuple; it is not a division. So, when I say one of the standard forms, it means it has a couple of other standard forms also,

Another way of representing is $A = \left\{ (x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)), \dots \right\}$

In this way also, we can have the representation of fuzzy set A. So now we will see something interesting that is called fuzzy set operations. What are the operations that can be performed in this fuzzy set? This is much similar to the classical sets, which means union, intersection, and complement. So-called fuzzy union, fuzzy intersection, and fuzzy

complement. So we are going to see here fuzzy union Set operations. Fuzzy set operations. What are they?

So, the fuzzy set operations, as I told. There are three. One is the fuzzy complement. The next one is Fuzzy intersection.

And the third one is Fuzzy union. Fuzzy union. Let us talk about, first of all, the fuzzy complement. So, the membership function, that is the fuzzy complement membership function,

$\mu_{\bar{A}}(x)$ of the complement of A, that is denoted by \bar{A} . Okay. So, $\mu_{\bar{A}}(x)$ is nothing but the complement membership function. So, that is defined by $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$ for all, x belongs to the universal set, capital X. The pictorial representation of this fuzzy complement is given by this schematic. Okay, so the x-axis is the variable, fuzzy variable x , and the y-axis is the membership function value.

So, we have the membership function value coming out to be like this, and we can have something like this. There, I can say that. OK. So, when I have this being $\mu_A(x)$ then I have the complement, which is given by this.

That is called $\mu_{\bar{A}}(x)$

which is nothing but $1 - \mu_A(x)$. So, this is the pictorial representation of the fuzzy complement. Next, we have the fuzzy intersection. Next, we have the fuzzy intersection. Fuzzy intersection operation.

The intersection of two fuzzy sets A and fuzzy set B is defined by $A \cap B = \{x \mid x \in A \wedge x \in B\}$. Which implies that it is an intersection; it is a common portion of that. We can also say further about this fuzzy intersection that there are two significant operators, two significant fuzzy intersection operators, which are minimum operator. Now, another one is the product operator.

Let us see first about the minimum operator: So, $\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x)$ the 'and' operator indicates the minimum of the membership function values in fuzzy.

Minimum of $\mu_A(x), \mu_B(x)$ for all X belongs to capital X.

Likewise, let's see the second important operator under fuzzy intersection, which is the product operator, that is given by $\mu_{A \cap B}(x) = \mu_A(x) \mu_B(x), \forall x \in X$

The schematic representation is given by 1.0, this one. So, So, this is one, and another membership function is this. Okay. So, the output of this intersection operator is this one.

That means this one. Okay, so that is given by $\mu_A(x) \cap \mu_B(x)$. Okay, so one here is $\mu_A(x)$, another one is $\mu_B(x)$. Okay, so accordingly, we get this as the output, which is again a membership function that corresponds to one value, and this is the variable, fuzzy variable. Okay, so what we have observed here, this, this, this, this is the output of the intersection. That is what I have drawn separately here. Okay, now we will see about the fuzzy union one, which is the last of the operations, fuzzy operation.

Okay, now let us see that fuzzy union. So, the union of two fuzzy subsets A and B or fuzzy sets A and B is defined by $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ for all x belonging to the universal set X. So, this is an OR operator. So, we are going to deal with something called the maximum. For the AND operator in the previous case, we were talking about the minimum operator.

So, here we are going to deal with maximization. So, using the maximum of that, we find the maximum among the membership function values. So, two significant operators are there under fuzzy union. The first one is the maximum operator, which is given by

$$\begin{aligned} \mu_{A \cup B}(x) &= \mu_A(x) \vee \mu_B(x) \\ &= \max\{\mu_A(x), \mu_B(x)\}, \quad \forall x \in X. \end{aligned}$$

Another important operator under fuzzy union is the sum operator. The sum operator, which is given by

$$\begin{aligned} \mu_{A \cup B}(x) &= \mu_A(x) + \mu_B(x) \\ &\quad - \mu_A(x)\mu_B(x) \\ &\quad \forall x \in X. \end{aligned}$$

Okay, so now this schematic representation is given for the fuzzy union, which is given by yes, this is membership. Okay, so let us consider 1.0 the maximum value and let us have two functions like this.

So, one function is represented by this symbol. So, one function is this, and another function is this. So, one function is μ_A of x, and another one is μ_B of x. So, $\mu_A \cup \mu_B$ of x is given by That is all. That is membership.

Membership that is, say 1.0 here, and this is your variable. So, as per this, we are going to also this one; it comes, it is reaching the Okay. So, that is what. So, this is union A union B. Okay.

So, this is the resultant of that. Now, let us move on to another important topic coming under fuzzy logic, which is nothing but fuzzy membership functions. Okay. So, typical fuzzy membership functions we are going to see here. For Z membership functions.

Let us see the first one as the L function that is defined by L of variable X with the parameters alpha and beta is equal to 1 when X is less than alpha. You will understand these expressions which we are going to get the values of the L function by a schematic representation quickly. So, the schematic representation is given here, x, this is your membership, say it is 1.0, okay. That is all. This is the representation of this L function such that this is your alpha, this is beta, this is beta.

So, when I have alpha and beta in the L function, I can define the value of the L function by $L(x)$, which is the input parameter with the function parameters alpha and beta, becomes equal to 1 if alpha or x is less than alpha. When x value is less than alpha, it is 1. It is going to be alpha. This range I am seeing, this range I am seeing, alpha minus x upon beta minus alpha, when alpha is less than or equal to x, and x is less than or equal to beta. And it is 0 when x is greater than beta. This is it. Now, let us talk about another function, which is a very important function called the S function.

The S function is given by $S(x)$ with the parameters alpha, beta, gamma, equal to 0 when X is less than or equal to alpha. Let us see the schematic here, the membership function. So, here it is gamma, the middle one is beta, the starting one is alpha. So, this maximum value corresponds to 1.0 membership grade.

So, this is the S function such that the S function gets the value 0 for X value less than or equal to alpha, and it is equal to 2 times of x minus alpha upon gamma minus alpha whole square for alpha less than x less than or equal to beta, and it is equal to 1 minus 2 times x minus gamma divided by gamma minus alpha whole square where x is in the range of beta and gamma, and is equal to 1 if X is greater than gamma in this case. So, this is the S function. Likewise, we have the gamma function, which is given by the next function, the gamma function.

Function the gamma function, that is gamma of x with the parameters alpha comma beta, is equal to 0 when x is less than or equal to alpha. Before that, we will see the function takes this profile. This is membership 1.0, okay. So, this is beta, and this is alpha. So, in this case, we can say that it takes, the function takes 0 value for x less than alpha, and the function takes the value which is x minus alpha upon beta minus alpha for alpha less than x less than or equal to beta, in this case. And it takes the value 1 for x greater than beta.

And two more importantly and commonly used functions are triangular and Gaussian functions, which we are going to see now.

So, triangular functions, triangular function, let us see triangular membership function, triangular membership function. Which is given by gamma of x such that with the parameters gamma of x variable with the parameters alpha, beta, and gamma equal to 0 for x less than alpha, where the profile is 0. Membership, this is alpha, the middle one is beta, and the last one is gamma. So, this is a triangular membership function. So, this takes a value 0 for x less than or equal to alpha.

So, for x in the range from this to this, it is given by $\frac{(x-\alpha)}{(\beta-\alpha)}$, and for this range, the function value is $\frac{(\beta-x)}{(\beta-\alpha)}$ in this range, and in the another range where $x > \beta$ That's all.

Triangular function membership function is finished now. Then we have Gaussian membership function, okay. So, Gaussian membership function, that is given by

$\mu(x; m, \sigma) = e^{-\frac{(x-m)^2}{2\sigma^2}}$ where the parameters sigma and m control the width and center of this Gaussian membership function, which is given by the schematic.

So, this is it. Now, let us talk about the crisp relation, which is the final topic of today's lecture. So, like that, we are going to see tomorrow fuzzy relation, fuzzy rules, and fuzzy inference.

And fuzzy compositions. Okay. So, coming to the crisp relation, let the Cartesian product of two universal sets be determined as $X \times Y = \{(x, y) | x \in X, y \in Y\}$

The crisp relation $\mu_R(x, y)$ is defined as $\mu_R(x, y) = \begin{cases} 1, & \text{when } (x, y) \in X \times Y \\ 0, & \text{when } (x, y) \notin X \times Y \end{cases}$ where

The value 1 indicates a complete relation, and the value 0 indicates no relation at all. So, when the sets are finite, it must be noted that when the sets are finite, the relation is represented by a relation matrix R, capital R.

Okay, now let us see an example to understand this crisp relation. So, take an example where we consider two sets X and Y. Okay, such that X equals 1, 2, 3, a set containing elements 1, 2, 3, and Y is a set containing A, B, and C. Now, for the above sets, two relation matrices R1, being a relation matrix, and R2, being another relation matrix, relation matrix.

Two relation matrices, R_1 and R_2 , can be defined as R_1 equal to X and Y . So, 1, 2, 3 first X related to YZ . So, the first one represents the row of the relation matrix. The second one represents the column of the relation matrix. Let us see that all the elements of this matrix being 1, 1, 1.

And let us have another relation matrix, let us say 1, 2, 3, A, B, C between these two universal sets. Let them be like this, which is 0, 1, 1, And 1, 0, 1, 0, 0, 0. Let us say like that. Okay, so here, which states that, so it states that R_1 , in R_1 , the relation matrix R_1 implies that each element in X is completely related to each element in the universal set Y .

Each element in set X is completely related to each element in universal set Y . Whereas, the relation matrix R_2 indicates that only the pairs 1, B, 2, A, 2, C, and 1, C are related. Other elements are not related. Now, let us see composition. Under crisp relation, we are going to see something called composition. What is that composition?

Let us see that now. Composition. Let us consider three sets, say X , capital X , capital Y , and capital Z . First, let R be a relation matrix that relates elements from set X to set Y . Similarly, let S be a relation matrix that relates elements from set Y to set Z . Now, let us have the relation which is let T be the relation that relates the same elements in X

that R contains. To the same elements in Z that S contains. Again, I repeat.

T is a matrix or relation matrix that relates the same elements in X that R matrix has to the same elements in Z set that the matrix S contains.

So. Given. R and S . T is determined. By the principle of. Composition.

By the principle of composition. We can obtain the. Final relation matrices. For a given. Relation matrices R_1 and R_2 .

So the composition can be min-max composition and max-product composition. It must be noted that For both crisp relations, both compositions, min-max composition and max-product composition, result in the same value. We are going to see that quickly. So if it is min-max composition then

If it is min-max composition, then μ_t of x, z is given by the maximum for y belonging to capital Y of the minimum of μ_r of x, y , μ_s of y, z . Likewise, if it is max-product composition, composition, then μ_t of x, z is given by the maximum for y belonging to

capital Y of μ_r of x, y multiplied by μ_s of y, z. Okay, so now, in the end, we are going to see a numerical problem to be solved. Let us take this example. Given the relation matrix R, say So, given the relation matrix R with x_1, x_2, x_3 and y_1, y_2, y_3, y_4 , that means it relates the elements of two universal sets X and Y, where the matrix elements are 1, 0, 1, 0, 0, 0, 0, 1, and 0, 0, 0, 0.

Likewise, we have another relation matrix, which is That is between Y and Z. Set Z has 2 elements, z_1, z_2 . X has 3 elements, x_1, x_2, x_3 . And Y has 4 elements, y_1, y_2, y_3, y_4 . R relates X and Y. That is why X is in the row.

And S relates Y and Z. That is why Y is in the row now. Now, to find T, which is a composition of R and S. So, T is found out to be in this way: T is R composition S. So, we can say that it relates X_1, X_2, X_3 and Z_1, Z_2 , it relates X and Z. So, we can go by max-min composition. So, in max-min composition, if you take it, you will have the first one coming out to be 0 here and 1 here, 0, 0, 0, 1. Due to time brevity, I am just giving the final value, and I ask you to solve this problem in the next.

Two methods, min-max and max-product, and any one you can take, say min-max. So, if you say by that, you will be having this one that is coming out to be max-min, sorry not min-max, it is by max-min you can go, so you can be getting. Maximum over y belonging to capital Y for the first element, I am saying minimum of μ_r of x comma y and μ_s of x y comma z, okay? So, in this way, you need to do so that gives for the okay, so we will be testing it for. Okay, so let me fill this element of 0 1 0 0. 0 1 0 0. So, in this case, R and S completely we have that is the input.

We need to find the relation matrix that is given by the composition of R and S. So, T is going to be the composition R and S that will be involving either max-min composition or max-product composition. So, let us say max-min composition of the first element. So, we go this and we go this. So, 1 0 1. Minimum of that and 0 0, minimum of that 1 0, minimum 0 0, minimum.

So, it is maximum of 0 0 0 0 like that. So, what is the first element maximum of 0 is 0, that is why. Similarly, we can go this one and go to this. So, 1 1 minimum of that 0 0 and 1 1 0 0 minimum, minimum find out. So, the minimum of these corresponding elements of the first row of R and the second column of S that gives max of 4 elements.

So, the maximum of 4 elements is going to be 1 here. So, like that, you can take the second row. And the first column of S to fill this element. Likewise, the second row of R and the

second column of S to get the second row, second column element of the T matrix. Likewise, for the third row of the T matrix, go by the third row of R.

With the first row or first column of S that gives this element, that is 3 comma 1, and for 3 comma 2 of your T matrix, go by the third row of R and the second row, second column of S so that you will be getting 1. One thing to be noted is that using max-min composition or max-product composition, you will get the same result, which is only the situation for CRISP. Relations, okay? So, by this, I conclude here our lecture today. Today, we have seen fuzzy logic introduction, introduction to fuzzy logics, then we have seen fuzzy sets, and then we have seen fuzzy set operations or fuzzy operations, which is nothing but the complement, union, and intersection. For Z operations, and then we have seen the membership functions, the membership functions, membership functions, and finally, we have seen the crisp relations that involve both composition and. We have seen the compositions based on min-max, max-min, and max-product relations. And through an example, we found that max-min as well as max-product compositions give the same result for two relation matrices.

With this, I stop my lecture today. We will see tomorrow fuzzy relations with a fuzzy rule base. Thank you so much for your patience.