Optimal Control

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Lecture - 13
Variational Approach To Optimal Control Systems (Continued)

Welcome to all of you. In today's lecture we will continue with the variational approach to optimal control system. Just to quick review in the previous class we have start our problem on the variational approach, it is applicable to optimal control system we considered a plant x dot with condition, initial point is given, final point is free, we consider our performance index as J equal to with terminal cost and integral cost which is given here.

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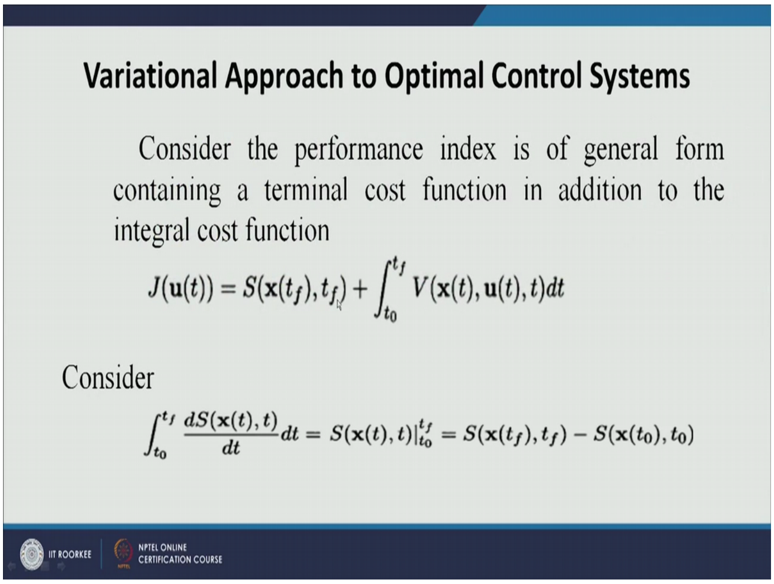

With a mathematical consideration we define our PIS, V plus dS by dt and plant is this. So, objective is to find out the optimal u which will minimize the J and satisfy the plant. Our approach is the Lagrangian approach.

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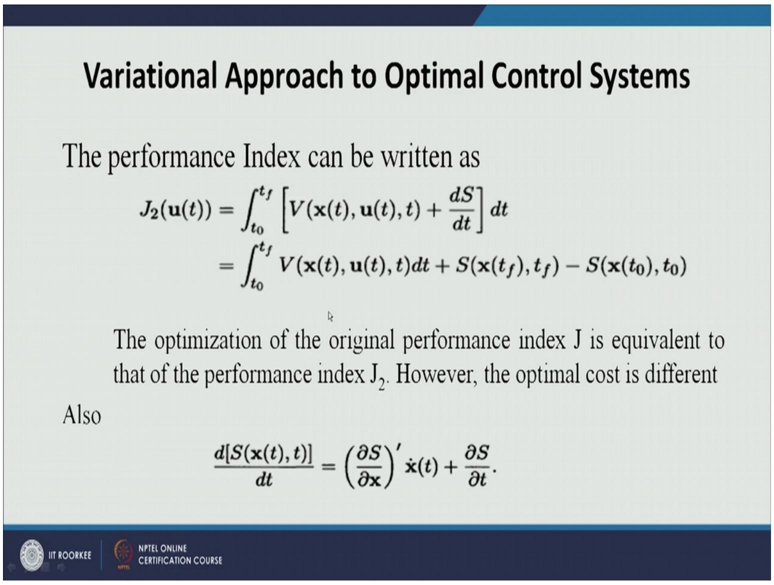

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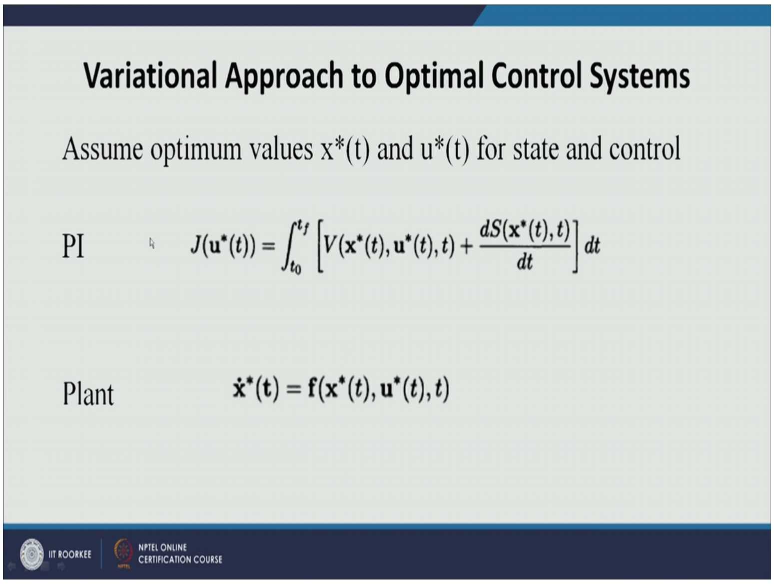

So, first we define our Lagrangian L as V plus lambda prime g, g is this is our condition which we have taken in the previous lecture.

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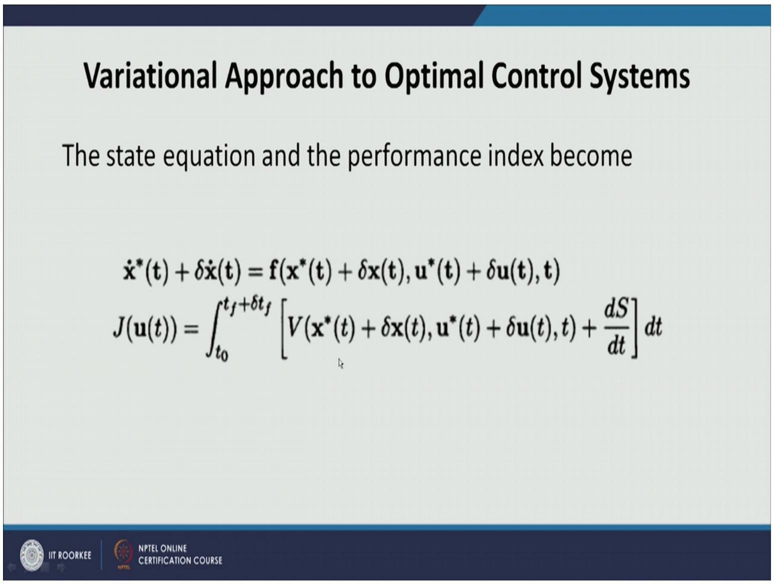

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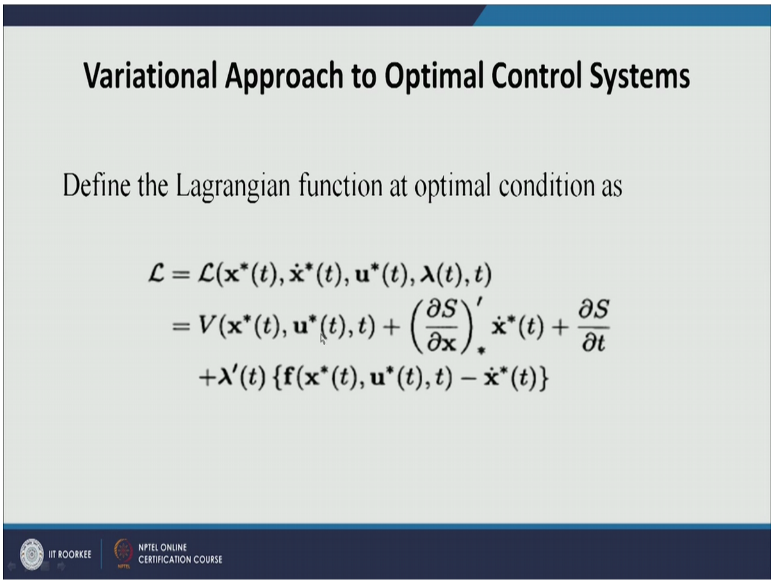

So, these wholes define the V plus lambda prime and this is my g. So, this is my Lagrangian at the optimal point, define Lagrangian at the variation point.

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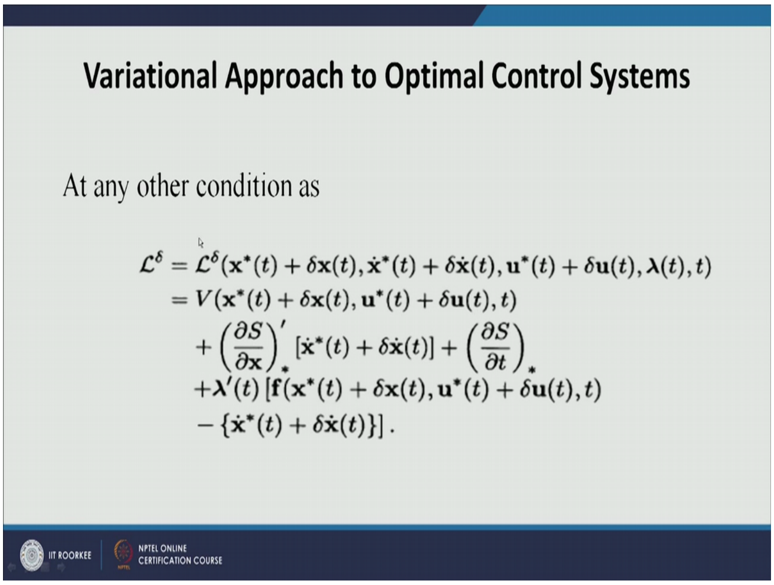

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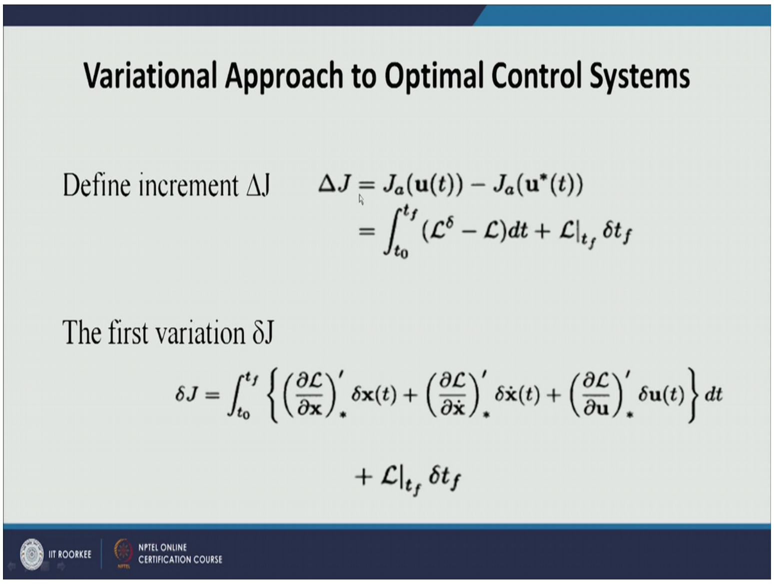

Find out the increment delta J as J u t minus J u star t which nothing but from this increment we find out the first variation delta J which is given with this expression plus L t f delta t f.

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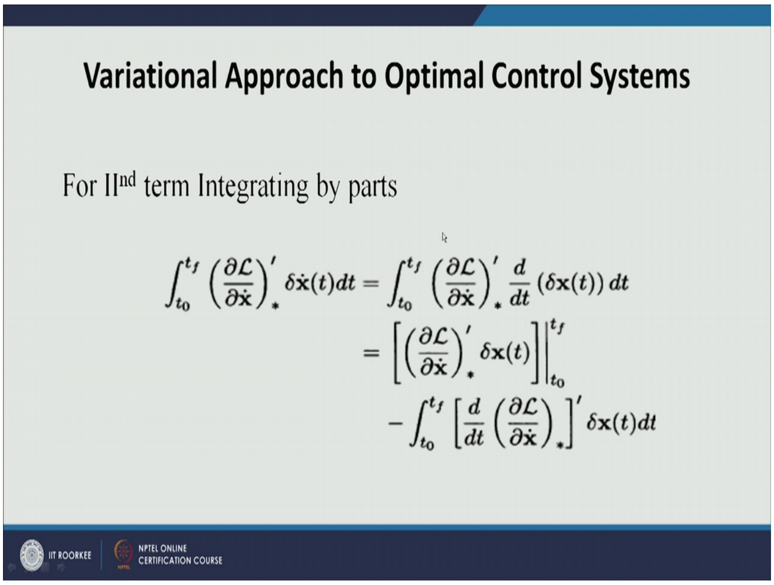

If the second term which is del L by del x, so delta x dot we expand by the integral by parts and we get the final equation is del L by del x minus d by dt del L by del x dot delta x t del L by del u delta u plus whatever is by terminal condition which is given.

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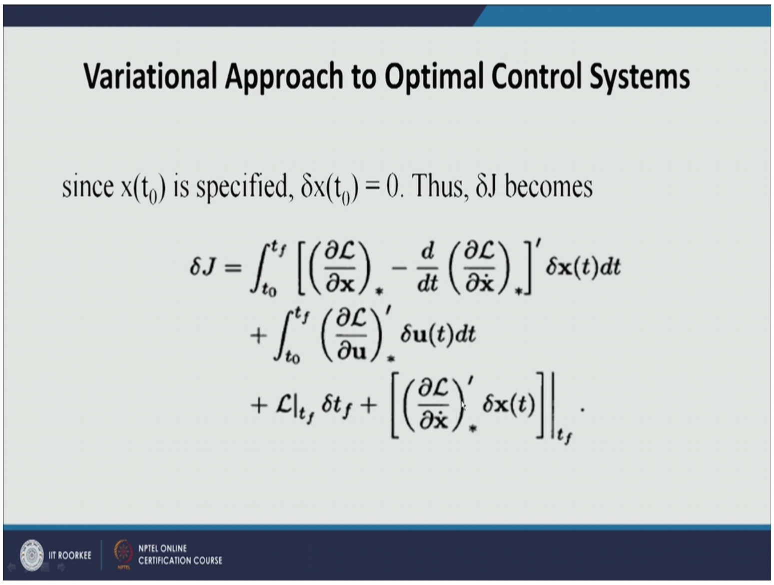

So, here the 2 variables delta x and delta u we treat u to be the independent, x to be the dependent, we select lambda such that the coefficient of the delta x will be 0.

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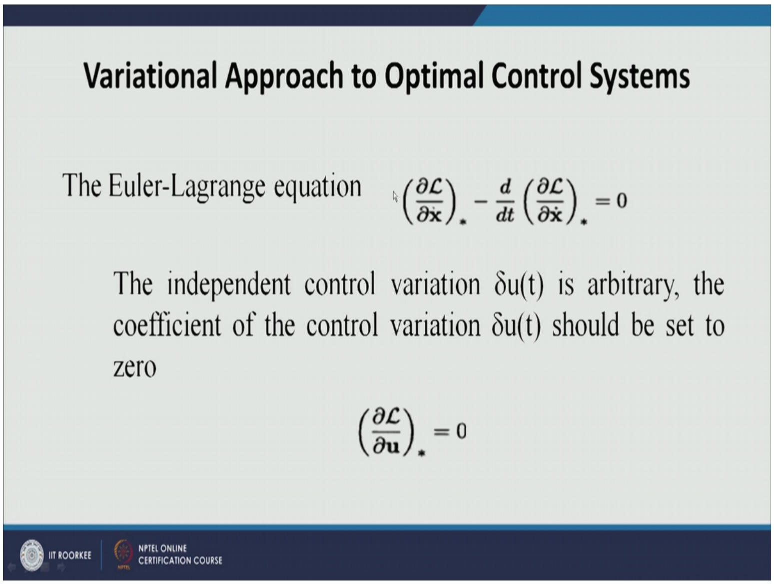

So, this give me the EL equation as del L by del x dot minus d by dt del L by del x dot equal to 0. So, if this is 0 then independent variable coefficient of delta u will also be 0. So, del L by del u is 0 and my terminal condition is L of t f plus del L by del x dot delta x t at t f point that equal to 0. So, up to this we have covered in the previous class.

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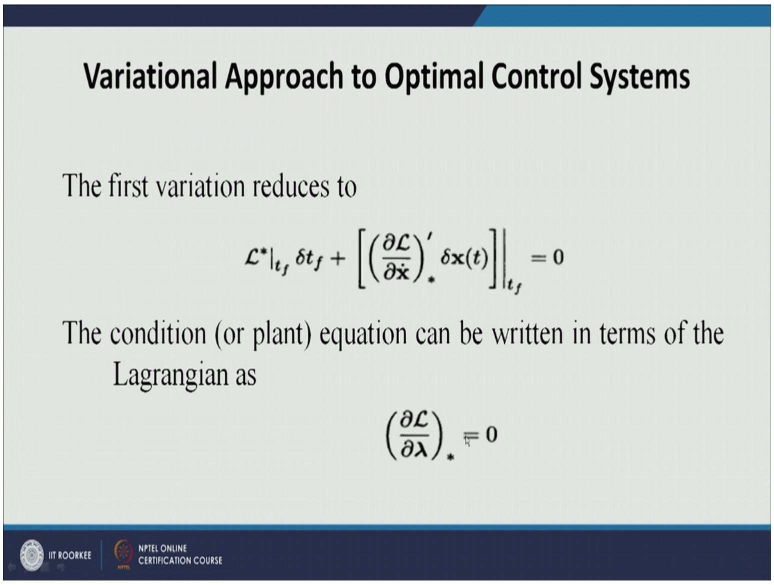

Now, the terminal condition is given with the delta t f which is the variation at the t f point and delta x of t f.

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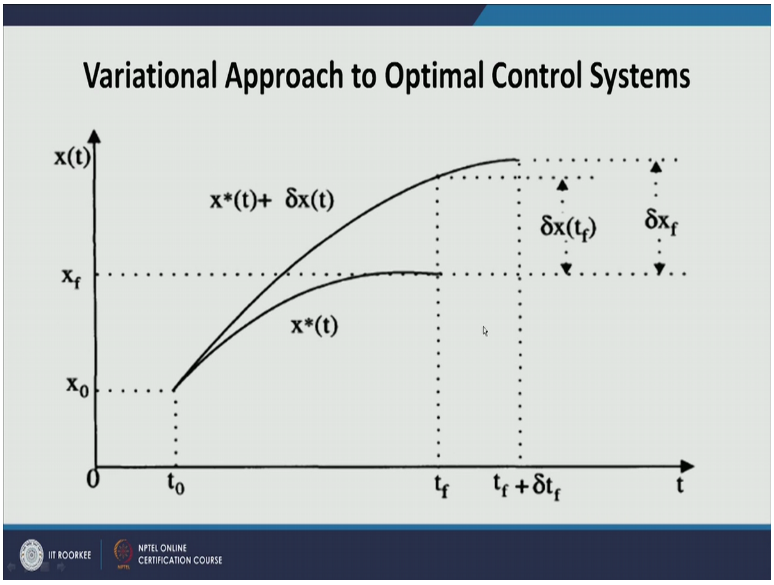

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Variational Approach to Optimal Control Systems

$$\dot{\mathbf{x}}^*(t_f) + \delta\dot{\mathbf{x}}(t_f) \approx \frac{\delta\mathbf{x}_f - \delta\mathbf{x}(t_f)}{\delta t_f}$$

$$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) + \{\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)\}\,\delta t_f$$

$$\delta\mathbf{x}(t_f) = \delta\mathbf{x}_f - \dot{\mathbf{x}}^*(t_f)\delta t_f.$$

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So, if this is the delta x of t f delta x f. So, what I can write? I can approximate x dot of t f plus delta x dot of t f is delta x of minus delta x of t f t f. What is this? Say delta x of t f is what value of the x t at optimal point plus delta x which is the variation point. So, this value will have t f a delta t f point is this value; x f is the value at the t f point and at t f point value of x star of t f del that will be this value. So, difference between these 2 is my delta x of t f and difference between the highest the final value m value of x t at that t f point is delta x f.

So, what we are saying, x dot of t f value of the optimal point plus what is the delta x dot of t f that we are approximating as delta x f minus delta x of t f at this because this is the value at t f point and this is the value of t f plus delta t f point. The difference between these 2 and divided by delta t f the difference between, so this is we are saying nothing but my x dt of t f plus delta x dot of t f. So, this I can approximate with the value delta x of f minus delta x of t f divided by delta t f. So, with this value if I will cross multiply this. So, I will get x dot of t f multiplied with delta t f delta x dot of t f t multiplied with the delta t f these 2 value and if I will take is the delta x of t f. So, this equation I can write in this particular form and from here I can find out the value of delta x of t f.

So, what actually we are doing with this expression, the final condition which we have taken as the L star at t equal to t f point multiplied with delta t f plus del L by del x prime multiplied with delta x t at t equal to t f point and I am saying my this condition is equal to 0.

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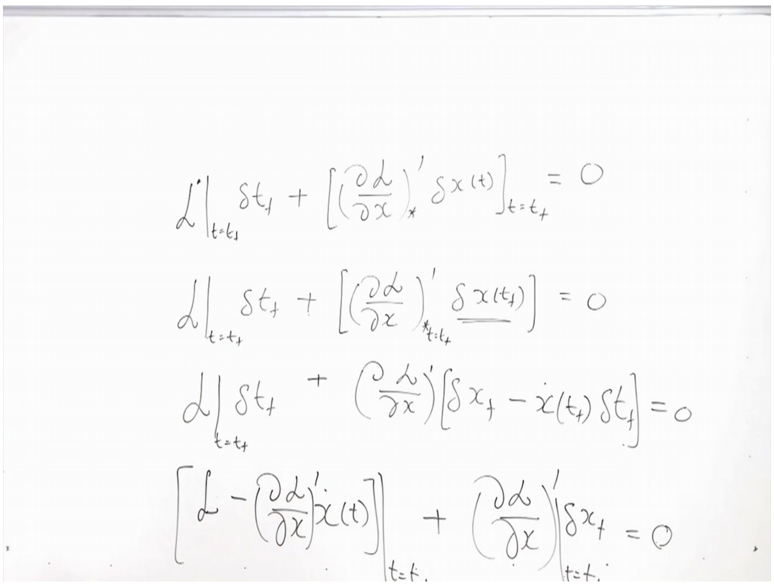

So, in this condition I will place delta x of t f S delta x f and t f t f point delta t f plus and delta x of t f which is this. So, I take delta L by del x prime and delta x of t f and this is also t equal to t f point, so this equal to 0. So, this delta x of t f I am replacing with delta x f this is my delta x f minus x dot of t f delta t f and this equal to 0.

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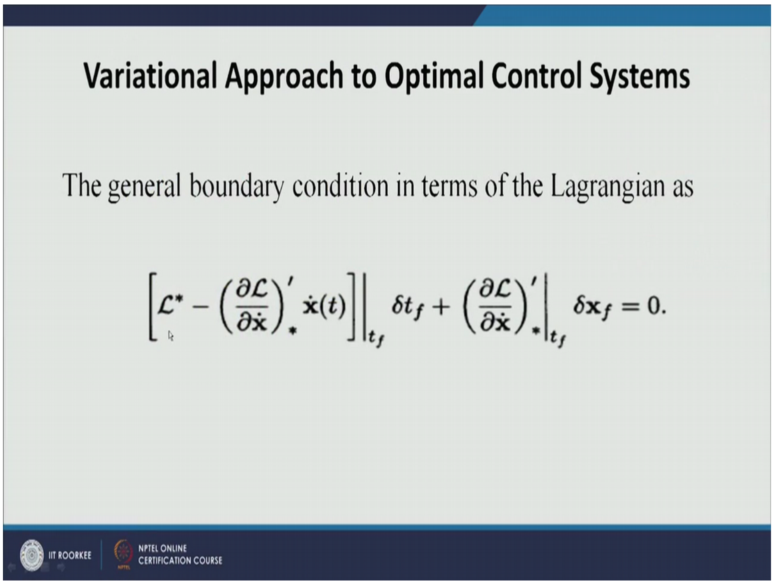

So, I can club this L delta t f, this delta t f term I can club together to get my final relation as L minus del L by. So, this is nothing but L minus del L by del x del, L by del x multiplied with x dot t this I will evaluate at t equal to t f point plus del L by del x prime delta x f and this sum I will integrate t equal to t f and this equal to 0.

So, my final condition will be can be represented by this equation which is L minus del L by del x prime delta x dot sorry; in this if I am taking this. So, this is nothing but x prime and this is evaluated at t f point similarly del L by del x dot evaluated at the t f point delta x f this equal to 0, is my final condition.

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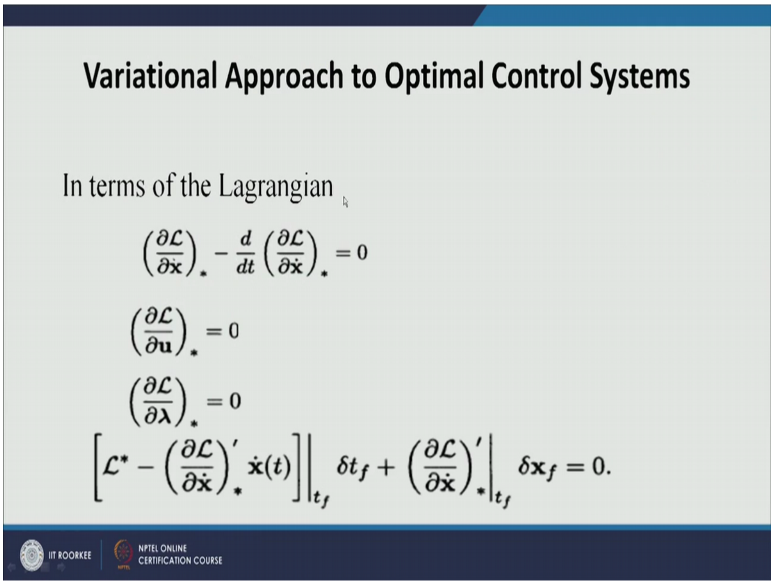

So, with this I have my Lagrangian equation says del L by del x dot minus d by d t. So, first condition I get as my EL equation, second is del L by del u 0 my control equation this is my del L by del lambda equal to 0 this nothing but give me my dl L by del lambda equal to 0 and this is my final boundary condition. So, these conditions can be used to solve any optimal control problem in which u is not constraint. This can further be simplified if we will define a term called Hamiltonian.

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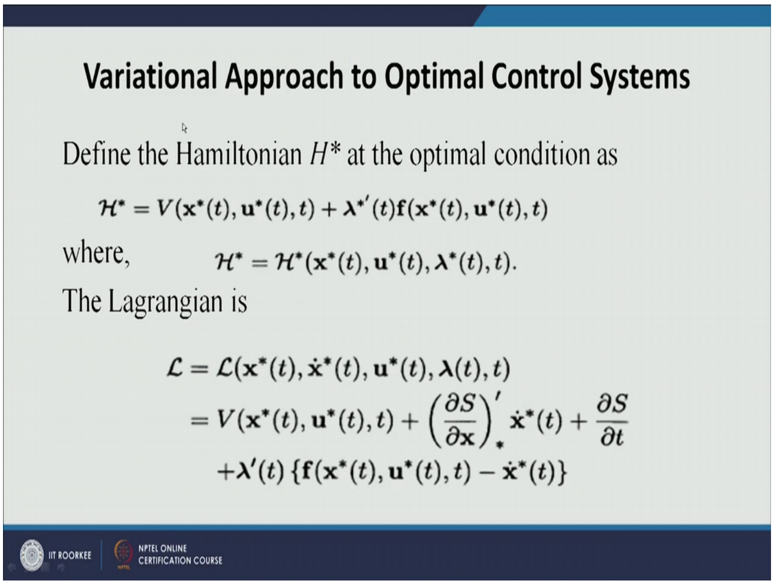

Hamiltonian is defined simply as my V plus lambda prime f, here V is the integrant of my performance index while f is nothing but my is state equation. So, what actually we have considered.

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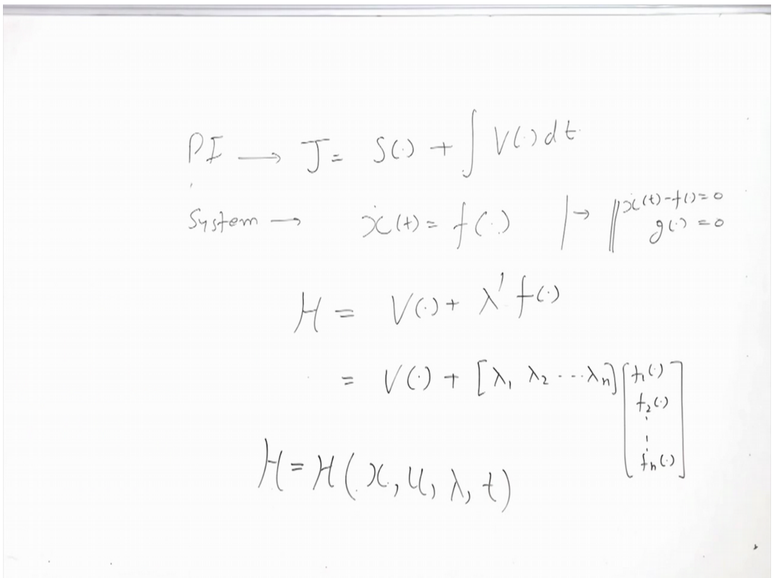

So, from my J which we have defined as the terminal cost plus integral of V dt this was my PI and my system was x dot t equal to f dot which is f x u t. In this I am defining my Hamiltonian simply as V plus sorry lambda prime f, because x is the n number of by states. So, I have x 1 as, x 1 dot t as f 1 x 2 dot as f 2. So, anyway my Hamiltonian simply is defined as lambda prime we are writing as lambda 1, lambda 2, lambda n; f 1, f 2, f n.

So, whatever is my system that I am taking as a condition my condition in case of Lagrangian we have taken the x dot t minus f equal to 0 which I am representing as the g equal to 0. So, this was condition in case of the Lagrangian, but in defining the H we take this f along with the V represented in this way. So, Hamiltonian H can be written as has a function of x u lambda t. So, we are defining our H as V plus lambda prime f where H is a function of x u lambda n t, we already have define our Lagrangian as V plus del S by del x, x dot t plus del S by del t which is nothing but my, integral inside the integral cos function plus lambda prime and this representing my g.

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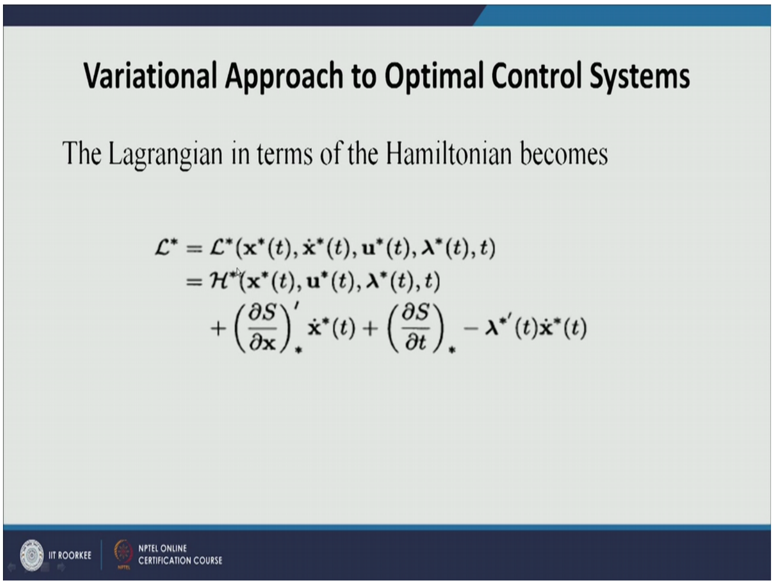

So, what we can do? We can define our L in terms of the H. So, in terms of Hamiltonian if I will define my Lagrangian which is nothing but H which will include V plus lambda prime f and what is rest left del S by del x x dot del S by del t minus lambda prime x dot t. So, H defining is my V plus lambda prime f plus these three terms are left out from this - number 1, number 2 and the third is x dot multiplied with lambda prime. So, we can define my L in terms of H, now, all the four equation which we have seen here, now these equations can be converted into the Hamiltonian form, so that we will see how we can do this.

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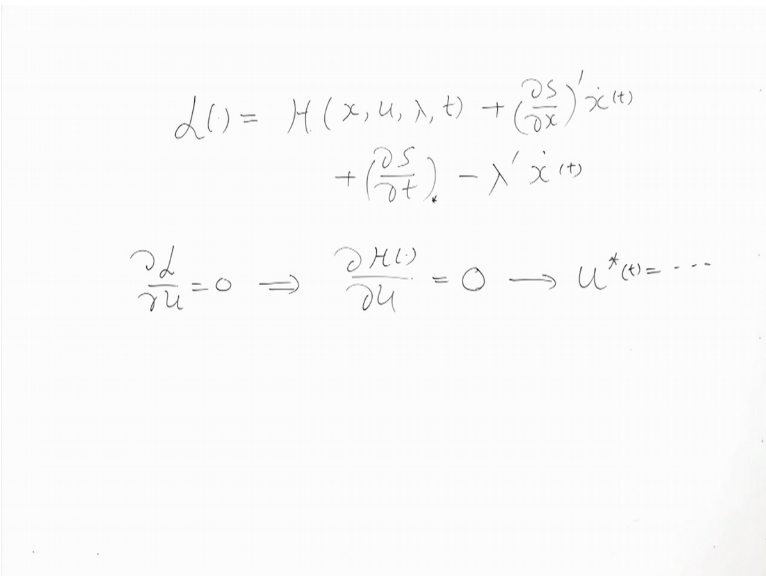

Let us write it on the board what actually is my L in terms of the H see what we are defining L has my H which is a function of x u lambda t plus my term with del S by del x prime d dot t plus del S by del t and minus lambda prime x dot of t. So, this is my L. Now if we will see what is my first equation, for u t that was for optimal control I have my question as del L by del u equal to 0.

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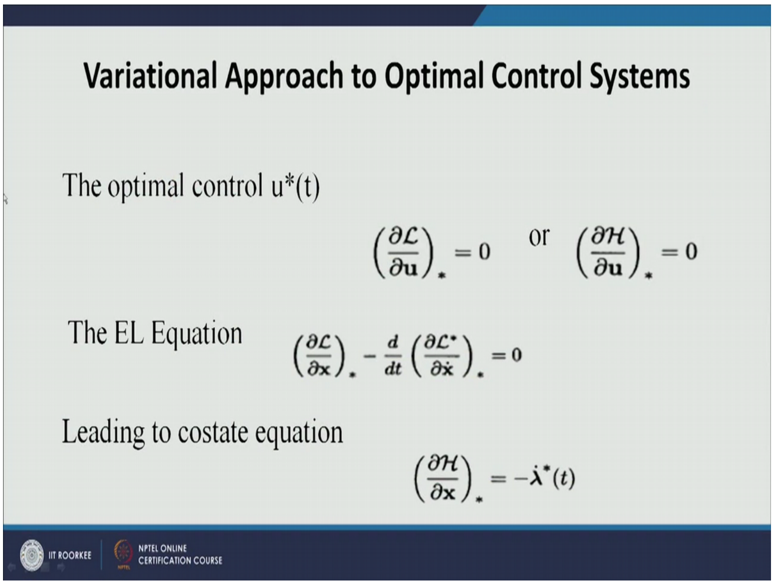

So, my first equation is del L by del u equal to 0, if I will use L as H plus del S by del x x dot del S by del t minus lambda prime x dot t. If we will see only H is a function of u this is independent of u, this term is independent of u, this term is independent of u. So, this nothing but give me del H by del u. So, del L by del u I can later write as del H by del u which is nothing but equal to 0. So, if I will take del H by del u equal to 0 this will give me the control. So, solution of this will give me the u star t whatever be the value I will get.

Then we have the EL equation which is defined as, this we will left out as such this is this was my del S by x dot of t and this was my sorry minus what actually we have lambda prime x dot of t.

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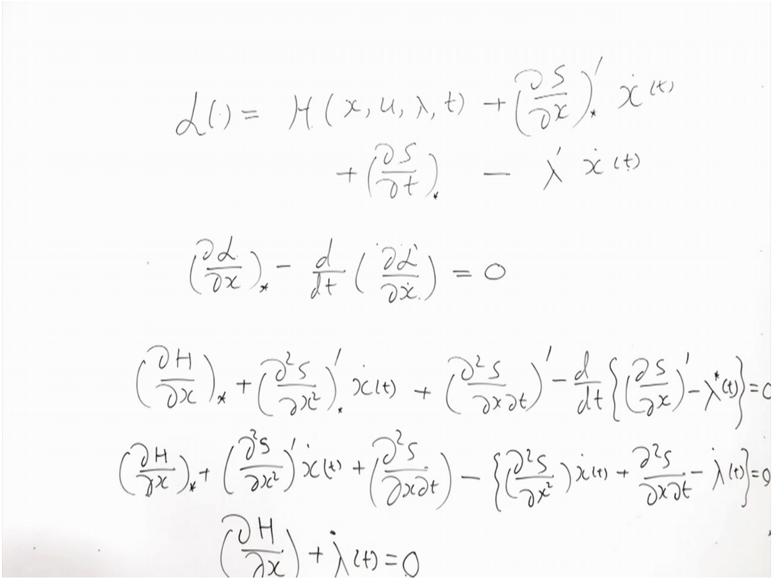

So, this was my L and I have my EL equation as del L by del x minus d by dt of del L by del x dot that is equal to 0. So, if I will take del L by del x this is. So, del H by del x if I will use this plus del 2 S by del x square x dot of t plus del 2 S by del x del t prime minus d by dt del S by del x prime minus lambda star t and this is equal to 0. Say what we are doing? We are taking del L by del x. So, with respect to x I have to differentiate all this term. H is a function of x, so directly we will have del H by del x, del S by del x is also function of x. So, this is del 2 S by del x square x dot of t and x dot if we will say. So, del 2 S by del x del t. So, my first term will give me the first three terms then d by dt of del L by del x dot H is not a function of x dot H is dot a function of x dot, I have a function of x dot as del S by del x prime and another function of x dot is lambda prime x, so minus lambda prime equal to 0.

So, this equation if I will further explain this part d by dt of this what actually we will get. So, I have this term as H del H by del x plus del 2 S by del x square prime x dot of t plus del 2 S by del x del t minus d by dt of del S by del x what actually we will get del 2 S by del x square x dot of t plus del 2 S by del x del t minus lambda dot of t that all equal to 0. Now if you will explain this my this term will cancel out, this time will cancel out and I will left only with del H by del x plus lambda dot of t equal to 0 and this will nothing but leads, so if I will expand my this EL equation in terms of the Lagrangian and Lagrangian I will use it in terms of the Hamiltonian.

So, this equation finally, leads to del H by del x equal to minus lambda dot t and here this is known as by costate equation. So, my first equation was the del L by del u equal to 0 which leads to del H by del u equal to 0. Second equation the EL equation, del L by del x minus d by dt del L by del x dot equal to 0, this will leads to del H by del x equal to minus lambda dot t. So, in this four equation, these 2 we have seen del L by del lambda is 0, from del L by del lambda my H is a function of lambda and this x dot is also a function of lambda.

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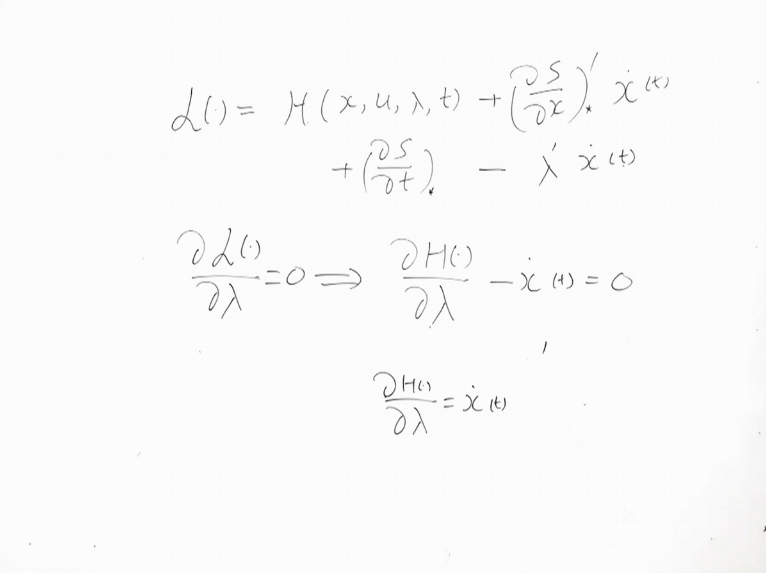

So, now we are seeing the third equation del L by del lambda equal to 0, del L by del lambda equal to 0, this will leads to what? The first H is a function of lambda del H by del lambda and last term minus H dot t that is equal to 0.

So this simply give me del H by del lambda equal to x dot and similarly we can place the value of the L in the terminal condition. So, del L by del lambda leads to this and once we will place the H into my terminal condition sorry L in the terminal condition I get H plus del S by del t delta t f del S by del x dot minus x dot t delta x f equal to 0 this will be my final condition.

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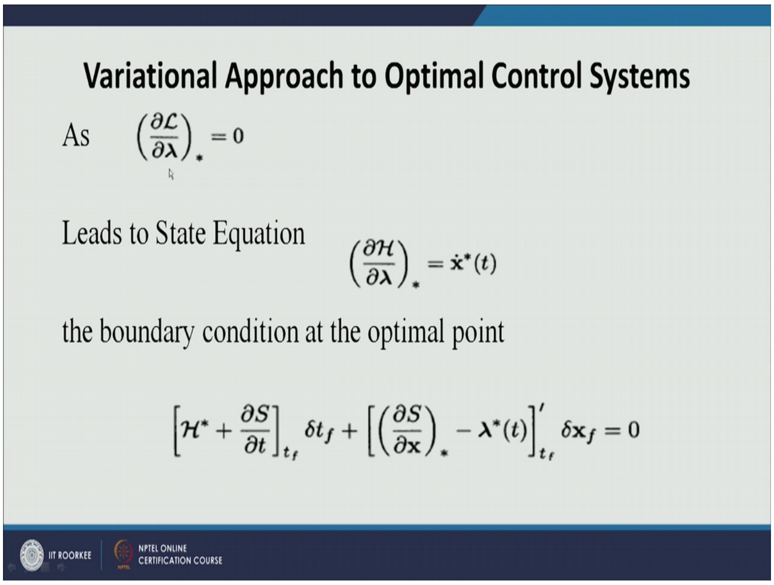

So, in terms of Hamiltonian my equations are del H by del u equal to 0, del H by del lambda equal to x dot which is called my state equation, del H by del x equal to minus lambda dot called my costate equation and boundary condition as given by H plus del S by del t delta t f, del S by del x minus lambda t delta x f that equal to 0.

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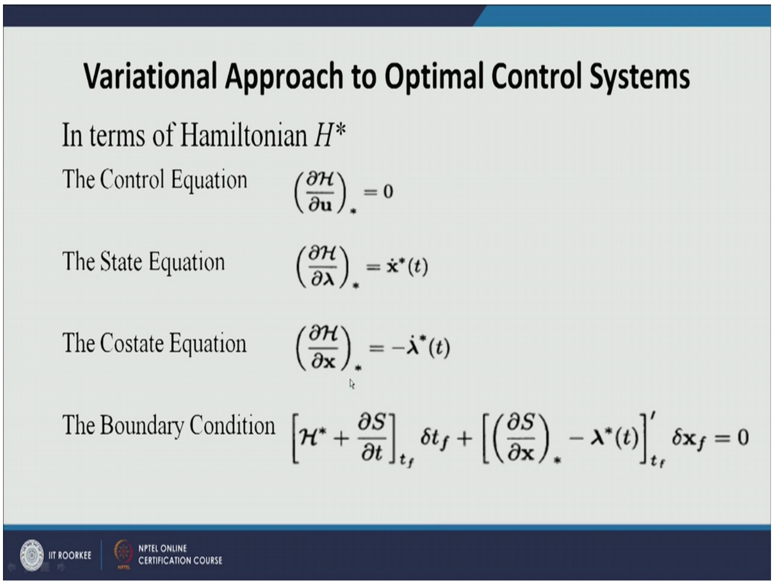

So, four equations in Lagrangian given here can be transformed in terms of the Hamiltonian as these four equations. So, our approach normally here is for a given system we define the H find del H by del u equating to 0 give me the optimal control, I take del H by del lambda, so I will have my state equation; del H by del x I will have my costate equation and these equations can simultaneously be solved using the final boundary condition.

So, what is the final boundary condition? As in this case we have assumed my boundary conditions to be initial point is fixed, but the endpoint is free. So, we have taken a journal case. So, we can have the different type of the system based on my terminal condition, these cases we can consider it as number one we have the fixed final time and the fixed final state. So, these are the equations which can simultaneously be solved to find out to find the optimal control for a given plant which is minimizing a given performance index. Based on the terminal condition we have the different cases in which my final condition is changing according to the x t f and the t f condition. So, these cases we will take up in the next class. So, this class we ending here.

Thank you very much.