Optimal Control

Dr. Barjeev Tyagi
Department of Electrical Engineering
Indian Institute of Technology, Roorkee

Lecture – 17
Linear Quadratic Optimal Control Systems (Continued)

Welcome class, we are discussing in the previous class about the Linear Quadratic Optimal Control System. We are designing a linear quadratic regulator in which we consider our reference input to be 0. So, my all the states are returning to the origin; they are coming to the 0.

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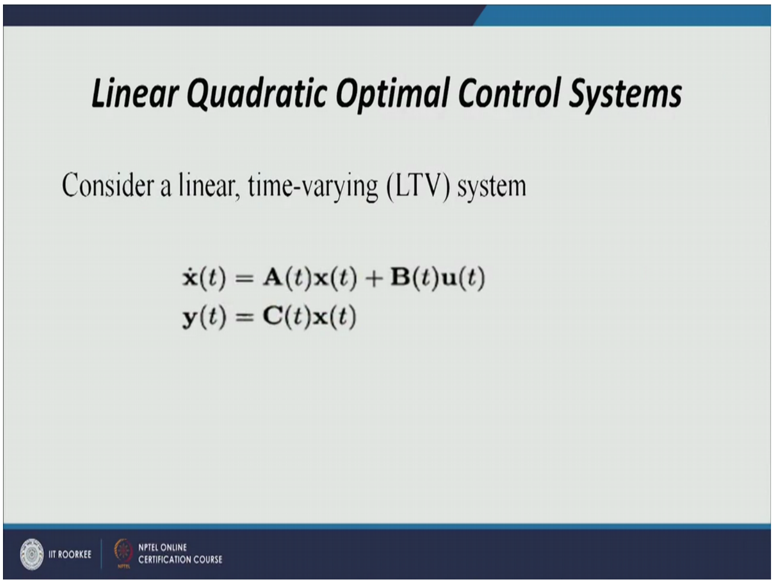

So, just to review what we have done: we have considered a linear system as x dot t equal to A t x t plus B t u t y t equal to C t x t, A, B, C are the function of time, this means we are considering the time varying system.

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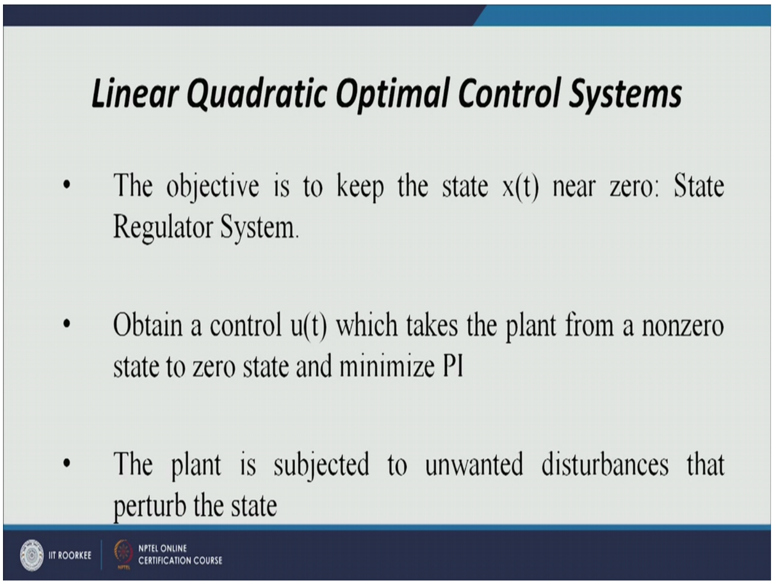

So, for a linear time varying system our objective is to obtain a control which will take the plant to nonzero state to 0 state and minimize the PI.

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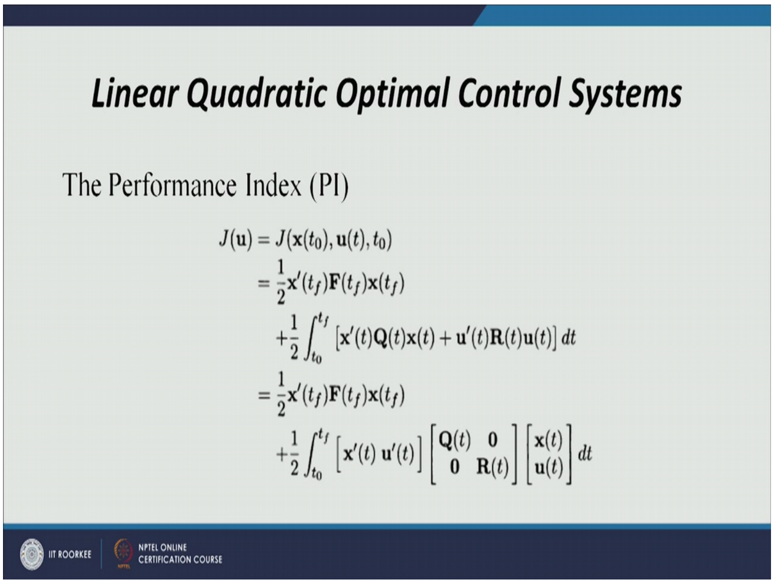

PI is a quadratic objective function which will contain the x transpose t f f t f x t f which is my terminal cost plus half of x prime Q x plus u prime R u. So, x Q x represent my error u R u u prime R u will represent my control efforts. So, I am trying to minimize the terminal cost, the error because my final destination is 0. So, any point will nothing but representing me the error. So, each state at any given instant representing the error and u prime R u is nothing but my I am trying to minimize the control efforts.

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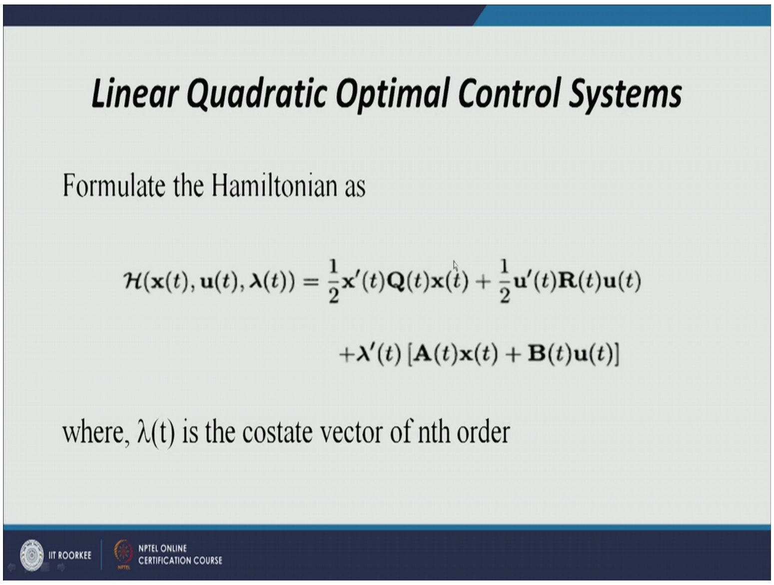

And this is LQR problem we solved using the Hamiltonian approach. So, first we represent a Hamiltonian which is nothing but my half of x prime qx plus u prime R u plus lambda prime a x plus B u.

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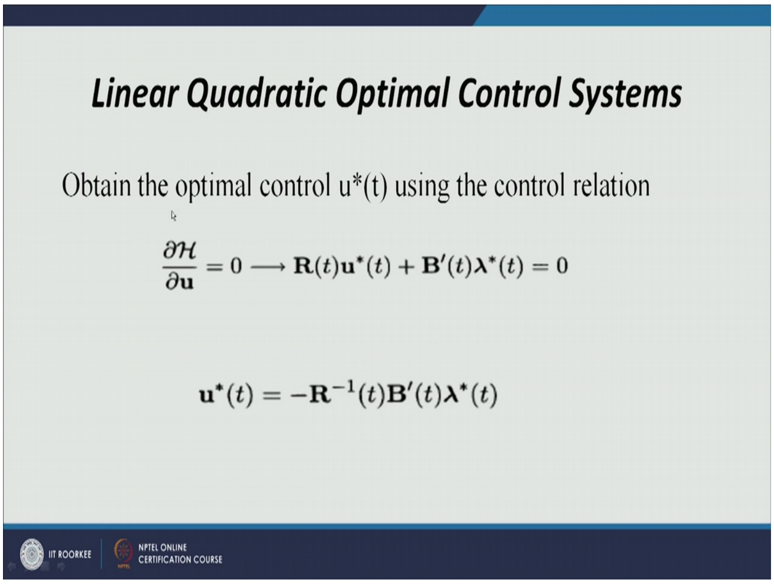

With this my first condition is del H by del u equal to 0 which is giving me R inverse B prime lambda.

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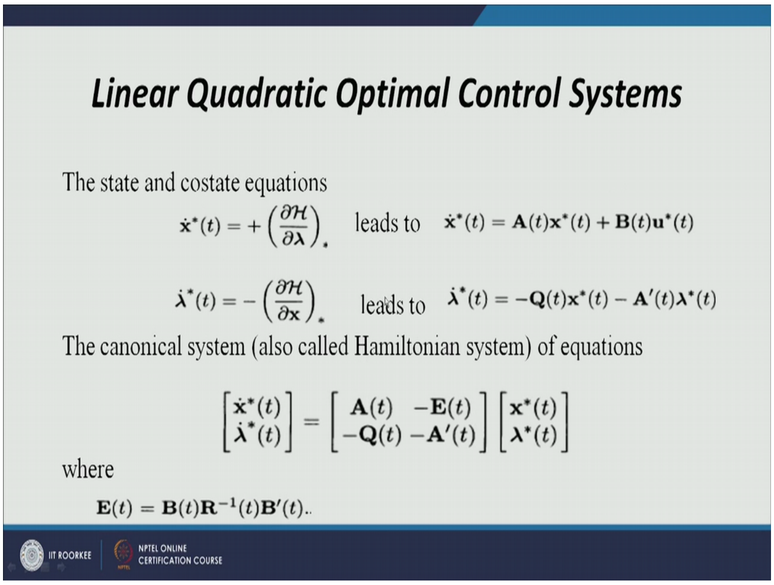

And then I will have the state and the costate equation which giving me the Hamiltonian system representing as x dot t equal to A t e t is B R inverse B prime Q minus Q minus a prime x and lambda.

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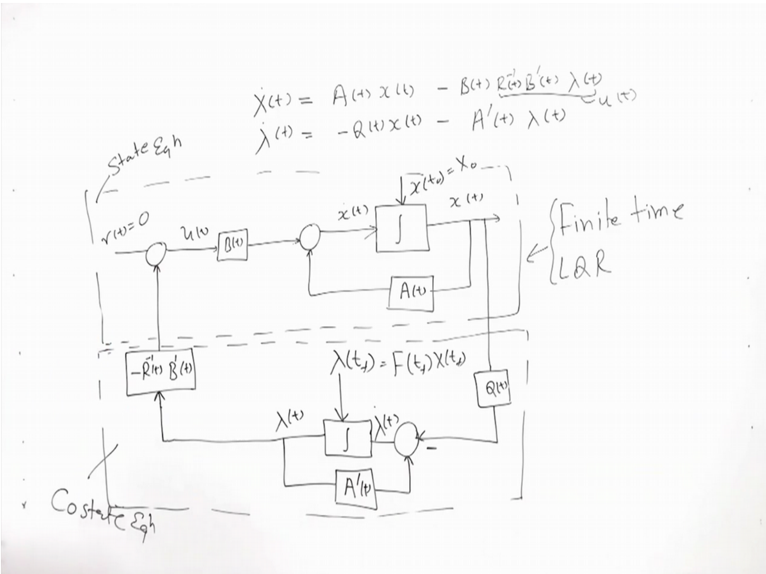

So, basically if we will see this Hamiltonian system is giving me the two equations: one is I am representing my x dot t in terms of the A t x t and another term is my e t; e t is B R inverse B prime t R is also the function of time into lambda t. So, this is my first equation: A t x t minus e t lambda t here e is B R inverse B prime. And my costate equation is x t as minus Q t x t minus a transpose t lambda t. So, I have the state equation as well as my costate equation.

If I will represent the system; so if this as my x t integral so this will be my x dot of t. So, A t x t plus B t and this is my nothing but ut, this R t we are considering to be 0. So, this is my state equation, because R inverse B prime lambda this is nothing but my ut. So, I am writing the first equation as x dot equal to A t x t plus B t u t. If I will see the second equation x t is multiplied with Q t. So, what I am getting? I am getting lambda t; sorry lambda dot t is Q t x t with negative sign so I consider my negative sign to be here integral, I got the lambda t multiplied with a transpose t. So, here this is my lambda dot t is Q t x t plus lambda t a transpose t.

Lambda t is multiplied with R inverse B transpose: R inverse t B transpose t to get my u. So, I am taking negative sign with this. So, this is minus R inverse B prime multiplied with this, so this block diagram completely represent by state and the costate equation. And if we will see this upper portion is representing my state equation, while the lower portion if we will see this part is nothing but my costate equation or say system anything we can call. So, what actually for a given plant we have to write an algorithm to find out the value of R inverse B prime lambda. This we are trying to determine.

And these are determined with, like in my system I am given with the x t 0 and what is given here: for lambda t if you recall in the previous class, if we will apply our terminal conditions so we got lambda t f equal to F of t f x of t f.

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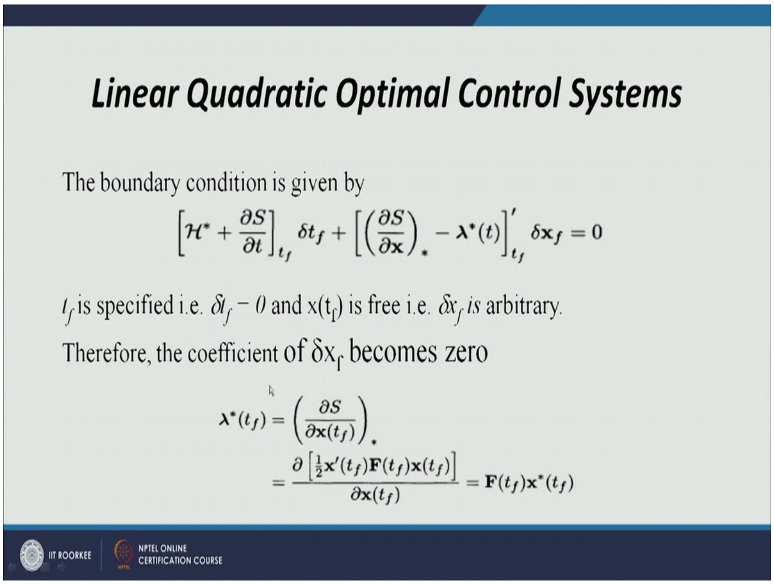

So, with respect to this, so this is at this normally my terminal point lambda t f is defined as sorry F of t f x of t f x t f is my terminal endpoint condition which is I have to transfer my all history to origin, because t f is finite. In a finite time we want to transfer our initial state to end 0 state. So, such kind of a regulator is nothing but my finite time LQR. So, this is nothing but my finite time LQR system is.

The process to solve this problem is we normally assume our lambda t as a matrix P t and x t.

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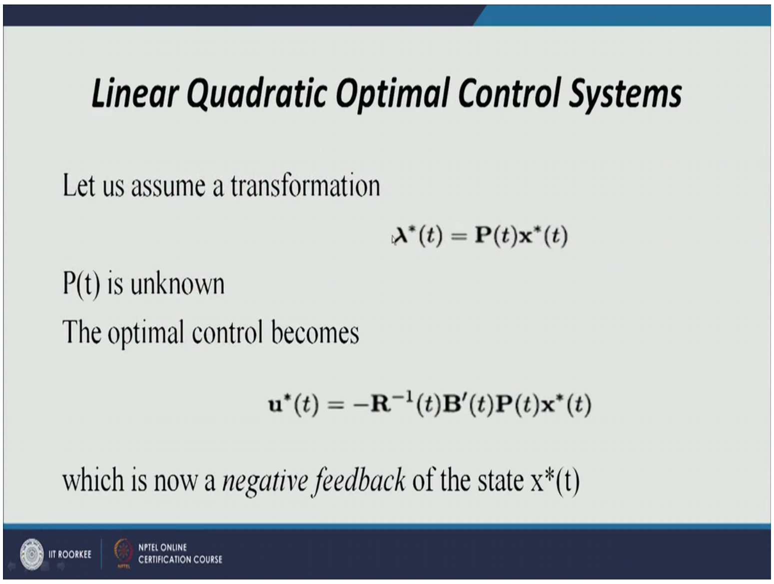

And this as we said before we get the intuition to consider lambda t as p t xt from my terminal condition, because this condition giving me F of t f x of t f: F of t f is my terminal cost and the nature of the F of t f is it is symmetry and positive semidefinite.

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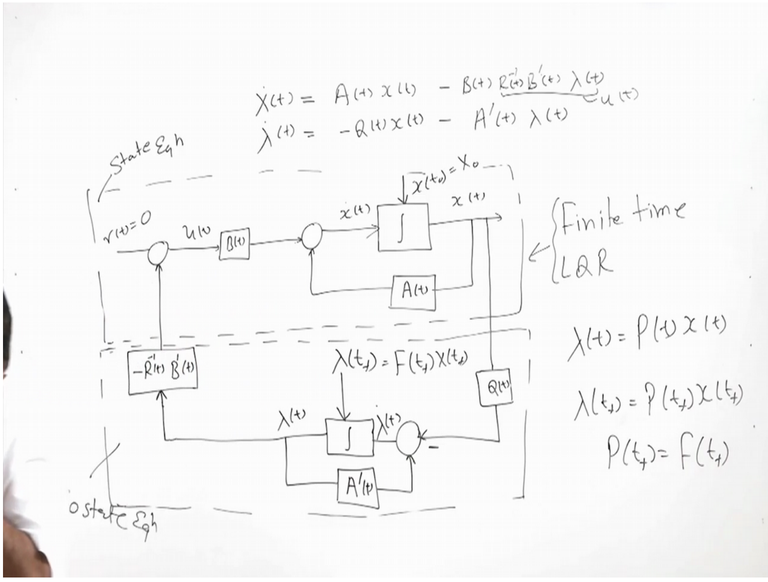

So, naturally my pf t f; if I will consider lambda t as p t x t as we are considering, so at the terminal point my lambda t f is P t f x of t f so this give me nothing but ptf equal to F of t f as F of t f is a symmetric positive definite matrix, so p will also be the symmetric. It will be positive semidefinite or positive definite that we will discuss later on, but we select ptf as a symmetric positive definite matrix. It is a symmetric because of the symmetry of my f.

So, this p we will select as symmetric positive definite matrix.

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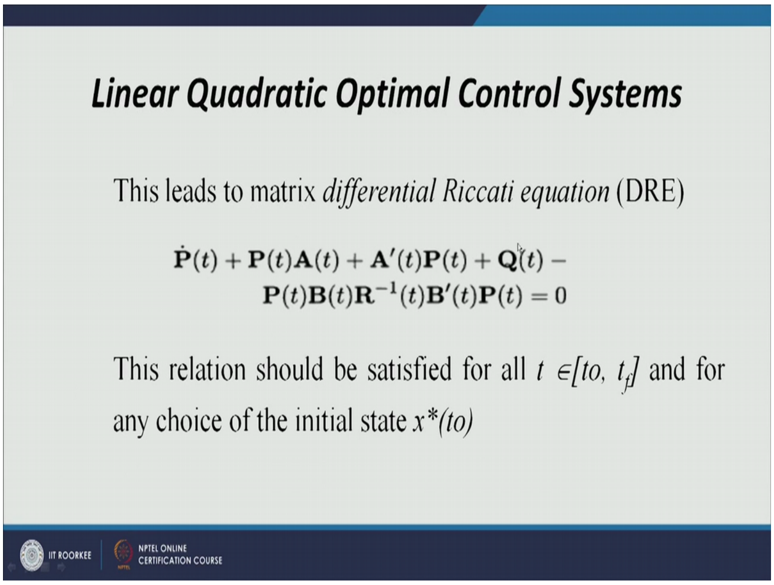

So, if I know t, I can directly implement my close loop control as minus R inverse B prime p t x t. And if we will just differentiate and in the previous class we have seen we have land up with this matrix differential Riccati equation.

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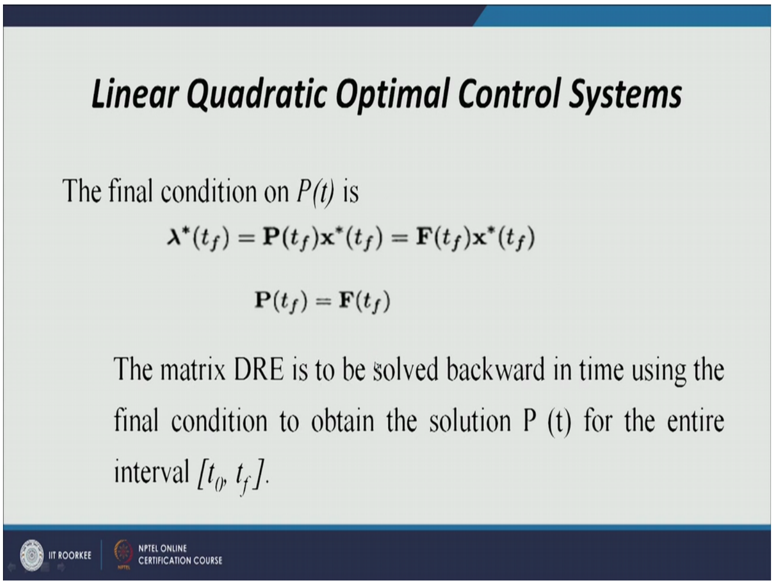

And by solving this equation subjected to p f t equal to f t f we can find out the p t and if we can place the p t here.

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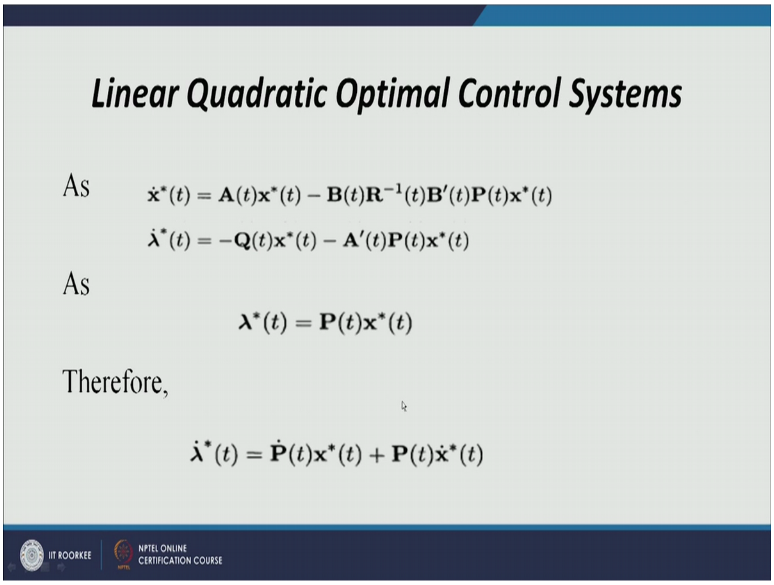

We can have the close loop feedback control.

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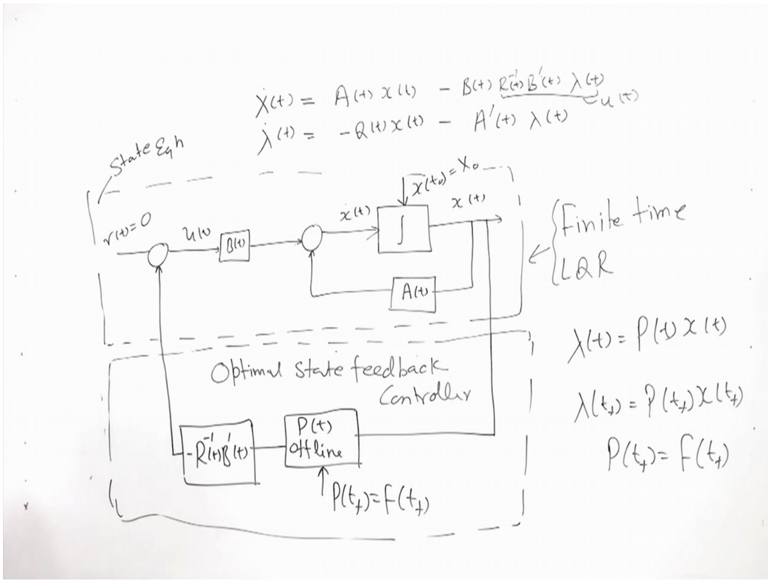

This means if in this because here all calculations we have to made to make it a close loop. So, if I will modify this. Now, directly I am taking the feedback, I have to find out the P t and this we can find out offline. So, this P t can be determined with the condition P t f equal to F of t f. And if I know the P t I can have my R inverse B transpose P t if I will and can directly give as by u t.

So, in a feedback case this is nothing but my optimal state feedback controller. So, by solving my matrix differential Riccati equation, by getting the value of the p which I can solve offline with this value of the P t if I will implement here so this is nothing but my optimal state feedback controller and the overall problem is my finite time linear quadratic regulator problem.

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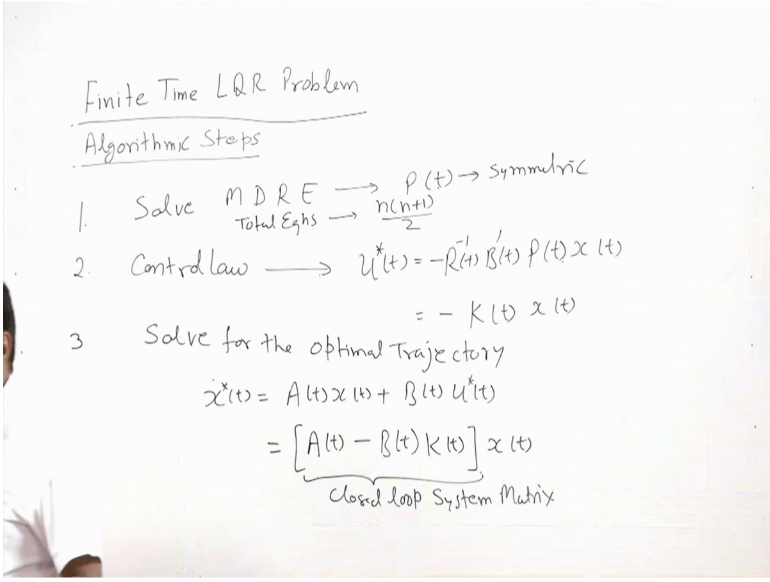

So, in this way we can solve this problem simply by solving this matrix Riccati equation. So, we can write the algorithmic steps required to solve a finite time regulator problem.

To get the optimal control using the finite time regulator problem for the final time regulator problem my algorithmic steps will be: first one I have to solve the matrix differential, Riccati equation; what we will get by this we will get the pt. So, for a P t is known; if p t is known then I can find out the control law. And what is my control law? My optimal control law will be minus R inverse B transpose P t into x of t. And this I can simply write my k t x t. So, my control law is k t x t; so we can determine the k t if we know the value of the p t. So, my optimal control law will be u t equal to minus k t x t.

My third step is solve for the optimal trajectory. So, what is the optimal trajectory? My system is x t x dot equal to A t x t plus B t u t. As if I will use u to be the optimal. So, my trajectory I can say will be the optimal. So, u star t if I will take this is nothing but A t minus k t sorry sorry; this is A t minus B t. And I am replacing u t by minus k t x t, so negative sign I have taken here B t k t x t.

So, this A t minus B t k t is my new; this is my closed loop system matrix. So, whatever be the state of the A t we will show later on my closed loop system matrix will be a stable matrix; if we are able to find out the value of the P t such that u t can be determined. So, by simple solution of the P t I can develop a complete state feedback control law to control my system. As we have seen in the previous lecture for a first order system we can easily solve manually my Riccati equation.

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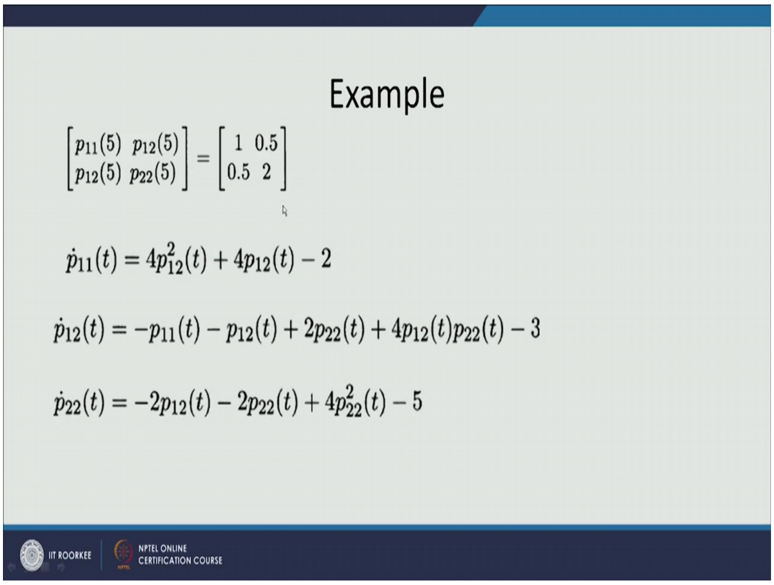

But in another example we have seen if my system is of the second order then I will get 3 non-linear differential equations. And these 3 non-linear differential equations has to be solved with the final condition given as my P t f equal to F of t f. This is my F of t f this is my p of t f. So, I know p 11 p 22 p 12; these 3 parameter I have to find.

So, for a total number of the equations if we will see for matrix d R e as p t is symmetric, so the total number of equations will be total equations will be n into n plus 1 by 2. So, for a second order system I will have the 3 equation and as the order of the system increases this number of the equations will increase. Now question here is how to solve this equation? These equations can be solved using any numerical technique, like your Runge-Kutta, Euler approach utilizing this we can solve this equation, but to solve them manually is not a easy task. So, any differential equation solution technique we can adopt to solve these equations.

So, next we will see the some silent feature of our Riccati equation as well as the regulator system.

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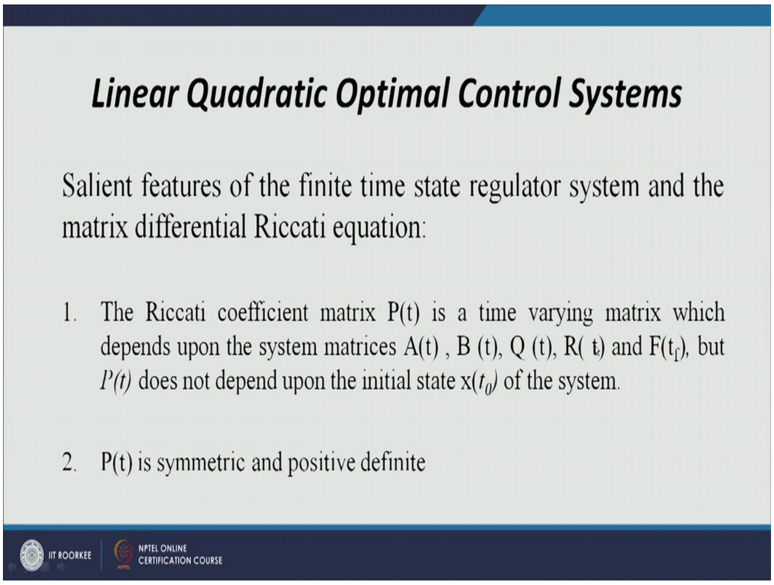

So, Riccati coefficient matrix is a time varying matrix which depends on the matrix A t B t Q R and F, but P t does not depend on the initial state x t 0. So, this means whatever be my state if I know my system matrix like my A B matrix Q and R matrix and the terminal cost matrix F. So, whatever be the initial condition of my system I can develop my state regulator to drive this state to the origin, because I have to shift my; sorry I have to transfer my all the initial state to my final state which we are taking near the 0. As we have seen before.

So, second feature is my P t should be symmetric as we have discussed this is symmetric because f of t P t f equal to F of t f and F of t f is a symmetric matrix so p t will be also be the symmetric matrix because p t is matrix so my Riccati equation.

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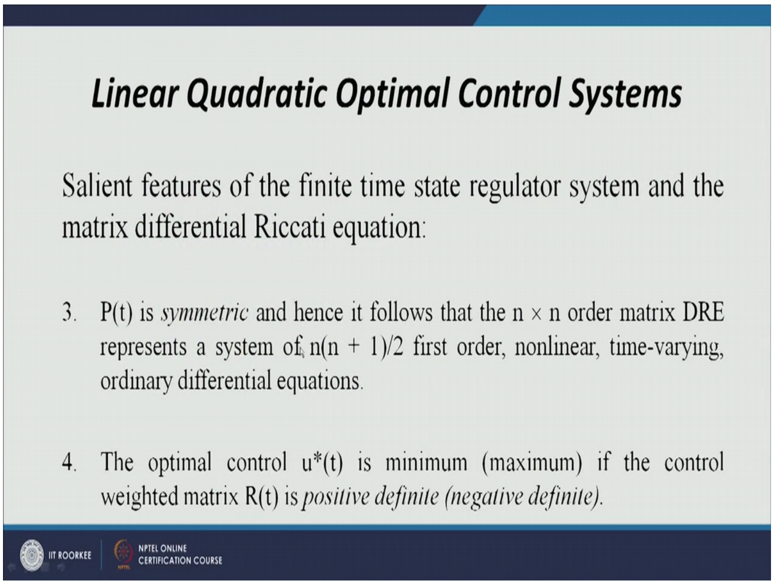

Which is p dot must have n into n plus 1 by 2 equation to get all the parameters of my p matrix. As we have seen for a second order case my P is p 11 p 12 p 12 p 22. So, total number of the Riccati equations will be having n into n plus 1 by 2. And these are the first order non-linear time varying ordinary differential equation and any differential equation solution algorithm. We can utilize to solve this equation like we have said Runga-Kutta, Euler methods can be used to solve these differential equations. U t is the optimal control u star t is minimum if the control weighted matrix is positive definite.

So, it is minimum if it is positive definite, R t is positive definite and maximum if it is negative definite. So, we will discuss in detail in the sufficient condition why this condition is there. So, actually is the R t.

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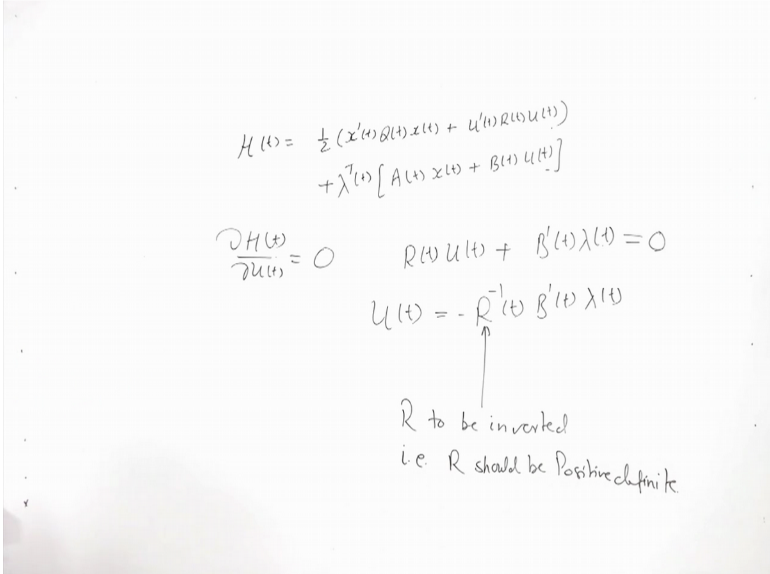

So, if we recall we have defined the H as half of x prime t Q t x t plus u prime R u plus lambda transpose A t x t plus B t u t. And for optimal control we have the condition del H by del u to be 0. And what was our condition, if we are solving this? This was R t u t plus B transpose lambda t equals to 0.

So, if I have to find out the u t I need to be R to be inverted R inverse t B transpose t lambda t. So, R to be inverted this means R should be positive definite: so R should be positive definite. So, my R should be positive definite and later we will show if R is positive definite then it will nothing but satisfy the condition of minima which is my sufficient condition, which in the later part of our discussion we will prove this.

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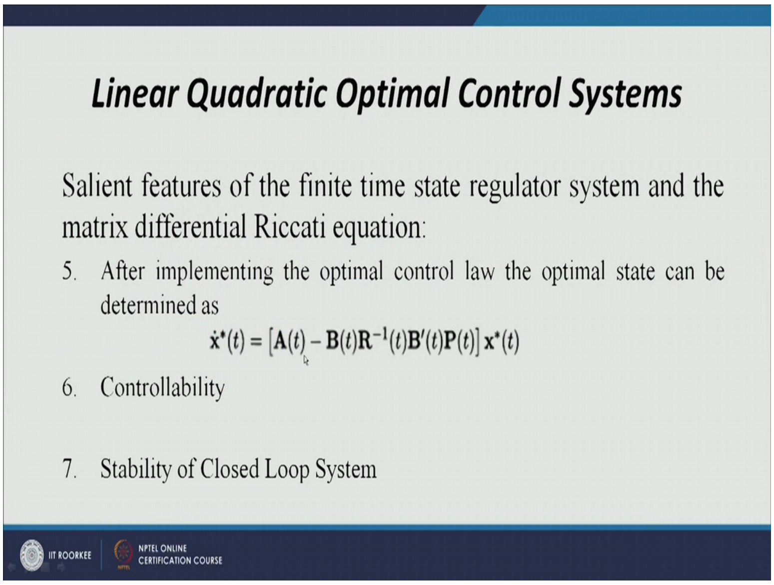

So, after implementing let like we have seen my closed loop system will be nothing but a transpose minus B R inverse B prime p t into x t. So, this B t R inverse B prime p this is showing nothing but my k t, so this is A t minus B t k t x t. So, this is my closed loop matrix. And we will also show that this matrix nothing but give me a stable closed loop system.

Next point is the controllability. So, in a finite time regulator do we really require my system to be controllable? So the answer for this is no, because in a finite time if I will solve my system. So, for a finite time I always get my all values to be the finite values, whether it is a state or control. So, this means till my time is finite, my all other variable all the states which I am finding, their value will also be the finite. So, if I will have the finite values this means I am satisfying my bounded input, bounded output condition.

So, whether my system is controllable or not I am able to find out the some numeric value for the given to states; sorry I can find out the some numerical value for my all the states. So, for a finite time regulator particularly it is not required my system to be the controllable system. But, in the later part as my t f will approach to infinity then to get the solution my system must be a controllable system.

Next is the stability of the closed loop system. As my another feature: as we said we can prove we will prove here that my closed loop system is a stable system. So, this lecture I will stop here. And in the next lecturer we will prove that my closed loop system is a stable system.

Thank you very much.