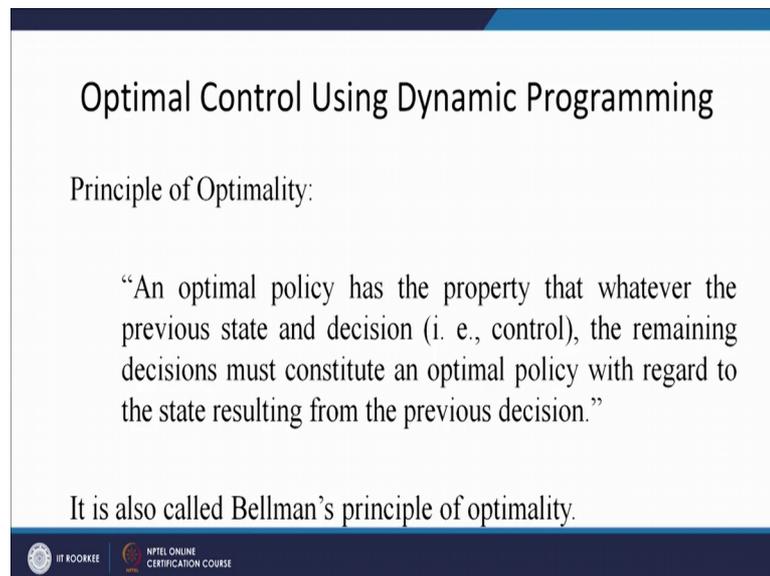**Optimal Control**
Dr. Barjeev Tyagi
Department of Electrical Engineering
Indian Institute of Technology, Roorkee

Lecture - 36
Optimal Control Using Dynamic Programming

Welcome friends in this session of our discussion we will discuss the Optimal Control using the Dynamic Programming. So, how we can utilize the dynamic programming to determine the solution of a optimal control problem. Say dynamic programming is an old approach to solve any system using the backward pass; it is principally based on the principle of optimality. What is the principle of optimality we can define it as an optimal policy that has the property, that whatever the previous state and decisions the remaining decisions must constitute an optimal policy with regard to the state resulting from the previous decision.

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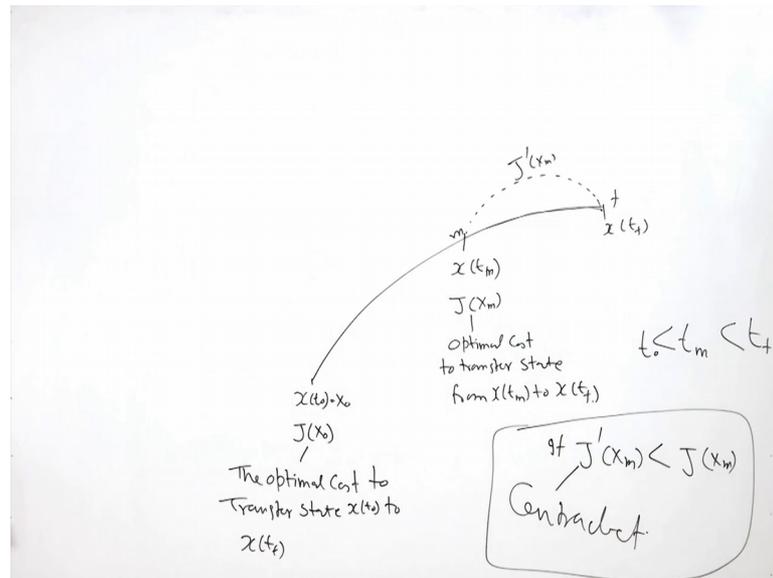This is also called the Bellman's principle of optimality. What actually is the meaning of this optimal policy has the property that whatever be the previous state and decision; the remaining decision must constitute an optimal policy understand in this way.

Let we have the optimal trajectory which is the starting at x t 0 and terminating at x of t f. So, this is my complete trajectory and let us say J x 0 or is the optimal cost to transfer state x t 0 to x t f. Let this is representing the optimal cost to transfer the state x t 0 to x t f. So, x t 0 is my initial state this is my optimal cost let we take a point in between this trajectory.

Let us say this is x of t m and this t m is less than t f and greater than t 0, and this is I am saying J X m, this is the optimal cost to transfer state from x t m to x of t f. So, what we are saying if we have J x t m as the optimal cost to transfer this is state x t m to x t f. So, this if x t 0 to x t f cost is optimal and this is my optimal trajectory, then the cost to transfer from x t m to x t f will be optimal. So, optimal policy that the property that whatever be the previous state and decisions, so whatever be my this state the remaining decisions must constitute an optimal policy. So, what the decisions I have to take from x t m to x t f, this will make my optimal policy.

So, this is the principle of optimality, suppose we say this is x t 0 to x t f is my optimal trajectory and let us say from x t m to x t f we have J dash x t m the cost to transfer through the dash line is J dash x m, if my J dash x m. So, if J dash x m is less than J x m this means let us consider this is my optimal path from point m to point f. This means if I am taking the optimal cost from x t 0 to x t f that will not be optimal, because then this

will be t 0 to m and m to f through the dash line this will be the optimal cost, and this will contradict power initial assumption that x t 0 to x t f my optimal cost is x 0.

So, it will it contradict. So, this will not be true. So, this means if this is the optimal trajectory then optimal cost will be J x m to transfer from m to x m only using this dark line. So, this is my principle of optimality on which my dynamic programming approach is based.

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So, now, let us consider a system given as x k plus 1 equal to f of x k, u k, x k. We are considering a general system with the initial condition is x k equal to 0 is x 0 and the cross functional is J i x k i equal to J s of k f k f plus sigma v k u k and this is my cost functional why we are saying the J. So, i is either 0 1 2 k f minus 1 this means in my each state I am writing the value of.
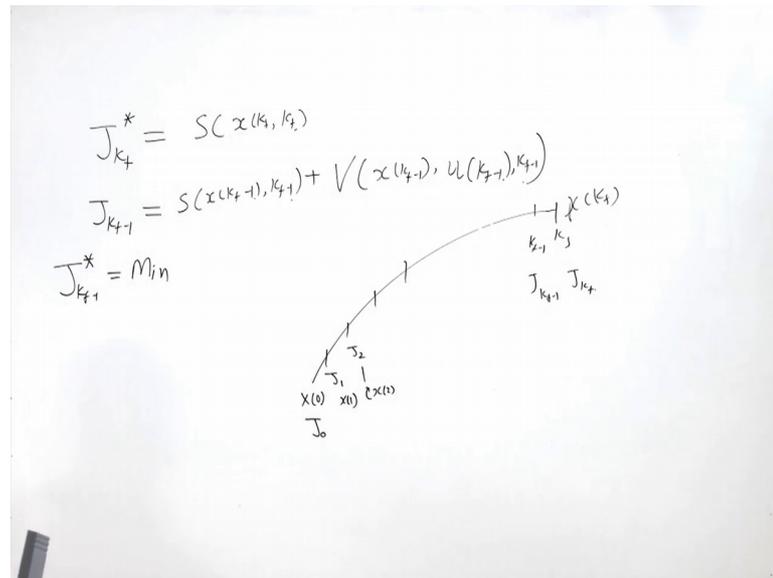
So, I am starting from x 0 to x of k f and i is wearing from 0 let us say 1 2 3 4 and upto k f k f minus 1.

So, I can say this is my J 0, J 1, J 2, J k f minus 1 J of k f. So, what this is representing? J 0 is the optimal cost to transfer state from x 0 to x f, J 1 is the optimal cost to transfer state from let say x 1 to x f, J 2 transfer the state from x 2 to x f. So, these are representing my optimal cost at the different value of this I. As I am saying my J i is equal to s x of k f k f plus sigma of this term. So, if I will write J of k f what is this? This is nothing but my terminal cost s x of k f k f. So, this is nothing but my s. So, this will be my last terminal cost and the whole cost we are considering as J i means J 1 J 2 up to so I is taking the value of. So, this I can represent as my end cost this is my terminal cost as J of k f.

So, if I will write the cost. So, this will be my optimal cost. So, if I will write the cost at the one point before at k of minus 1. So, my this value will be S x of k f minus 1, k f minus 1 plus if I is k f minus 1. So, this will be V k f minus 1 u k f minus 1 plus v I have only the one term k f minus sorry x of k f minus 1, k f minus 1, this is my V and this cost to be minimum. So, the optimal cost if I want to write this is the minimum for the value of sorry, I have miss the u term here x of k f minus 1 u of. So, the optimal cost is with u of k minus 1.

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$$J_{k_f}^* = S(x(k_f, k_f))$$

$$J_{k_f-1} = S(x(k_f-1), k_{f-1}) + V(x(k_{f-1}), u(k_{f-1}), k_{f-1})$$

$$J_{k_f-1}^* = \min_{u(k_{f-1})} \left[V(x(k_{f-1}), u(k_{f-1}), k_{f-1}) + J_{k_f}^* \right]$$

$$J_{k_f-2} = S(x(k_f), k_f) + V(x(k_{f-2}), u(k_{f-2}), k_{f-2}) + V(x(k_{f-1}), u(k_{f-1}), k_{f-1})$$

$$= V(x(k_{f-2}), u(k_{f-2}), k_{f-2}) + S(\cdot) + V(x(k_{f-1}), u(k_{f-1}), k_{f-1})$$

$$J_{k_f-2}^* = \min_{u(k_{f-2})} \left[V(x(k_{f-2}), u(k_{f-2}), k_{f-2}) + J_{k_f-1}^* \right]$$

So, this I am rubbing I can write this simply as V x of k f minus 1 u of k f minus 1 and k f minus 1.

This term plus what is the s k f minus 1 k f minus 1 I can write as k f. So, J of k f. So, with respect to u if I will find out the minimum cost of this, this is giving me the minimum cost of J of k f minus 1. To find this what we can do del of J of k f minus 1 by del u of k f minus 1. So, this is giving you the minimum cost now if we will go one step more below J k f minus 2, this I will write for me here. So, this is k is k f minus 2 to k f minus 1. So, it will have the 2 terms plus S x of k f k f. So, if I will explain this, this is S x of k f my terminal cost plus if k equal to k f minus 2. So, this is V x of t k f minus 2 u k f minus 2, k f minus 2 plus. So, I am expanding this summation term, I am only the 2 terms k f minus 2 and k f minus 1 plus V x of k f minus 1, u of k f minus 1, k f minus 1.

So, this I can write as V x of k f minus 2, u of k f minus 2, k f minus 2 plus if I will yet s of k f minus 1, v of k f minus 1 this term and this term will constitute nothing but my optimal value or I can write this as S plus V as x of k f minus 1, u of k f minus 1, k f minus 1 and I will get the minimum value of J k f minus 2, the optimal value will be nothing but minimum value with respect to u, k f minus 2 this is V x of k f minus 2, u of k f minus 2, plus J k f minus 1 star. So, similarly we can keep on expending with the different value of the I taken as the k f is starting from the k f, k f minus 1 if I will take.

So, I will get this term k f minus 2, I will get this term. So, if I will go for a general value of the k.

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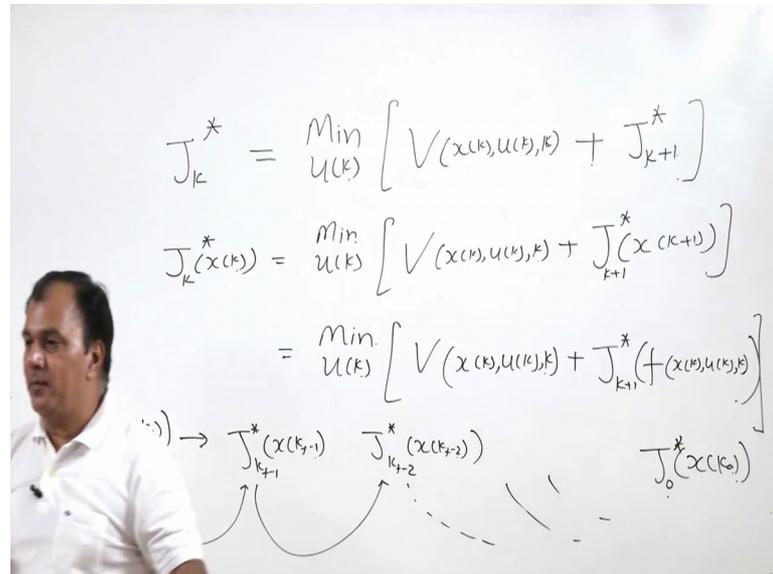
I equal to k, the optimal value of the J k can be written as minimum with u k which will be nothing but my V x of k, u of k, k plus say if we will see for k f minus 1 it was the k f k f minus 2, k f minus 1. So, this I can write as simply optimal value at the point J k plus 1. So, general relation I can write in this form.

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Optimal Control Using Dynamic Programming

At any time or stage k, using the principle of optimality , PI is

$$J_k^*(\mathbf{x}(k)) = \min_{\mathbf{u}(k)} \left[V[\mathbf{x}(k), \mathbf{u}(k)] + J_{k+1}^*(\mathbf{x}^*(k+1)) \right]$$

This relation is the mathematical form of the principle of optimality as applied to optimal control system.

If one had found the optimal control, state and cost from any stage k+1 to the final stage k_f then one can find the optimal values for a single stage from k to k+1.

So, this means at anytime k using the principle of optimality, my performance index will be J k star x k minimum with respect to u k as V of k, u k plus J k plus 1 star x star of k plus 1.

So, this means with respect to whether at the point k this is x of k, this is the optimal cost to transfer the state from x k to k f which is equal to minimum of u k, V k, x k u k o J star k plus 1 x star of k plus 1. In other words more general form these relations we are writing as J star x of k minimum of u k, V x k u k, k and at this point we already obtain the optimal value of the u. So, g x of k plus 1 we are writing. So, this is at the point k and this is at the point k plus 1. And as we know we have considered our system as x k plus 1 f of x k u k, k. So, in terms of k I can write these minimize u k, V x k, u k, k plus optimal value of now this is nothing but the function of x k, u k, k.

So, in this solution if we will start from the k f point, using the backward pass we can reach up to the J k 0 this means our steps will be find out the J k f at k f point, J x of k f because this is the terminal cost I already know this. So, I can find this then find J k f minus 1, x of k f minus 1. So, I know this optimal, if I know this optimal I can find this if I know this I can find J star, k f minus 2 and by this in this way I can finally, find J 0 which is J x of k 0 can be find out. So, using this I can solve this problem utilizing the dynamic programming approach, how actually we will do that we will see in one example.

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Optimal Control Using Dynamic Programming

Example

For the given plant, $x(k+1) = 4x(k) - 6u(k); x(0) = 8$

determine the optimal values of u(0) and u(1) which minimizes the PI

$$J = J_0 = \left(x(2) - 20\right)^2 + \frac{1}{2}\sum_{k=0}^{1}\left(2x^2(k) + 4u^2(k)\right)$$

Let us considered a simple first order system as x k plus 1 equal to 4 of x k minus 6 of u k where x 0 is given as 8. My objective is to determine the optimal control and optimal states 4 u 0 and u 1 which is minimizing my performance index as J equal to J 0, x minus 2 minus 20 square plus half of sigma k is varying from 0 to 1. So only the two steps, we will see how it can be solve. So, for any number of the sequence then we can develop an algorithm to solve this, plus half of sigma k 0 to 1 2 of x square k plus 4 of u square k.

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$$J_2^*(x(2)) = (x(2)-20)^2$$

$$J_1^*(x(1)) = \min_{u(1)} \left[\frac{1}{2}(2x^2(1) + 4u^2(1)) + (x(2)-20)^2 \right]$$

$$= \min_{u(1)} \left[\frac{1}{2}(2x^2(1) + 4u^2(1)) + (4x(1) - 6u(1) - 20)^2 \right]$$

For minimum $u(1)$

$$\frac{\partial H}{\partial u(1)} = 0 \Rightarrow u^*(1) = \frac{12x(1) - 60}{19}$$

$$J_1^*(x(1)) \longrightarrow \text{using } u^*(1)$$

So, we will see how to solve this problem, as we know what is this value; J 2 star this is my J 0 is given my performance index is given as x 2 minus 20 square. So, this means at the point x 2 this is my optimal cost. So, I can write J 2 star x of 2 was x 2 minus 20 whole square. So, this is nothing but my whole square, what I will write for my next cost J 1. So, this is minimum of u 1. So, what we have? We have the V k, k is 1 now. So, V x of 1 u of 1 what is the V x of 1 u f 1 is nothing but my this course. So, my V is half of 2 x square k plus 4, u square k this is my v.

So, I can write this half 2 x square 1 and 4 plus 4 u square 1 this is my v term plus J star plus J star x of 2 and for this J star x of 2 I write x 2 minus 20 square, and this is my J 1. Minimum value with respect to u 1 we have to find x 2, I can write in terms of the exponent u 1 from my plant equation is the x k plus 1. So, if I am writing the 2 x 2 this is the 4 of x 2 minus 6 of u 2. So, this means I can write this as minimum u 1 half 2 x 2 square 1 plus 4, u square 1 plus in place of x 2 I will write my plant as 4 x 1 minus 6 u 1.

So, minimum of u 1 means this I have to find the minimum with respect to u 1 of this quantity, let us say my this quantity is h for minimum of u 1 my del h by del u 1 will be 0; this means to get this value I have to find this means del h means this whole quantity in the minimum with respect to u 1 I have to find this to be 0. So, this will be 0. So, this will give me the minimum with respect to u 1. So, if we will differentiate this. So, this will give you I am directly giving the result you can try yourself u 1 as 12 of x 1 minus 60 by 19.

This means we are representing u 1 in terms of the x 1, and till this state we do not know what is the u 1 and what is the x 1 this we have to find out. With this value of the u 1 my let say this is u 1, this is the optimal value. So, I can find out the optimal value of the J 1 x 1 using u 1 star, means if I will substitute this value in this relation this will give me the optimal value of the J 1. So, at u point one this is my optimal u.

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So, in the same way we will write for the J 0. So, we will write this J 0 x 0. So, this will be minimum of u 0 and this will be half of what the v we have concerned? Considered here half of 2 of x square 0, plus this is my 4 u square of 0, plus J 1 star x 1, and x 1 I can write in terms of the x 1 in terms of the x 0 and u 0.

So, I can write this as minimum of u 0, 1 by 2, 2 x square 0 plus 4 u square 0, this bracket will not be here. So, 2 x square 0 4 u square 0 plus like we have seen it here value of the u 1 we are placing in terms of the J 1 star. So, my this will be nothing but my

f of which is f is a function of x 0 and u 0. So, my this all expression which will be my h 0 now is a function of x 0 and u 0. So, my this whole function will comes out to be as a function of my x 0 and u 0, and this we have to solve for minimum of u for minimum of u 0, we can write this as del h 0 by del u of 0 equal to 0.

So, this we are getting directly and what is the J f x 0 u 0 when we are substituting the value of the optimal value of the u 1 in the expression of my J 1 star which we have drive before. So, my this whole equation will be equation in terms of the x 0 and u 0 form. So, I can directly differentiate this with respect to u 0 to get the optimal value of the u star 0, this means the optimal value of the. So, once you will solve this equation you will get the value as 4.81 using my x 0 which is given as 8, because this whole equation is a equation in terms of the u 0 and x 0, x 0 is already given if I will place the value of the x 0 in this relation I can find the value of the u star 0 as 4.81.

So, at this point I know what is my optimal value of the u 0, I also know what is my x 0 given as 8, if these 2 value is known to me I can find out what will be my x 1 my x 1 is 4 of x 0 minus 6 of u 0, and if you will place this value this is nothing but 3.14 we will get once the x 1 is known.

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$$u(1) = \frac{12\,x(1) - 60}{19} = -1.175$$

$$x(2) = 4\,x(1) - 6\,u(1)$$
$$= 19.61$$

$$\frac{\partial H_0(\cdot)}{\partial u(0)} = 0 \qquad \begin{cases} u^*(0) = 4.81 \\ x(0) = 8 \end{cases} \text{using}\ x(0) = 8$$

$$x(1) = 4\,x(0) - 6\,u(0)$$
$$= 3.14$$

So, what the x 1 is known and I have represented my u 1 was 12 of x 1 minus 60 by 19. So, I already have determined the x 1 with this value my u 1 will confirm to be minus 1.175. So, now, I know x 1 and x 1 and u 1. So, easily I can find out what actually will be

my x 2 which will be nothing but my 4 of x 1 minus 6 of u 1 and if this value is known then my x 2 will be 19.61.

So, by this example we can see how we can apply the dynamic programming to determine optimal value of the u and optimal value of the state, and with this value at the each step we can also calculate what will be my optimal cost. So, all the parameters can be determined as we go from the backward pass, you can see we have started with the J 2 star. So, we will find out the J 2 star J 1 star, J 0 and once we will reach to the next last step there x 0 is defined. So, we can find out the u 0 from u 0 to u 1, u 1 to x 1, x 1 to again u 1, u 1 to x 2. So, all optimal values of the control optimal value of the state and the optimal cost function we can determine using the dynamic programming approach.

So, I stop my discussion here in the next class we will discuss how this dynamic programming approach can be applied to a continuous time system either it is possible or not and if possible then what actually will be our approach.

Thank you very much.