

INTELLIGENT CONTROL OF ROBOTIC SYSTEMS

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Lecture 05: Lyapunov Stability

Good afternoon, everyone. Today, we are going to learn about Lyapunov stability analysis of robotic systems. The outline of this lecture is as follows. First, we will cover the Lyapunov stability theory.

In that, we will explore the concept of stability. Then, we will be going to see the indirect method, also known as the first approach of Lyapunov stability. Next, we will explore the second approach of Lyapunov stability analysis, called the direct approach. Then, we will examine examples associated with these two methods. After that, we will move on to Lyapunov function-based control of robotic systems.

In that, we will explore two examples. One is the single-link manipulator. The other is general n-degree-of-freedom robotic systems. Now, coming to the concept of stability analysis. Let us talk about that.

Let us consider a non-linear system that is given by

$$\dot{x} = f(x(t)) \Rightarrow f(x, t), \quad x(t_0) = x_0$$

That means the initial state is represented by x of t naught, which is x naught, where x belongs to the n -dimensional Euclidean space. It is a state vector, and t is greater than or equal to 0.

Let $x = x_e$ be an equilibrium pt.
s.t. $f(x_e) = 0$
 $f(0) = 0$



Let us consider the origin as the equilibrium point of the system.

So, in that case, f of 0 equals 0, and \dot{x} equals 0 at that equilibrium point. So, which implies that the system will continue to be in the equilibrium state $x_e = 0$ for all time. Okay. So, This equilibrium point is said to be. I want to create a new slide here.

Okay. So, let me erase this and write again. Okay. Let me write here. So, this equilibrium point is said to be stable in the sense of Lyapunov.

If you are able to select a bound on the initial condition so that it will result in the state trajectory $x(t)$ that remains within a chosen finite bound. I repeat, the equilibrium point is said to be stable. In the sense of Lyapunov, if you are able to select a bound on the initial condition x_0 , that will result in the trajectory of the system $x(t)$ to be $x = f(x(t))$ lying within a chosen boundary. Okay, so this is the definition or basic concept of stability in the case of Lyapunov stability analysis. Now coming to the point that we can explain this definition more formally that the system described by the equation 1, which is is stable in the sense of Lyapunov.

At $x = x_e \Rightarrow x = 0$ if for every real number that is $\epsilon > 0$

there exists a real number delta greater than 0.

Such that the $\|x(0)\| < \delta$ results in the $\|x(t)\| < \epsilon$ for all $t \geq 0$

That means the initial for any initial state $X(0)$ which is bounded by delta, which is the radius here, such that there exists a real number delta and epsilon such that the bounded $X(0)$ will result in the trajectory $X(t)$ to be within the stated boundary epsilon for all t . That means it will not cross this radius epsilon.

So, that means the system is stable at this equilibrium point in the sense of Lyapunov. Okay, so now let us talk about the asymptotic stability. What can we conclude about asymptotic stability? Here we can say that the system is said to be asymptotically stable at x equal to 0 if the state $x(t)$ approaches 0 as time approaches infinity. Then we can say that the system is asymptotically stable at the equilibrium point x_e , which is 0 or the origin in this case.

If the trajectory system trajectory stays within the boundary, which has the finite boundary limit of epsilon, then we can say the system is stable, but for an initial condition which is bounded. The state trajectory tries to come and settles in the equilibrium point x_e , which is the origin at time as time t approaches infinity, then we can say that the system is asymptotically stable in the sense of Lyapunov. Okay, now let us see. The types of or the approaches that come under Lyapunov stability theory. Lyapunov's first method is also called the indirect method. We are going to see now, but before that, the Lyapunov

first approach as well as the second approach gives an answer to the question of how we can determine the stability of the equilibrium point x_e without explicitly solving the dynamics of a system.

So, let us see about the first or the indirect approach. Now, let us consider a nonlinear system which is given by \dot{x} equal to f of x . Now, expanding in the Taylor series around the origin or the equilibrium point x_e , we get \dot{x} equal to A into x plus g of x , where A is the Jacobian matrix and In the Taylor series, around the origin or the equilibrium point x_e , we get \dot{x} equal to A into x plus g of x , where A is the Jacobian matrix and of the function f of x that is given by A equal to $\text{dof by dou } x \text{ at } x \text{ equal to } x_e$ and g of x contains the higher-order terms that is $\lim_{x \rightarrow 0} \text{mod } x \text{ tends to } 0 \text{ mod } g \text{ of } x \text{ upon mod } x \text{ equal to } 0$.

That is the case. Now, this is the case we started for the first approach method in Lyapunov. We can say that the first approach utilizes the Taylor series method in order to solve the question or in order to analyze the stability of a system. So, when we have linearized the nonlinear system linearized to give \dot{x} equal to $A x$ plus g of x because of the Taylor series expansion, then the nonlinear system

is asymptotically stable if and only if the linear system, that is \dot{x} equal to ax , is stable. That is If all the eigenvalues of matrix A have negative real parts, then we can conclude that the system \dot{x} equal to f of x is asymptotically stable. And the advantages associated with this First approach of Lyapunov stability analysis is It is easy to apply.

This approach is easy to apply. When we say advantages, Definitely we have disadvantages, Also associated with this first approach. So disadvantages,

If in case the eigenvalues, of matrix A , some of them are 0, then a concluding remark on the stability of the system is difficult. is difficult, and also we can say that this approach is valid only if the initial conditions are close to the equilibrium point x_e . Okay, only if the okay. So we can say that if this another point is if the initial conditions are close to the point x_e , equilibrium point, but the question again raises how close is the term close is okay. So these are the difficulties associated with the first approach. Then we will see the next approach which is called Lyapunov's second approach or also it is called the direct approach method or direct method.

Now, let us consider a non-linear system given by $\dot{x} = f(x)$ Suppose that there exists an energy-like function called Lyapunov function, say $V(x)$ with the following properties.

- ① $V(x_e) = 0$
- ② $V(x) > 0$, *positive definite*

Such that $V(x)$ is greater than 0 for x not equal to x_e .

Likewise, we have the third property of this Lyapunov function candidate.

$$\dot{V}(x) < 0 ; \text{ Negative definite}$$

Which means negative definite. Then we can conclude that x_e is asymptotically stable.

Okay, so this is the mere intro about the second approach. Now, having given $V(x)$, then we need to go for finding the \dot{V} of X , the time derivative of the Lyapunov function candidate, which is given by

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} \\ &= \frac{\partial V}{\partial x} \cdot f(x) \\ &= \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + \dots + \frac{\partial V}{\partial x_n} f_n \end{aligned}$$

Now, this method has advantages which are given by it answers the question of stability of non-linear systems without explicitly solving the dynamic equations. Question.

How to solve the Stability of non-linear systems without explicitly solving the dynamic equation of the system, okay. Also, it has another advantage that it can easily handle time-varying systems, that is $\dot{x} = f(x(t))$

and This approach can determine the region of asymptotic stability or the domain of attraction of this system. It can determine the region of asymptotic stability or the domain of attraction of the equilibrium point.

Now, coming to the disadvantages. The main disadvantage of the direct method of Lyapunov's stability theorem is there is no systematic way to find or obtain a Lyapunov function candidate. Another point is Lyapunov's stability criterion provides only sufficient condition for stability Okay So that's the case.

Now. Let's see. An example. That involves. Both these approaches.

Consider. The non. Linear. Given by $\begin{aligned} \dot{x}_1 &= -x_2 + a x_1 x_2^2 \\ \dot{x}_2 &= x_1 - b x_1^2 x_2 \end{aligned}$ with $a \neq b$

Now, our goal for this problem is to investigate the stability of the equilibrium point of the system. In order to investigate the stability of the equilibrium point of the system.

Given a non-linear system, we need to investigate the stability of the equilibrium point of the system. Now, the solution to this problem is at the equilibrium point. We have minus.

$$\begin{aligned} -x_2 + a x_1 x_2^2 &= 0 \rightarrow \textcircled{1} \\ x_1 - b x_1^2 x_2 &= 0 \rightarrow \textcircled{2} \end{aligned}$$

Okay.

Now, multiply equation 1 by x_1 and multiply equation 2 by x_2 and sum them.

That gives $x_1^2 x_2^2 (a-b) = 0$

From which $x_1 = 0$ or $x_2 = 0$

Here, which implies $x_1 = 0$ or $x_2 = 0$ that is the Unique equilibrium point. For the system. Okay. So now let us talk about the Linearized system. At this equilibrium point. So.

On linearizing this System. At this equilibrium $x_1 = 0$ or $x_2 = 0$ point. Will give the linearized system.

The linearized system at $x_1 = x_2 = 0$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\Rightarrow |sI - A| = 0 \Rightarrow s^2 + 1 = 0$$
$$\Rightarrow s = \pm j\omega$$

which means that the characteristic roots are purely imaginary. Purely imaginary. So, we cannot draw any conclusion on the stability of the non-linear system.

Using the first method of Lyapunov approach. So, we need to go for using the second approach here. So, the second approach here is the direct method that must involve using a Lyapunov function candidate.

So, using Lyapunov. The direct method lets the Lyapunov function candidate be chosen as

$$V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2$$

Which is a positive definite function and $V(0)$ equal to 0. So, the time derivative of this Lyapunov function candidate,

$$\dot{V}(x) = x_1(-x_2 + ax_1x_2^2) + x_2(x_1 - bx_1^2x_2)$$

Which implies, upon simplification, \dot{V} equal to $(a-b)x_1^2x_2^2$

Therefore, we can say that if a is less than b , \dot{V} is negative. So, the system is stable. If a is greater than b , \dot{V} becomes greater than 0 for all x that belongs to the n -dimensional Euclidean space, so the system is unstable for a greater than b . Unstable for a less than b , we can also comment on the asymptotic stability of the system with this condition a less than b . Because we can say that \dot{X} , X_1 equal to 0 and X_2 equal to 0 in this case. So, that leads to the equilibrium point being stable. So, we may say about that point, that condition case 1 can also be asymptotically stable. Now, let us see Lyapunov stability analysis of linear systems. Now consider a linear system of the form

Let us choose a Lyapunov function candidate. In the quadratic form. Given by $V(x) = x^T P x$

Where P is a symmetric. Positive definite. Matrix then we have
Let us have it like this.

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P A x \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + P A) x \\ \dot{V}(x) &= -x^T Q x\end{aligned}$$

Where what I did, $A^T P + P A = -Q$. I have represented it as minus Q . Let us see. Here,

$$A^T P + P A = -Q$$

As per the previous equation. So, one thing we need to note is that the system is asymptotically stable if the matrix Q is positive definite. One thing we have to note is that

$A^T P + P A$ is also termed as the Lyapunov matrix equation.

The solution to this equation is the P matrix. If the P matrix, after solving, is positive definite, then we can say that the system is asymptotically stable. So, to check P , if it is positive definite, one can use Sylvester's approach. Okay.

P must be positive definite, which means one of the conditions for P to be positive definite is that all the eigenvalues associated with the P matrix should be positive. Okay. Now, Let us see an example associated with this problem. Okay.

So, using the direct method, that is, the Lyapunov method, determine the stability of the system. Using MATLAB's M-Script, $\dot{x} = Ax$

where

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0.5 & 0 & 2 \\ -3 & -2 & -5 \end{bmatrix}$$

Also, find the corresponding Lyapunov matrix P. So, the solution to this problem is a MATLAB script. We know that Lyap is a function to solve the Lyapunov equation to get P. The Lyap command in MATLAB solves the equation

$$AX + XA^T = -C$$

where the input is A transpose instead of A for this inbuilt Lyap MATLAB command.

So, the code is going to be A matrix which is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0 & -3 \end{bmatrix}$

And let us say that $Q = \text{eye}(3, 3);$
 $P = \text{lyap}(A', Q);$

And then we get the eigenvalues of P. Let us solve. Okay, so we get the eigenvalues of P coming out to be all greater than 0, which implies that the system is asymptotically stable. Okay, so that is all about this method. Now, let us see about the examples containing a single-link manipulator and an n degrees of freedom robotic system. Okay, so now let us see how we can solve the Lyapunov function-based control problem of robotic systems.

Let us see here. Let us have the example of a single-link manipulator, which is a problem of trajectory tracking. Okay. So, now here we have the single-link manipulator's dynamic model coming out to be

$$ml^2\ddot{\theta} + k\dot{\theta} + mgl \cos(\theta) = \tau$$

The τ is the joint torque. M is the mass of the link. L is the length of the link. Okay. And K is the control gain. Okay. So, now let us see what is going to happen to the system. Okay. All right. Let us see that.

Okay, so here we can see that we have to design a controller that will make the trajectory of this single-link manipulator θ to track theta d. Okay, so we need to design a controller via stability Lyapunov stability analysis so that the actual trajectory of the single-link manipulator tracks theta d this side trajectory.

So, we define the error $e = \theta - \theta_d$

So, in that case, $\dot{e} = \dot{\theta} - \dot{\theta}_d$

And $\ddot{e} = \ddot{\theta} - \ddot{\theta}_d$ So, the above equation number 1 can be written in terms of the error as

$m\ddot{e} + \dot{e} + mgl \cos \theta = \tau$ Which is the error dynamic equation of the system. Now, let us choose a Lyapunov function candidate given by $V = \frac{1}{2} m\dot{e}^2 + \frac{1}{2} k_p e^2$

So, if we have this Lyapunov function candidate which is having two terms V equal to 1 by 2 ML square E dot square and half 1 by 2 KP E square then we can have the derivative that is a time derivative of this function Lyapunov function to be coming out as

$$\dot{V} = m\dot{e}\ddot{e} + k_p e \dot{e}$$

$$\tau = m\ddot{e} + \dot{e} + mgl \cos \theta \Rightarrow \dot{V} = \dot{e}(m\ddot{e} + \dot{e} + mgl \cos \theta) + k_p e \dot{e}$$

where That is the controller that we are going to choose in order to make V dot being less than 0 or negative definite. So, let us choose

$$u \Rightarrow mgl \cos \theta - k_p e - k_d \dot{e}$$

as the control law in order to fit into the above equation for

$$\dot{V} = -(k_d + k) \dot{e}^2$$

with the assumption that $(k_d + k) > 0$ then we have $\dot{V} < 0$

Which implies that $\dot{e} \rightarrow 0$ & $\ddot{e} \rightarrow 0$. but it does not imply that e tends to 0.

So, substituting this control into the error dynamic equation, we get $m\ddot{e} + (k + k_d)\dot{e} + k_p e = 0$

This implies that $k_p e = 0$ which means that $e = 0$. Okay. Because this error dynamic equation that we are substituting the control law, e equal to 0. So, error tends to 0. Finally, today we are going to see an n degrees of freedom problem.

In 2 minutes, we will finish that. Let us take the n degrees of freedom system, which is given by the manipulative dynamics $\tau = M(\theta)\ddot{\theta} + (C(\theta, \dot{\theta})\dot{\theta} + h(\theta))$

This is equation 1, which Represents the dynamic model of an n degrees of freedom robot manipulator. And take a control law. $\tau = k_p E - k_d \dot{\theta} + h(\theta)$ with k_p and k_d or diagonal matrices.

Okay. Now, it is a regulation problem. Regulation problem. That means we are not forcing the robotic system to track a trajectory. Rather, we are making the robot reach a goal point and stay there.

That goal point is in the trajectory path. Okay. So, we are considering a regulation problem here. So, in that case, now substituting this controller 2 into the dynamic model will give the closed-loop system dynamics given by

$$M(\theta)\ddot{\theta} + (C(\theta, \dot{\theta})\dot{\theta} + k_d \dot{\theta} + k_p \theta) = k_p \theta \Rightarrow \ddot{\theta} = 0$$

which is equation number 3.

So, having chosen the Lyapunov function candidate as, $V = \frac{1}{2} \dot{\theta}^T M(\theta) \dot{\theta} + \frac{1}{2} E^T K_p E$ that is equation number 4.

So, let us consider this as the Lyapunov function candidate where the First term is having the system parameters, which involves the inertia matrix, and the second term has the error parameters. This function candidate is a scalar function, and it is positive definite. This function is always positive or 0 because the mass matrix M of theta and Kp are positive definite matrices. Now, differentiating the Lyapunov function candidate gives

$$\dot{V} = \frac{1}{2} \dot{\theta}^T M(\theta) \dot{\theta} + \dot{\theta}^T M(\theta) \ddot{\theta} - E^T K_p \ddot{\theta}$$

which is again simplified into $\frac{1}{2} \dot{\theta}^T \dot{M}(\theta) \dot{\theta} - \dot{\theta}^T K_d \dot{\theta} - \dot{\theta}^T C(\theta, \dot{\theta}) \dot{\theta}$ Thus, it becomes $-\dot{\theta}^T K_d \dot{\theta}$

Now, we end up with only one term on the right-hand side, that is $\dot{V} = -\dot{\theta}^T K_d \dot{\theta}$

From this equation, we are coming to this because there is a property that $\dot{M} - 2C$ is a skew-symmetric matrix. Here, the skew-symmetric matrix is present.

So, as per the properties of skew-symmetric matrix, that $\dot{x}^T S \dot{x} \Rightarrow 0$

Because of that, we have $\dot{\theta}^T (\dot{M} - 2C) \dot{\theta} \Rightarrow 0$ in the previous equation.

That is why we got so we end up $\dot{V} = -\dot{\theta}^T K_d \dot{\theta}$

So, we can come to a conclusion that V dot is negative definite as long as K_d is positive definite. So next, we need to investigate a final point that here we need to investigate whether the robotic system can get stuck with non-zero error. We need to check.

That means the error must be zero. We need to confirm that. Because V dot can remain zero only if theta dot equals to 0 based trajectories, and the trajectory must also have theta dot double dot equal to 0, which means that V dot can remain 0 provided with the trajectories that are along theta dot equal to 0 and theta double dot equal to 0, and because K_p is non-singular, we have from Equation 2 that E equals to, that is, K_p into E equals to 0, which implies, because K_p is non-singular, we can conclude that the error E equals to 0.

Thus, the control law, which is given by equation 2, we can state that it can globally stabilize the system. So, we can say that the system is asymptotically stable by the controller given by equation 2. And K_p , this conclusion we have come from the system

closed-loop dynamic equation, that is, basically from equation 3, not 2; it is from equation 3. We can say that this is the trajectory associated with $\dot{\theta}$ equal to 0 and $\ddot{\theta}$ equal to 0. So, we end up with KP error equal to 0 from equation 3, from which we can conclude that error $A\dot{\theta} + PA = -\theta$ equals to 0. Okay. So, coming to the conclusion, I can say that in this lecture, we have seen Lyapunov stability theory with its methods, indirect and direct approach, and then we have seen examples associated with that. Then we have taken a linear system, and by using the Lyapunov matrix equation we could solve the stability of the linear system. And then we have seen the Lyapunov-based control of robotic systems, which involves an n degrees of freedom manipulator and a single-link manipulator. In the next class, we will be seeing about robot dynamics as well as the programming part. Thank you very much.