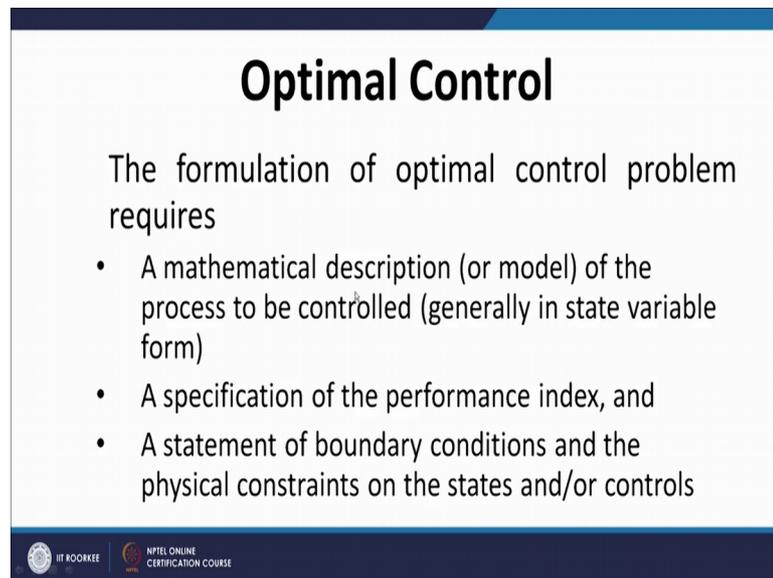


Optimal Control
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Lecture – 02
Basic Concepts of Calculus of Variations

Welcome friends to this session of our discussion. In the previous class we are discussing about the performance index. So, we have discuss about the optimal control problem formulation we are discussing in which the first we have seen the mathematical description of a model.

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Optimal Control

The formulation of optimal control problem requires

- A mathematical description (or model) of the process to be controlled (generally in state variable form)
- A specification of the performance index, and
- A statement of boundary conditions and the physical constraints on the states and/or controls

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For a linear (Refer Time: 00:45) system we have seen how we can represent a model general model, we will discuss little bit later on and then we are discussing about the; what is the specification of the performance index.

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Performance Index

Minimization of the integral of the squared error
Then Performance criterion is

$$J = \int_{t_0}^{t_f} x'(t) Q x(t) dt$$

$x_d(t)$ desired value, $x_a(t)$ actual value, and
 $x(t) = x_a(t) - X_d(t)$, is the error.
Q is a weighting matrix, *positive semi-definite*.



And we have seen that the performance index; we can take it as a minimization of the integral of the square error.

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Performance Index

For minimum control efforts the performance measure is

$$J = \int_{t_0}^{t_f} \sum_{i=1}^m u_i^2(t) r_i dt$$

or in general

$$J = \int_{t_0}^{t_f} u'(t) R u(t) dt$$

R is a *positive definite* matrix



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Performance Index

Minimizing the error between the desired target position $X_d(t_f)$ and the actual target position $X_a(t_f)$ at the end of the maneuver or at the final time t_f

The terminal (final) error is $x(t_f) = x_a(t_f) - x_d(t_f)$

$$J = x'(t_f)F(t_f)x(t_f)$$

This is called the *terminal cost function*
F is a *positive semi-definite matrix*



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Performance Index

General Optimal Control System:

Combining the above formulations, we have a performance index in general form as

$$J = x'(t_f)F(t_f)x(t_f) + \int_{t_0}^{t_f} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)] dt$$

Or

$$J = S(x(t_f), t_f) + \int_{t_0}^{t_f} V(x(t), u(t), t) dt$$

R is a positive definite matrix, and Q and F are positive semi-definite matrices, respectively. Note that the matrices Q and R may be time varying

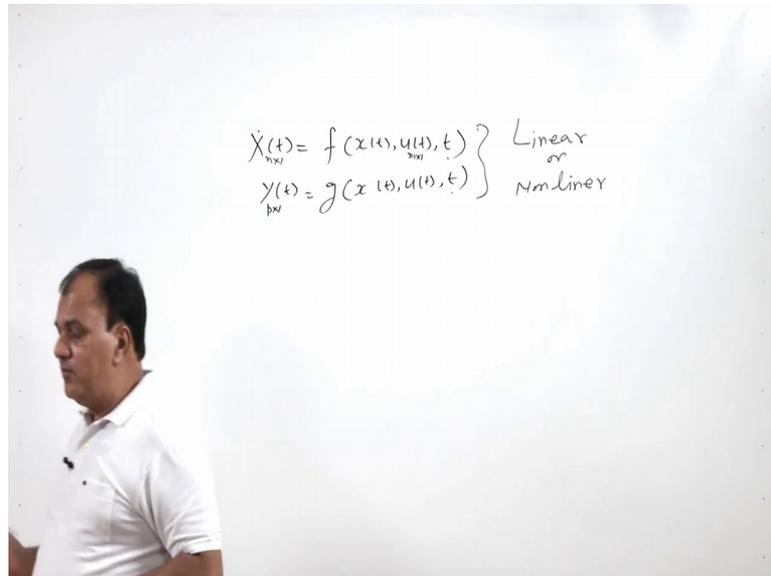


Let us say minimization of my control efforts, minimization of the terminal cost and the overall minimization of the terminal cost plus integral square error plus my control effort, I can represent my performance index as given as $J = x' f x + \int_0^t x' Q x + u' R u$.

We are first time representing the terminal cost when t equal to t_f ; I am trying to minimize my error. So, this is giving me the terminal cost and in a interval from t_0 to t_f

I am trying to minimize the integral of the square error as well as I want to minimize the control efforts. So, this is when I am representing my system in linear form.

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In general we represent our system if I will take $\dot{X}(t) = f(x(t), u(t), t)$. So, this is my and $Y(t) = g(x(t), u(t), t)$ where X is my $n \times 1$, $p \times 1$ and u is $n \times 1$. So, this is the general representation of my system where my this system maybe linear or non-linear the linear case representation we have seen in the previous class that is $\dot{X}(t) = aX(t) + bU(t)$ and $Y(t) = cX(t) + dU(t)$. So, this is my general representation. So, this means this is N number of the first order differential equation if we are representing then that will be my general representation.

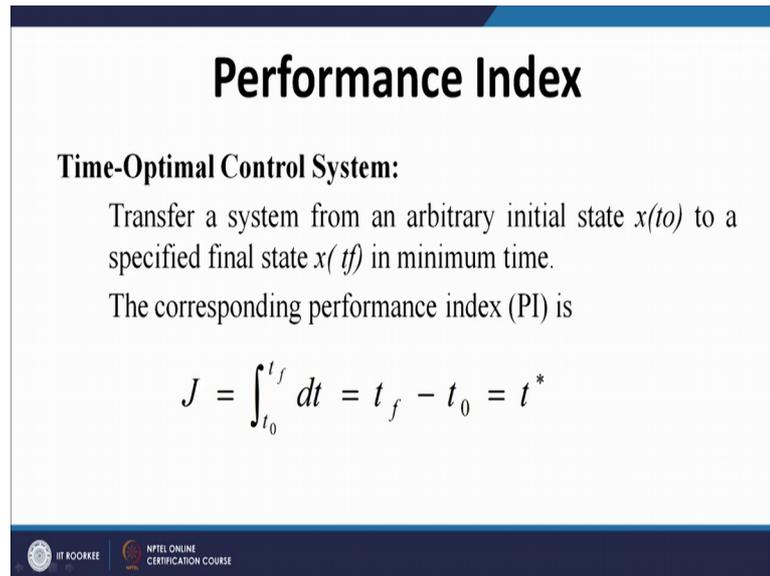
In such a case, we can have the general representation of my performance index as given as $J = \int_{t_0}^{t_f} v(x, u, t) dt$ which representing my terminal cost at $t = t_f$ point and integral t_0 to t_f $v(x, u, t)$, this is defining my integral cost. So, in linear case the same I can represent by this in this case R will be a positive definite matrix as we have discussed Q and f what is the Q ? Q is giving the way to my error and in case of this I can give a 0 way some of error, but I cannot give the negative weights.

So, my Q will be Q maybe a positive semi definite matrix similarly the same logic we can apply to the f . So, Q and f will be my positive semi definite matrix and Q and R are my time varying because as a general system I am considering time varying system it may be linear it may be non-linear when it is a time invariant system then my a, b, c, d

matrix will be the constant matrix Q R and f these matrices will also be the constant matrix otherwise they will be the function of time in case of the time varying.

So, in this way, we are representing our performance index for a general system and for a linear time varying system.

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Performance Index

Time-Optimal Control System:
Transfer a system from an arbitrary initial state $x(t_0)$ to a specified final state $x(t_f)$ in minimum time.
The corresponding performance index (PI) is

$$J = \int_{t_0}^{t_f} dt = t_f - t_0 = t^*$$

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Sometimes the some other form of the performance index we can have depending upon the nature of the problem like we take that time optimal control system in which our objective is to transfer the state of the system from its initial value to final value in minimum time there we are not concentrating on the error or control effort or some other parameter. Here our objective is to minimize the time to transfer a state from its initial value to final value such a system we called the time optimal control system and the performance index can simply be defined as t_0 to t_f because our objective is to minimize the time.

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Performance Index

Fuel-Optimal Control System:

In order to minimize the total expenditure of fuel, the performance index is

$$J = \int_{t_0}^{t_f} |u(t)| dt$$

For several controls, PI may be write it as

$$J = \int_{t_0}^{t_f} \sum_{i=1}^n R_i |u_i(t)| dt$$

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We can also have the fuel optimal control system where total expenditure of the fuel we are taking as a sum of u t which is my input.

So, we are trying to minimize this which is multiplied with some weight matrix. So, my performance index will be J equal to mod of u t d t for a single and if I will have the several p I then this is nothing but the summation of R i mod of u i t . So, depending upon the nature of the problem basically we have to select what actually will be our performance index. In this particular course, in our most of the discussion we will consider a performance index given as J equal to S x of t f t f plus t 0 to t f v x of t u d t d t , this performance index once we are discussing the problem in general and once we are discussing the linear systems then we will consider the performance index given as J equal to x prime f x plus integral t 0 to t f x prime Q x plus u prime R u time interval again depend upon the what type of the problem, we are considering if my problem goes to the infinite horizon then my t f can also approach to the infinity.

So, in this way, we can define our system as a linear system or by the general representation of the system we can select the performance index and next they are may appear that constraints on my input and the states.

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Constraints

The control $u(t)$ and state $x(t)$ vectors are either **unconstrained** or **constrained** depending upon the physical situation.

Constrained can be represented as

$$U_- \leq u(t) \leq U_+ \quad \text{and} \quad X_- \leq x(t) \leq X_+$$

+, and - indicate the maximum and minimum values the variables can attain.

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Normally these constraints we represent as $u(t)$ in between U_{\min} and U_{\max} where U_{\min} is the minimum value of the u and U_{\max} is the maximum value of the u . Similarly for the states can adjust they cannot go below a certain level and they cannot go above a certain limit. So, my U and X may be constraint. So, in this particular case, we are discussing the problem initially which are the unconstrained problem in an constant problem we can say my limits for the U_{\min} and the U_{\max} goes to infinity means you can take any value in this, but most of the case, later, part of the course we will consider that the my u are also constrained. So, if you will place the constraint then the nature of the problem changes.

So, in general we can formulate a optimal control problem in the following manner.

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For a given plant:

equality constraint $\rightarrow \dot{X}(t) = f(x(t), u(t), t)$

determine optimal $\underline{u}(t)$ which minimizes the

PI $J(x(t), u(t)) = S(x(t_f), t_f) + \int_{t_0}^{t_f} V(x(t), u(t), t) dt$

$u(t)$ & $x(t)$ are constrained as

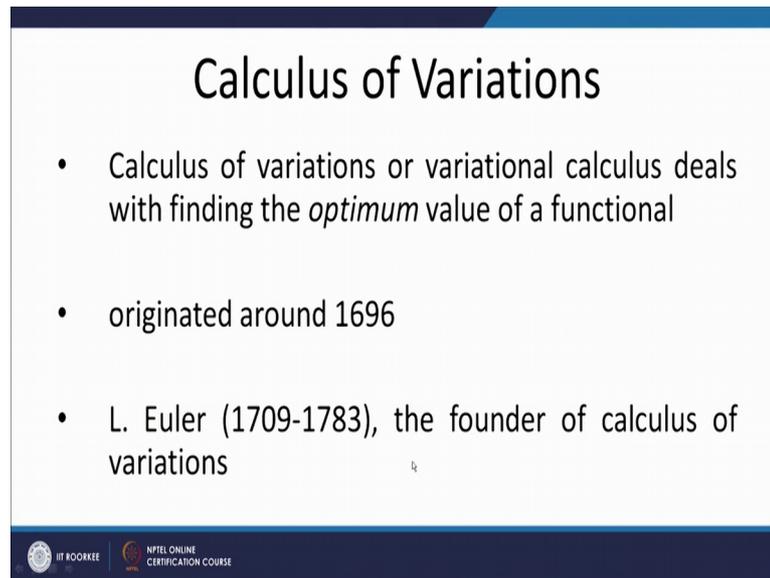
inequality constraint $\rightarrow |u(t)| \leq U \quad |x(t)| \leq X$

Say for a given plant, I am taking the general nature of the plant $\dot{X}(t) = f(x(t), u(t), t)$ determine optimal u which minimizes the performance index $J = S(x(t_f), t_f) + \int_{t_0}^{t_f} V(x(t), u(t), t) dt$. So, for a given plant my objective is to find out the optimal value of the $u(t)$ which is minimizing my given performance criteria and further we can also extend $u(t)$ and $x(t)$ are constrained as more $u(t)$ is less than equal to U and more $x(t)$ is less than equal to X . So, is the complete statement of a optimal control problem this is my plant objective is to find out the $u(t)$ which is minimizing the J u and x are constrained.

So, we will start our discussion simply by taking how we can get the minimization of J and as a optimization problem if I will take my objective is to minimize this J which is subjected to the constraint. So, this I can treat as my equality constraint and this is my inequality constraint. So, this is the formulation of my control problem. So, this is the formulation of our optimal control problem next we will see how to solve this problem. So, in this course we will use the 3 approaches to solve this problem the first one is the calculus of variation which we will start discussion. Next to this another approach is the dynamic programming approach we will discuss about the dynamic programming after the calculus of variation and in these 2 normally we do not consider the constraint on the controller state then third approach is the pontryagin minimum principle in which we will consider the constraints and solve the problem using the pontryagin minimum principle how it can be utilized to develop the time optimal control problem.

So, we start our discussion by considering the solution of the optimal control problem utilizing calculus of variations what is calculus of variations. So, calculus of variation or the variational calculus deal with finding the optimum value of a functional what is functional that we will discuss, it is a very old technique originated around 1696.

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Calculus of Variations

- Calculus of variations or variational calculus deals with finding the *optimum* value of a functional
- originated around 1696
- L. Euler (1709-1783), the founder of calculus of variations

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Then L Euler was the founder of this calculus of variation and till today we are using the Euler equation to solve the variational problem. So, first we will discuss the few basic concepts and the definition of the calculus of variation before coming to the minimization problem our first definition comes which is called the function.

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Calculus of Variations

Function: A variable x is a *function* of a variable quantity t , if to every value of t over a certain range of t there corresponds a value x

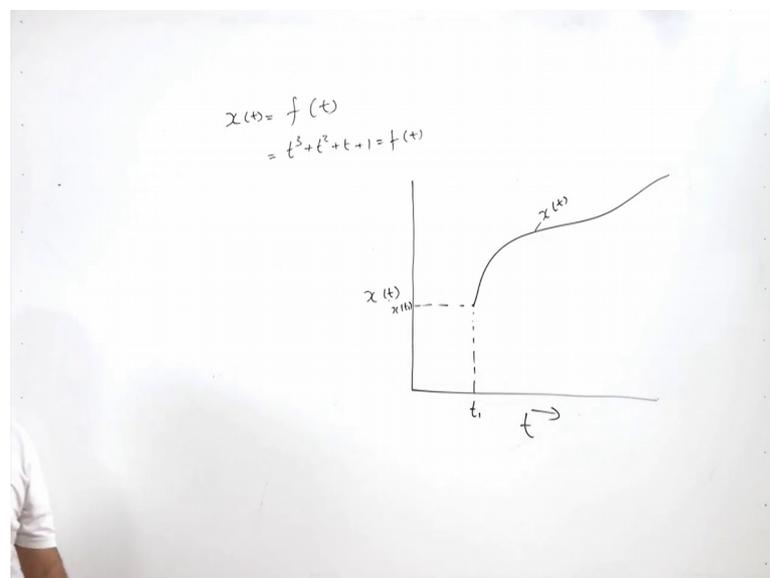
Functional:

- A variable quantity J is a *functional* dependent on a function $f(x)$
- written as $J = J(f(x))$, if to each function $f(x)$, there corresponds a value J
- Functional depends on several functions.

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What is a function? A variable x is a function of the variable quantity t if to every value of the t over a certain range of t there corresponds a value x .

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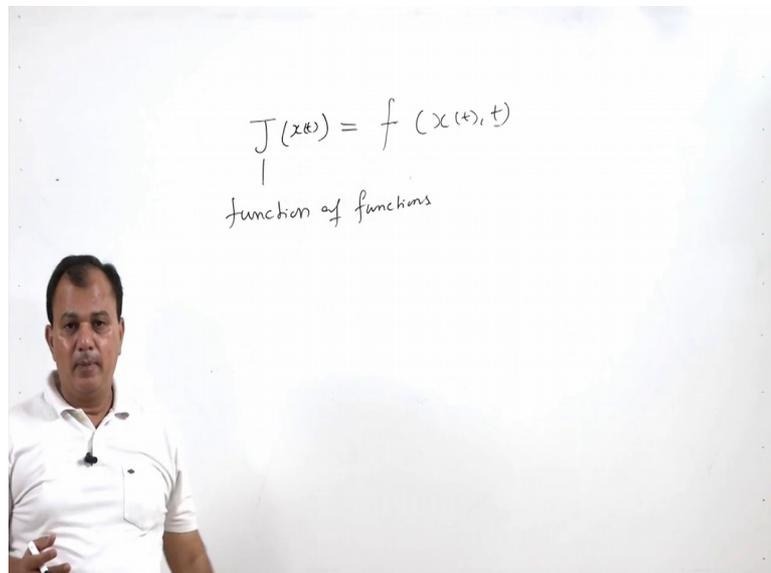
This means if we define let $x(t)$ to be the function of time t this means for each value of the t there corresponds a value x . So, so if I will take a point which is giving me the value of the t say t_1 and this will have the value of the x which we write as x of t_1 .

So, if I will treat this. So, $x(t)$ will be nothing but a function for each value of the t there exists a value of the x . So, I can say this x is a function of t for example, we can write

this like say $t^3 + t^2 + t + 1$. So, if $x = t$ equal to this, so this is nothing but my $f(t)$ is $t^3 + t^2 + t + 1$. So, for each value of the t , I know there will be a value of the x which I can make a plot here and that whole is represented as a function of x , this normally we already have studied at our in many subjects we use the function as a x .

Our next definition is comes out to be the functional if variable quantity J is a functional dependent on the function x like in this case my x will depend upon t . Similarly I can have a function which will be the function of the different function.

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So, I say $J(x(t))$ if I will write is a function of $x(t)$. So, what I am saying my J is a function of x . So, functional will also call the function of a function this is also called the function of functions. So, J may be the function of a single function x or it may be the function of the several other functions like say the x_1, x_2, x_3 or even sometimes we say J is a function of $x \cdot x \cdot x$. So, if for t there exist a value of the x then for each x there exist a value of j . So, a variable quantity J is a functional dependent on the function $f(x)$ it is written as $J = J(f(x))$ if to each function $f(x)$ there corresponds a value j . And functional depends on several functions it may be the function of single function or it may be the function of different functional.

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Calculus of Variations

Increment of a Function:

The increment of the function f , denoted by Δf , is defined as

$$\Delta f = \Delta f(t, \Delta t) = f(t + \Delta t) - f(t)$$

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Then we define the increment there may be the increment in a function and increment in a functional by definition increment of a function denoted as a delta f and delta f is simply defined as $f(t + \Delta t) - f(t)$.

So, what is the meaning of the increment?

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Increment $t \rightarrow t + \Delta t$

$$\Delta f = f(t + \Delta t) - f(t)$$

If t ; if we change the value t by any small amount say Δt then increment of a function is simply defined as value of the f at $t + \Delta t$ times minus value of the f at t point.

So, this is called the increment of a function similarly we can define the increment of a functional.

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Calculus of Variations

Increment of a Functional:

The increment of the functional J , denoted by ΔJ , is defined as

$$\Delta J = \Delta J(x(t), \delta x(t)) = J(x(t) + \delta x(t)) - J(x(t))$$

Here $\delta x(t)$ is called the *variation* of the function $x(t)$.

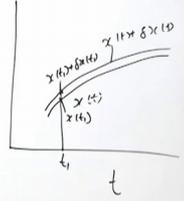



So, the increment of a functional J denoted by ΔJ is defined as $J(x(t) + \delta x(t)) - J(x(t))$.

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Increment of a functional

$x(t) \rightarrow x(t) + \delta x(t)$

$$\Delta J(\cdot) = J(x(t) + \delta x(t)) - J(x(t))$$


So, the increment of a functional we simply defined if x let say t will change as $x(t) + \delta x(t)$ what we said x is a function of time. So, this means if let this is my $x(t)$. So, what is the variation in $x(t)$? So, if the whole and this is say my $x(t) + \delta x(t)$. So, if my x

will change by any small amount then x plus Δx at each instant of the time there is a change in this; this means at each t , there will be a change in the value of the x by a small amount. So, I can say this is at this is x at t , this is same as the x at t plus Δx of t . So, this means the Δx is also changing with respect to time.

So, this is if x will change by an amount Δx then increment in functional J is simply defined as the value of the J at $x + \Delta x$ point minus J of x . So, this is the defining here the value of the J at $x + \Delta x$ point minus J of x . So, this is the increment of a functional and here this Δx is called the variation of the function this means we are saying the function x will have the variation Δx at each instant of the time and we are representing the whole as $x + \Delta x$ then we define the differential of a function. So, first we define the increment.

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Let us consider $t = t^*$
 $t = t^* + \Delta t$
 increment in function $f(t)$

$$\Delta f = f(t^* + \Delta t) - f(t^*)$$
 Expand with Taylor series

$$\Delta f = f(t^*) + \frac{\partial f(t^*)}{\partial t} \Delta t + \frac{1}{2!} \frac{\partial^2 f(t^*)}{\partial t^2} + \dots - f(t^*)$$
 If Δt considered to be very small $\Delta t \rightarrow 0$ Neglect second & higher order terms

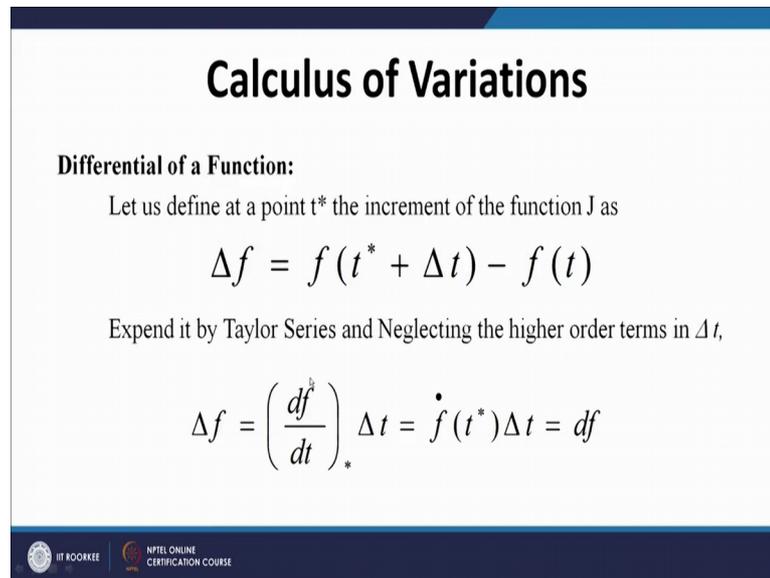
$$\lim_{\Delta t \rightarrow 0} \frac{\Delta f}{\Delta t} = \frac{df}{dt}$$

So, let us consider a point t equal to t^* then t will change from t^* plus Δt then we can define the increment in function f as Δf as the f at $t^* + \Delta t$ minus f of t^* .

So, if I will explain this with Taylor series then I have Δf to be f defined as the t^* point plus Δf by Δt plus $\frac{1}{2!} \frac{\partial^2 f}{\partial t^2} + \dots - f$ at t^* . So, I can cancel out this. So, so this is basically the expansion of f plus Δt using the Taylor series. So, if I will approximate to the first order term if Δt considered to be very small as Δt approaches to 0. So, I can say

delta f by delta t. So, I can put the limit t tends to 0. So, I can define this as my d f by d t which is called the differentiation or delta f can be defined as del f by d f by d t into delta t if I will neglect second and higher order term.

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Calculus of Variations

Differential of a Function:

Let us define at a point t^* the increment of the function J as

$$\Delta f = f(t^* + \Delta t) - f(t)$$

Expand it by Taylor Series and Neglecting the higher order terms in Δt ,

$$\Delta f = \left(\frac{df}{dt} \right)_* \Delta t = \dot{f}(t^*) \Delta t = df$$

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So, in this case my delta f can simplify be written as d f by d t delta t which is nothing but my f dot of t which is at the t star point delta t and this I can approximate as my d f which is nothing but the first order approximation of the increment. So, if I will get the first order approximation of my increment which is in this way. So, if we have taken the partial derivative if t is constant I can also replace it by my exact derivative d by d t.

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Calculus of Variations

Differential of a Function:

- df is called the *differential* of f at the point t^*
- $f'(t^*)$ is the *derivative* or *slope* of f at t^*
- the differential df is the *first order approximation* to increment Δt

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So, we call df to be the differential of f at point t^* $f'(t^*)$ is nothing but called the derivative or the slope at t^* point and the differential df is nothing but the first order approximation of the increment Δt .

Similarly, we can also define the variation of a functional.

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Calculus of Variations

Variation of a Functional:

Consider the increment of a functional

$$\Delta J = J(x(t) + \delta x(t)) - J(x(t))$$

Expanding $J(x(t) + \delta x(t))$ in a Taylor series

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So, variation of a functional is nothing but the first we define the increment of a functional as we have seen before how we define the increment the value of the J at $x(t)$

plus delta x t point minus J of x t what we are doing we are expanding J x t plus delta x t using Taylor series. So, what this will be. So, we are expanding J x t plus delta x t.

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Calculus of Variations

Variation of a Functional:

$$\begin{aligned} \Delta J &= J(x(t)) + \frac{\partial J}{\partial x} \delta x(t) + \frac{1}{2!} \frac{\partial^2 J}{\partial x^2} (\delta x(t))^2 + \dots - J(x(t)) \\ &= \frac{\partial J}{\partial x} \delta x(t) + \frac{1}{2!} \frac{\partial^2 J}{\partial x^2} (\delta x(t))^2 + \dots \\ &= \delta J + \delta^2 J \end{aligned}$$

where $\delta J = \frac{\partial J}{\partial x} \delta x(t)$ $\delta^2 J = \frac{1}{2!} \frac{\partial^2 J}{\partial x^2} (\delta x(t))^2$

δJ and $\delta^2 J$ are called the first variation (or simply the variation) and the second variation of the functional J, respectively.



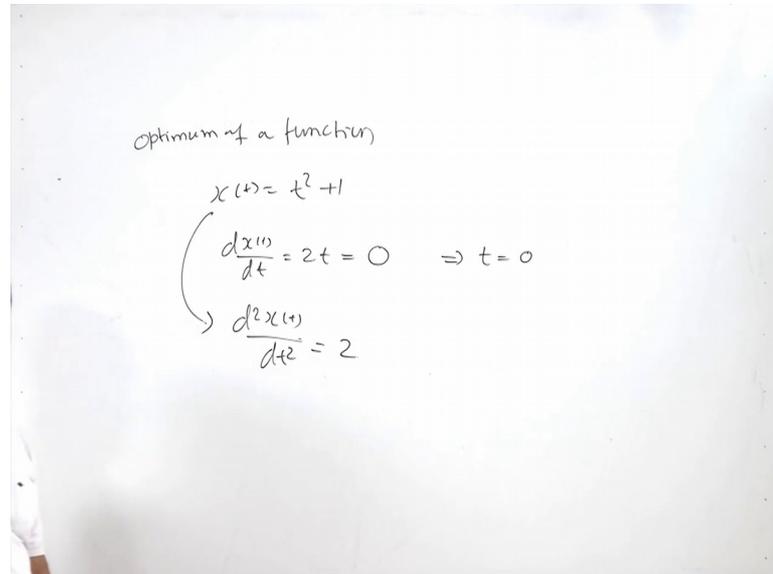

So, my first term will be J of x t second term will be del J by del x delta x next term is 1 upon factorial 2 del 2 J by del x square delta x t square then we have the higher order term minus J of x t this J of x t this J of x t will be cancelled out we are left with del J by del x delta x t plus 1 by factorial 2 del 2 J by del x square delta x t square plus higher order terms.

So, if delta x is considered to be very small then the first time if you will see this is nothing but the first order term del J by del x if you will see the second term this is a second order term which is del x t of square this means if x t is a small my all these term keep on reducing next term will be del x t cube which will be smaller than of the delta x term.

So, normally in our optimal control over first 2 term is of our interest where the first term we will define is delta J which is called the first variation and the second term we define as del square J which is nothing but my called the second variation. So, the increment of a functional if you will see the increment of a functional is nothing but the submission of my first variation plus second variation plus third variation and so on. So, in the increment of the J if I will see what actually will be the sign of my delta J if my first variation is not 0 then its sign will be governed by what is the sign of my first variation.

Suppose we say my first variation is 0 then sign of the delta J is governed by my second variation.

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The image shows a whiteboard with handwritten text and equations. At the top, it says "Optimum of a function". Below that, the function is given as $x(t) = t^2 + 1$. A curved arrow points from this equation to the first derivative $\frac{dx(t)}{dt} = 2t = 0 \Rightarrow t = 0$. Another curved arrow points from the first derivative to the second derivative $\frac{d^2x(t)}{dt^2} = 2$.

So, to find out the optimal value of a function like I will see $x(t)$ is equal to $t^2 + 1$ optimum of a function. So, I know how to get the optimal value of this. So, this I will take $\frac{dx(t)}{dt}$ which is nothing but $2t$ and I normally equate it to 0. So, in this case my optimal point is when t equal to 0 and then I take $\frac{d^2x(t)}{dt^2}$ which is nothing but my 2 and from my previous knowledge I know this is the optimal value is t equal to 0 and this is $\frac{d^2x(t)}{dt^2}$ is positive. So, this value will be my minimum value.

Similarly, in case of the functional the first variation and the second variations are used to determine the optimal values and the point of $x(t)$ means whether this is maxima or minima by using the first variation and the second variation. So, this we will discuss in the later part of our course. So, the first variation and the second variations are the important parameter which we will utilize in the later part of our course on the basis of the first variation and the second variation we define the optimum of a function optimum of a functional. And before discussing the optimum of a functional we will discuss the optimum of a function like we have seen in this example, how we can get these values that we will discuss in optimum of a functional function, and then we will start our discussion on the optimum of a functional.

So, I stop this class here and in the next class we will continue our discussion on optimum values how to find it.

Thank you very much.