Optimal Control
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Lecture – 11
Optimal of Functional with Conditions

Welcome friends, today's class is on the optimum of functional with condition. So, we will find out the optimum value of a given functional which is subjected to the condition, we can formulate the problem in the following manner.

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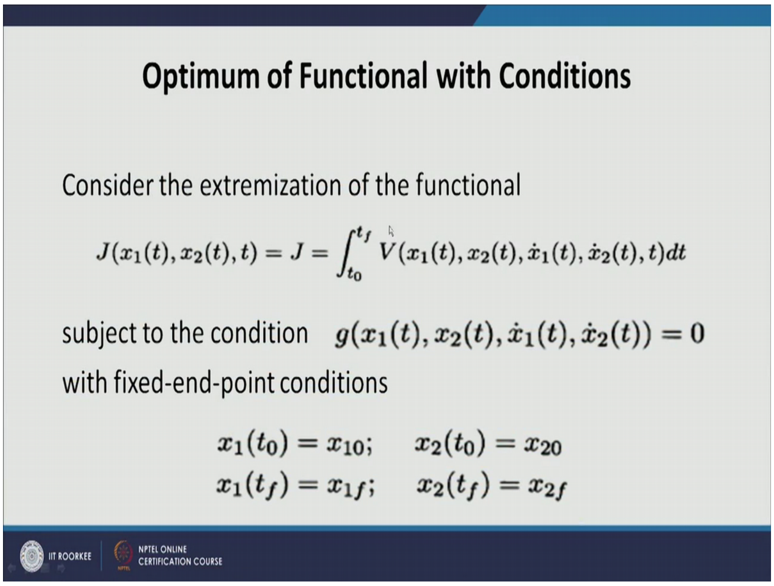

We consider the extremization of the functional J which is the function of functions x 1 t and x 2 t, can be written as t 0 to t f integral V of x 1 t, x 2 t, x 1 dot x 2 dot t, d t. So, objective is to find out the optimal value of this subjected to the condition J of x 1, x 2, x 1 dot, x 2 dot equals to 0.

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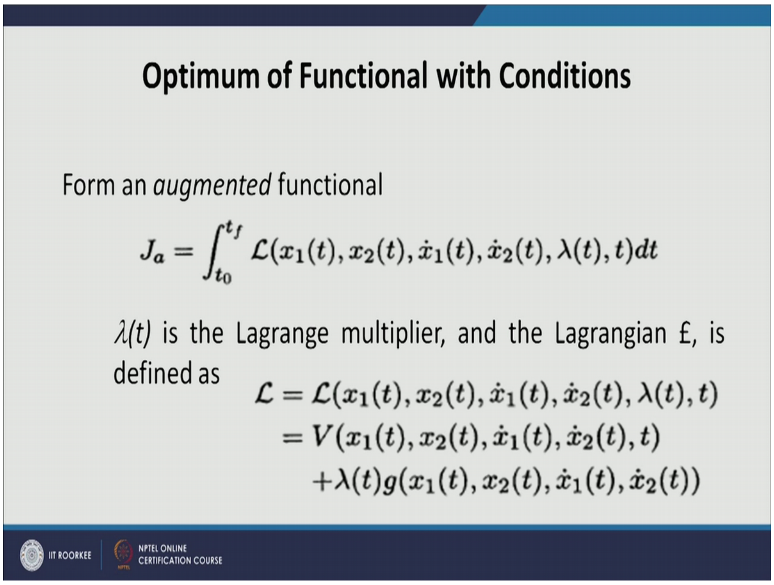

So, for this case, we will consider the fix endpoint problem in which my initial condition for the x 1 and x 2 are given, similarly the final condition for the x 1 and the x 2 are given. So, this problem we will use a Lagrangian approach, first we will define the augmented function J which is nothing, but the Lagrangian which can be defined as V plus lambda g. So, if we define the Lagrangian L as V plus lambda g.

So, we can define our augmented functional as t 0 to t f L of dt. Here L is a function of x 1, x 2, x 1 dot, x 2 dot, lambda n t. Say as our main problem was to minimization of the J, if you will see the minimization of the J this is nothing, but the minimization of J because L we are defining as V plus lambda times g, and this g is nothing but 0. So, if I am minimizing the J again I am minimizing the V as a integrants; from t 0 to t f where you place here take g as a 0. So, this is nothing, but by V dt which is similar to my functional J. So, the minimization of the J or the minimization of the J will give me the same results.

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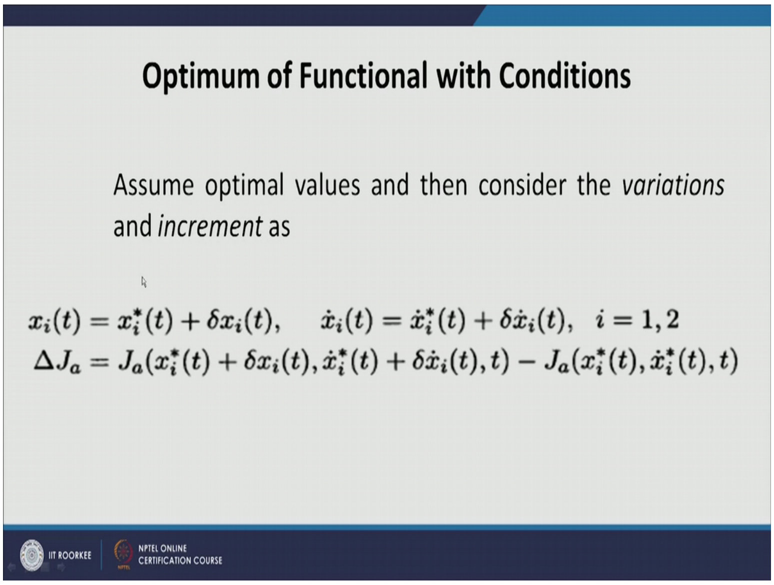

So, please start our problem first we consider the optimum value for x 1 and x 2 say x 1 star and x 2 star. If there is a variation of delta x 1 and delta x 2 in this so, at a variation point I can write my x 1 as x 1 star plus delta x 1, x 2 t s x 2 star plus delta x 2 similarly x 1 dot s x 1 dot t plus delta x 1 dot t, x 2 t as x 2 dot t delta x 2 dot t. With these variation point I define my J similarly at the optimal point I define my J, the difference of these 2 is giving me nothing but the increment.

So, increment I can write as delta J equal to J at the point, at the variation point x plus delta x minus J at the optimal point. So, these differences give me nothing, but my increment.

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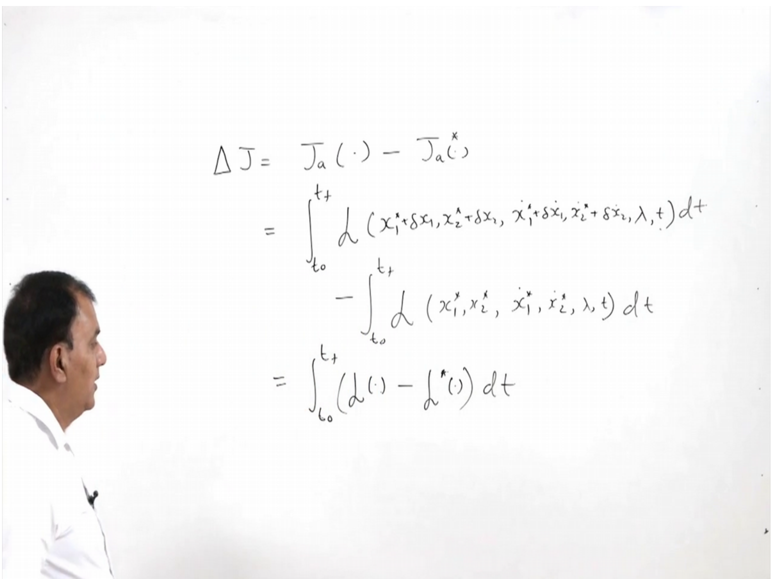

So, this increment if you will write some writing my delta J as, J minus J at optimal point, this I am writing as a star, and what is my J if I will see this J is nothing but my L integral of t 0 to t f L d t.

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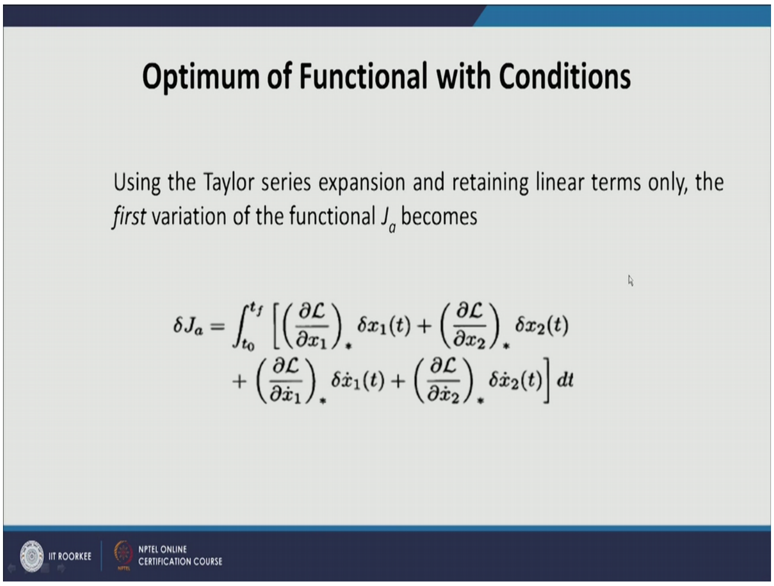

So, I can write this is t 0 to t f L at the point, x 1 star plus delta x 1, x 2 star plus delta x 2, x 1 dot star plus delta x 1 dot x 2 dot star plus delta x 2 dot lambda and t. So, this is my J at variation point minus t 0 to t f, L at the optimal point which is nothing but my x 1 star, x 2 star, x 1 dot, x 2 dot, lambda and t d t and naturally this is also d t.

So, if I will write in a simple term this is nothing but t 0 to t f, L at variation point minus L at optimal point d t.

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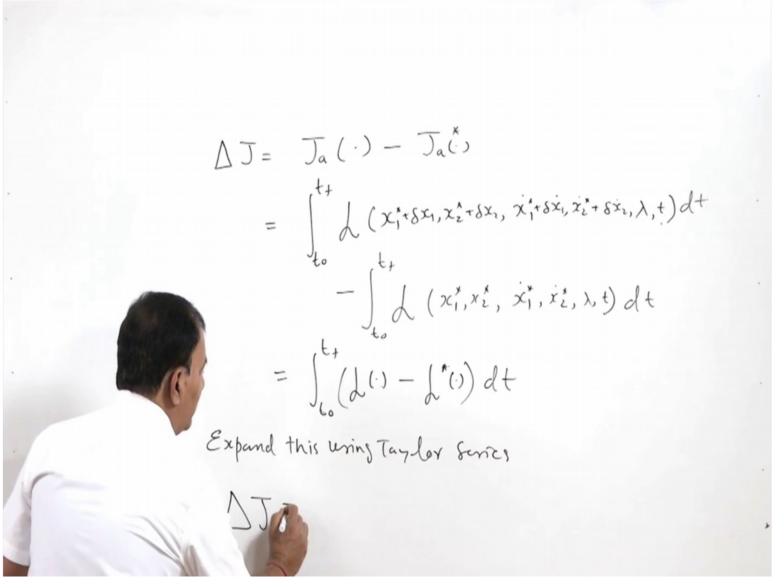

It will explain this by using Taylor series. So, what I will get? My delta J.

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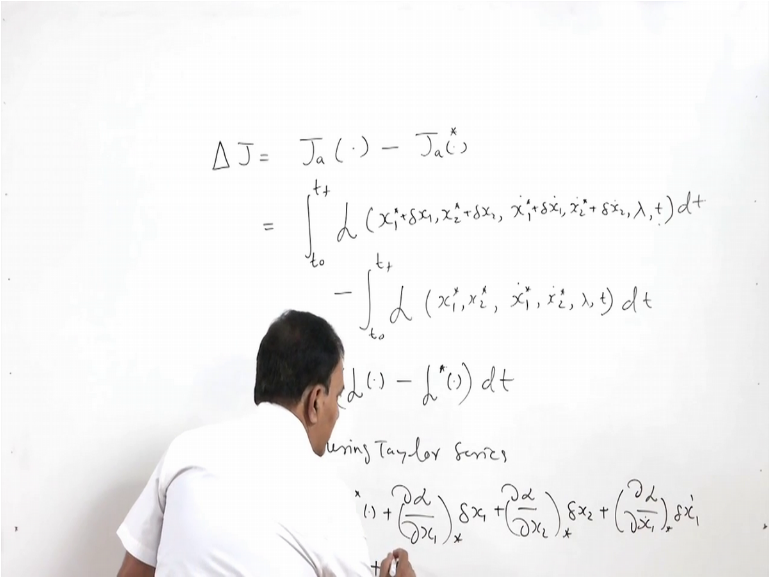

So, this is the same as we have done before nothing but t 0 to t f, L at optimal point plus del L by del x 1, delta x 1 del L by del x 2, delta x 2, and similarly for x 1 dot and x 2 dot, del L by del x 1 dot, delta x 1 dot plus del L by del x 2 dot, delta x 2 dot plus.

Now, I will get some higher order terms into d t; d t we can also include L star into this minus this L star into d t. So, this L star will cancelled out we are left with the first order term and the higher order term.

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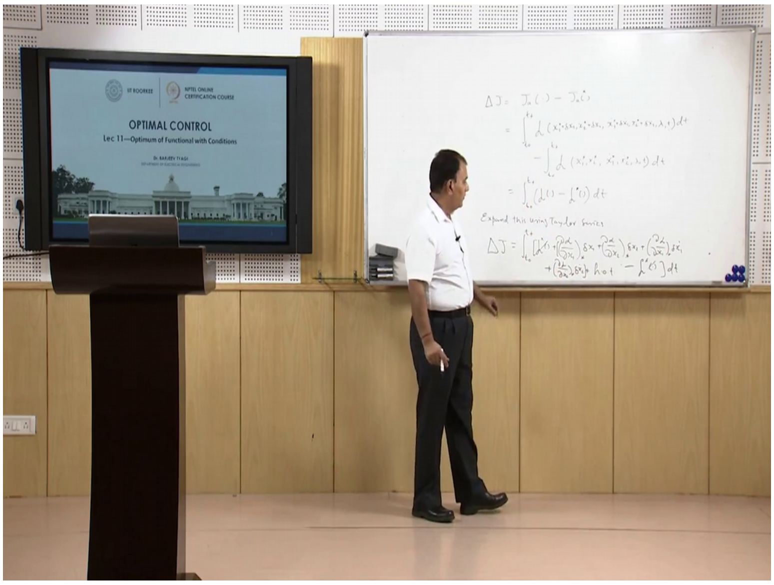

So, if we will write the first variation. So, first variation I can simply write as del L by del x 1 delta x 1, del L by del x 2 delta x 2, del L by del x 1 dot delta x 1 dot, plus del L by del x 2 dot delta x 2 dot d t. So, this means I am simply collecting my first order terms, because this L is canceled with this I am getting up to this my first order term which will be nothing, but my first variation, higher order terms we are neglecting. So, my first variation is given by as given here.

Now, in this case we have coefficients of the delta x 1 delta x 2 and delta x 1 dot, delta x 2 dot. So, if we will consider the third term del L by del x 1 dot delta x 1 dot using integration by parts I can convert my delta x 1 dot into delta x one. So, we are considering the third term of this expression del L by del x 1 dot delta x 1 dot.

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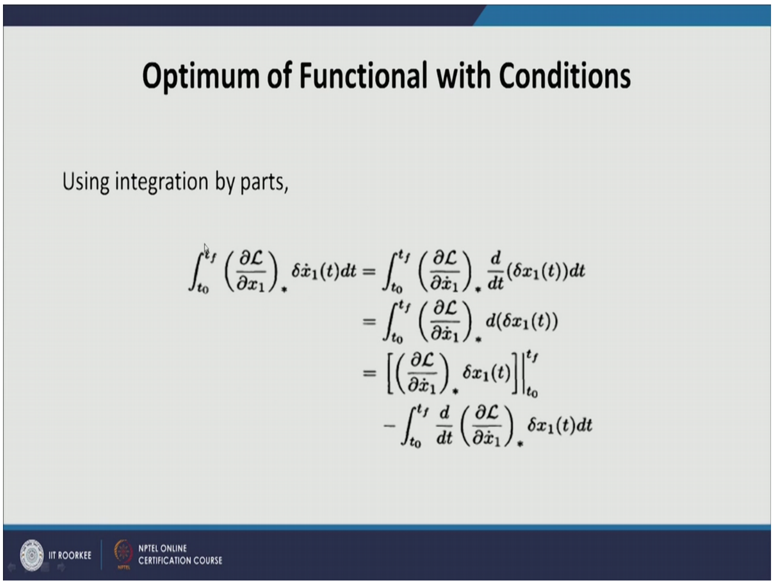

So, integral of del L by del x 1 dot into delta x 1 dot d t.

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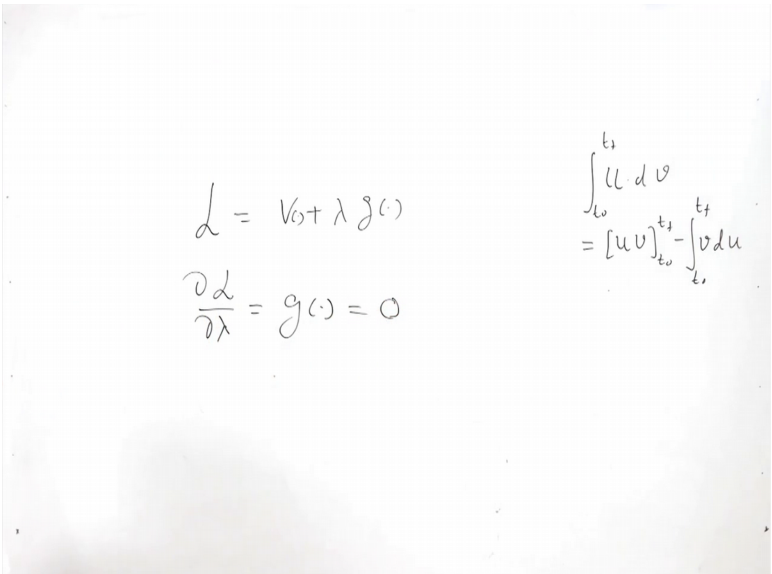

This we are integrating by parts using the same expression integral of as we have seen before u d v is nothing, but my u v, t 0 to t f is my integral limit is t 0 to t f, minus integral t 0 to t f, v d v.

So, if we will see here my u del L by del x 1 dot, and V is del x 1 dot we are writing as d by d t del x 1 t. So, I am taking d L by del x 1 dot as my u, del x 1 t as my V. So, if my expression will become this is my u d v, this u d v is u into v t 0 into t f as we have seen here minus t 0 to t f v and d u d del L by del x if t will cancelled out. So, my u is del L by del x dot. So, this is my u, so d of u. So, this means the third term which we have taken as del L by del x 1 dot, delta x 1 dot t this can be written as del L by del x 1 dot, delta x 1 dot t 0 to t f, minus integral of d by d t del L by del x 1 dot, delta x 1 t d t.

Similarly, the last term here which is in terms of the delta x 2 dot can be expressed in terms of the delta x 2. So, if we will substitute the value of third term and the forth term in terms of the delta x 1 and delta x 2.

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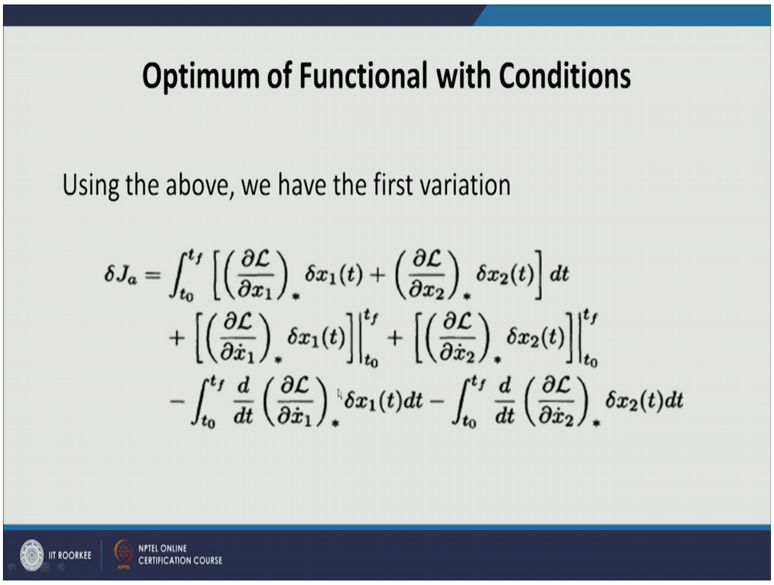

So, we will get our expression as delta J. So, my; these 2 terms representing the third term and these 2 terms are representing my last term. If the boundary conditions are considered this means t 0 to t f, if I will place in this, this is nothing but delta x 1 of t f, delta x 1 of t 0, delta x 2 of t f delta x 2 of t 0 and what we have considered; this all conditions I am given with as we can see in my problem x 1 t 0, x 2 t 0 are given, x 1 t f and x 2 t f are given.

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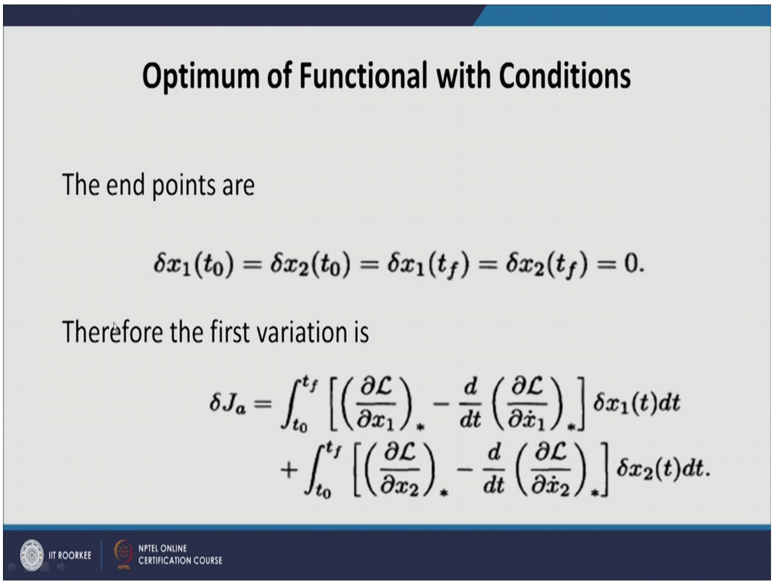

So, this means my delta x 1 at t 0 delta x 2 at t 0, delta x 1 at t f, delta x 2 at t f this all values nothing, but will be 0.

So, in this what is by limiting terms? This time will become 0 and I am left with the del L by del x 1 minus d by d t, del L by del x 1 dot delta x 1 dot, del L by del x 2 minus d by d t del L by del x 2 dot delta x 2. And if you will say this will have nothing, but the structure of my Euler equation. So, like in the previous case we have taken as the coefficient of the delta x 1 to 0 and delta x 2 to 0, in this case because x 1 and x 2 are dependent variable they are not independent. So, we cannot read this as already we have discussed in the previous class, because these 2 are dependent. So, this means delta x 1 and delta x 2 will also be the dependent variable.

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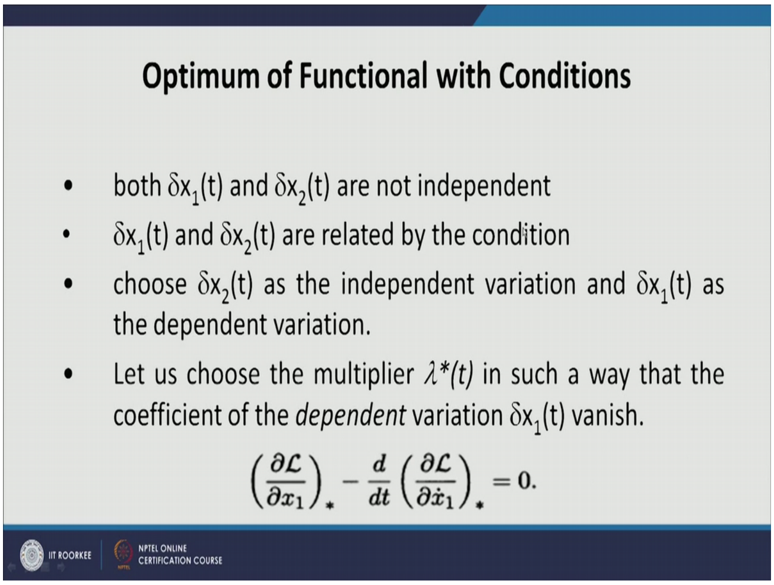

So, x 1 delta x 1 t delta x 2 t are not independent they are related with a condition. So, what we can do? We can choose one of the variable as the independent variable and other as the dependent variable. So, in this case we are treating delta x 2 as my independent any variable I can treat as independent and other as the dependent. So, if delta x 2 is independent then delta x 1 will be the dependent variable. As previously we have seen in the Lagrangian approach Lagrangian multiplier delta we have to select. We can select a delta sorry we can select a lambda in such a way that the coefficient of the dependent variable will 0, means we have taken x 2 as the independent x 1 as the dependent variable and if you will see if a function of lambda. So, we can choose a lambda which is making the coefficient of delta x 1 which is nothing, but my del L by del x 1 minus d by d t del L by del x 1 dot 0.

So, I am making this term equal to 0 by selecting a proper value of the lambda, let us say this is lambda star. So, by selecting this lambda as the lambda star coefficient of the delta x 1 is going to be 0. If this coefficient is 0 so I am left only with the delta x 2 term.

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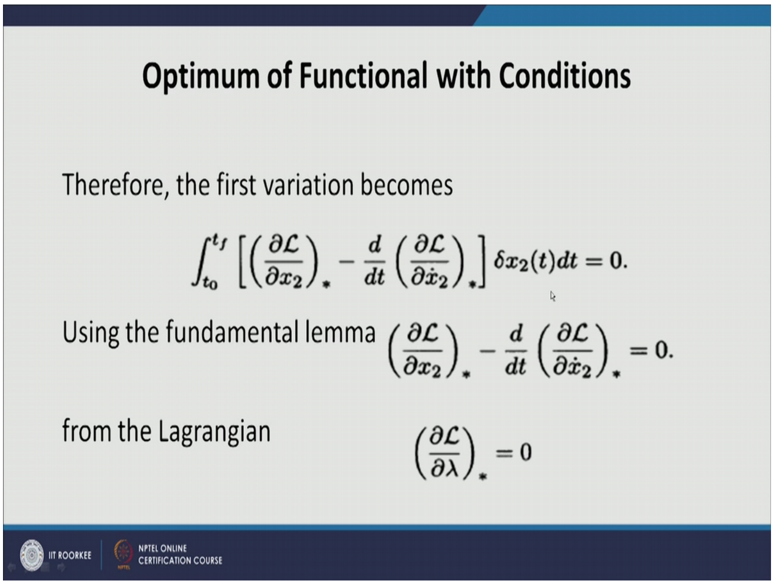

So, now, delta x 2 is independent and delta x 2 is also arbitrary. So, the coefficient of this also must be 0 this is our fundamental lemma which we have discussed before. So, due to the fundamental lemma I can write del L by del x 2, minus d by d t del L by del x 2 dot equal to 0. And L as already we have treated as my V plus lambda g. So, if I will take del L by del lambda this is nothing, but my g and the condition already I have considered as 0.

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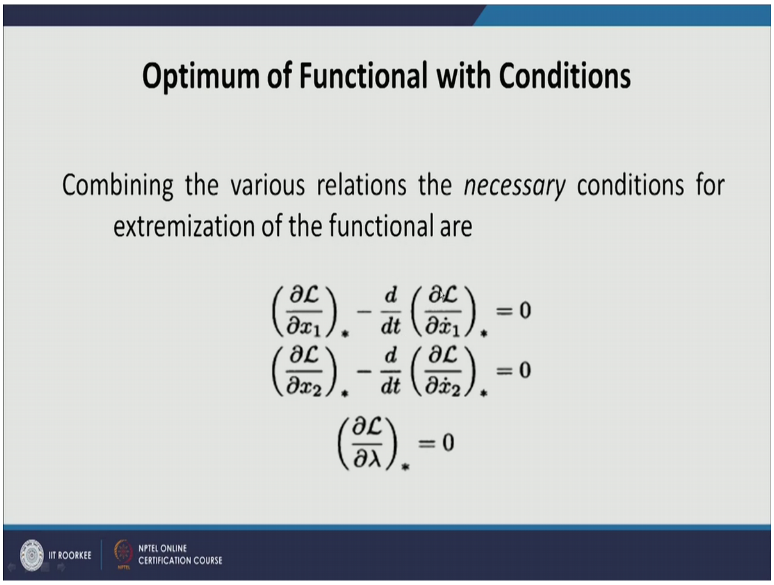

So, this means my del L by del lambda will also be 0. So, all this analysis basically lead to the necessary condition as del L by del x 1 minus d by d t, del L by del x 1 dot is equal to 0, similarly del L with respect to the other variable x 2, d by d t del L by del x 2 dot equal to 0, del L by del lambda also equal to 0. So, if you are finding the optimum of a value we can write this equation. As we can see here we have only 1 condition. So, del L by. So, we have the three equations, and we are the three unknown x 1 x 2 and lambda. So, these three equations will give us the three differential equations which simultaneously can be solved to determine the value of x 1 x 2 and lambda, and these values at this point will be nothing, but my optimal values.

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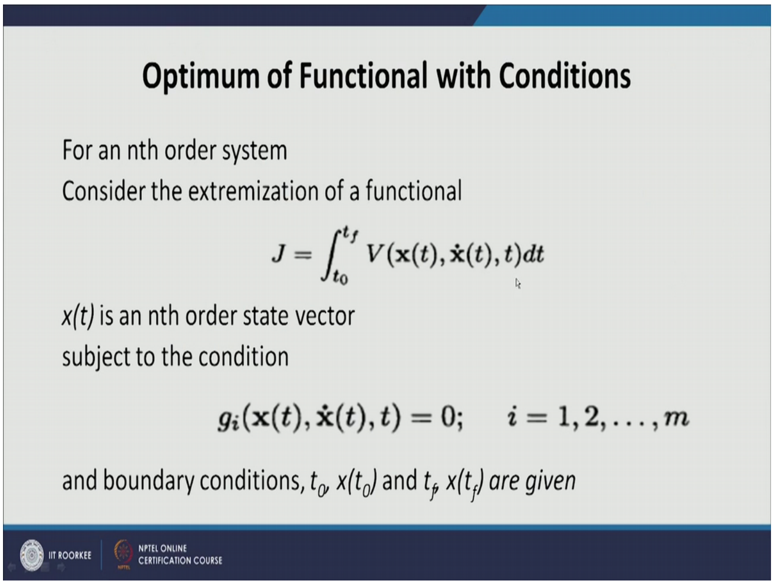

If we will journalize the relation so we can consider a nth order system in a way that J is my functional of V x t, x dot t t and here x is nothing but a nth order is state vector. So, this means we are considering my x as x 1, x 2, x n. So, if we are seeing the V x it is a V x 1 to x n similarly x 1 dot to x n dot and t.

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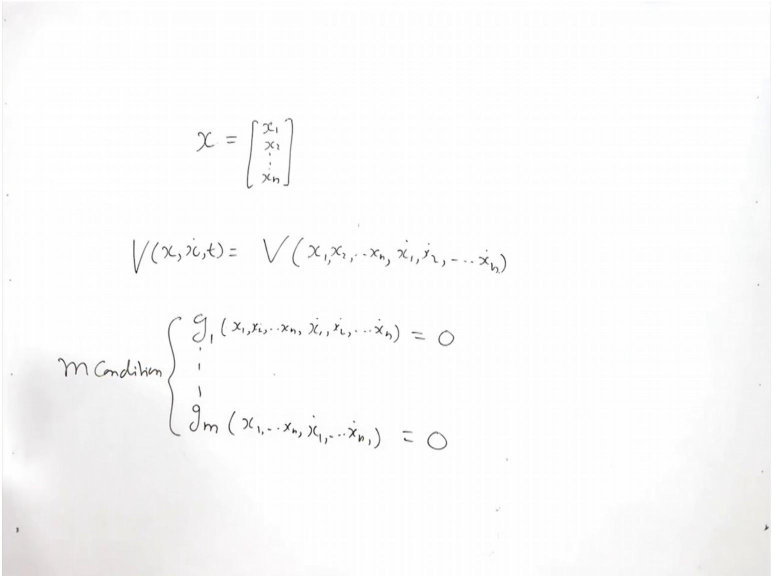

So, this means V x x dot t is nothing, but my V x 1, x 2, x n, x 1 dot, x 2 dot, x n dot, x 2 dot n, x n dot. So, this is nth order state vector x we have considered here and this. So, our problem is to find out the optimal value of the J, which is subjected to the condition g i x, x dot t where i is varying 1 to m, this means we have the m number of the condition. So, my condition is g 1, x 1, x 2, x n, x 1 dot, x 2 dot, x n dot, this must be equal to 0 and this is nothing, but g 1 to g m we are considering; x 1, x n, x 1 dot, x n dot and this must be equal to 0.

So, we have m condition, in the first case we have taken only the 2 variables x 1 and x 2 and 1 condition as the g. So, in general formulation of this problem we are considering the optimization of the J, which is subjected to m number of the conditions; and again we are considering my boundary conditions t 0 x t 0 and t f x t f are given.

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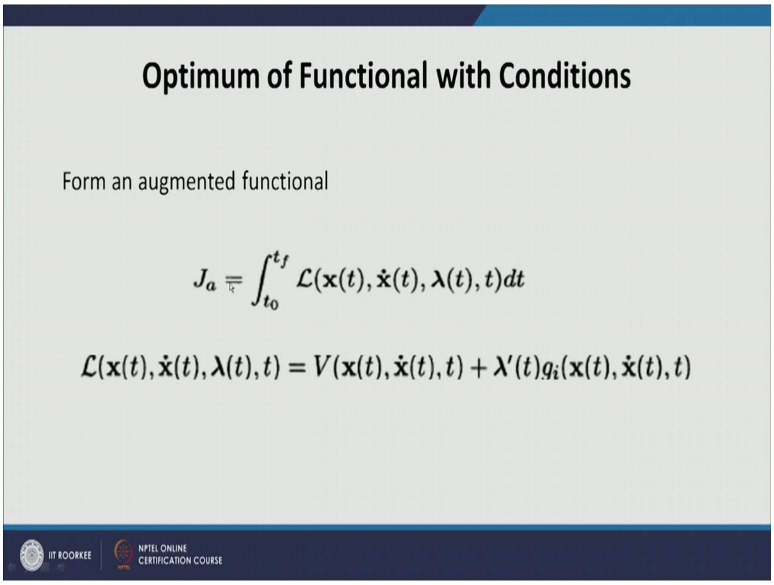

So, again this is a fixed endpoint problem as we have considering the previous case. So, this fixed endpoint problem can be journalized in the form that first we will write a J which is nothing, but the Lagrangian times d t, and Lagrangian is define nothing but as my V plus lambda prime g. Why lambda prime because for the m condition we have the m number of the Lagrangian multiplier.

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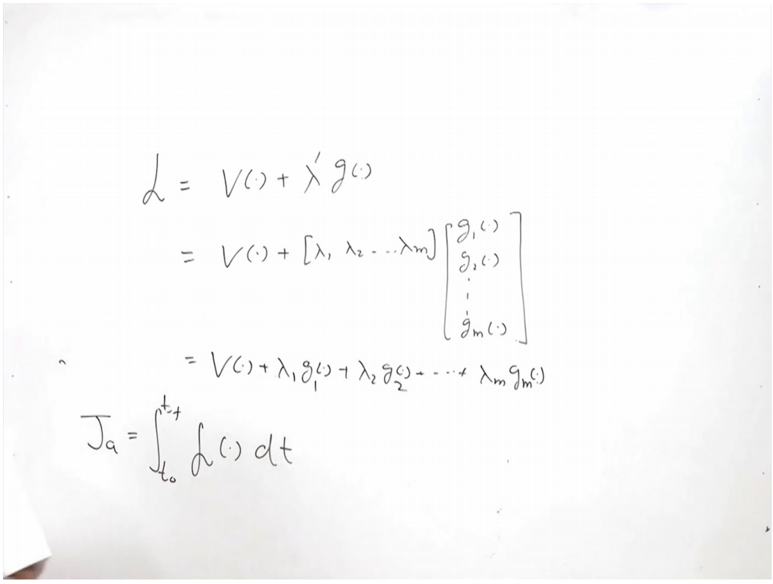

So, if you will see what actually will be my Lagrangian and we are writing as V plus lambda prime g. G is a vector now or we are writing this as V plus lambda 1, lambda 2, lambda m as g 1, g 2 and g m. So, my Lagrangian is nothing, but V plus lambda 1, g 1, plus lambda 2, g 2, and lambda m, g m. So, this is my Lagrangian. And what we are seeing my augmented function will be t 0 to t f, L d t because all my g sorry this is g 1, g 2, gm all my g 1 equal to 0, g 2 equal to 0, g m equal to0. So, L minimization of the L is same as the minimization of the V.

So, the J which we have considered either I will take the minimization of the J or the minimizations of the J both are the similar as we already have discussed, and if you will follow the same approach.

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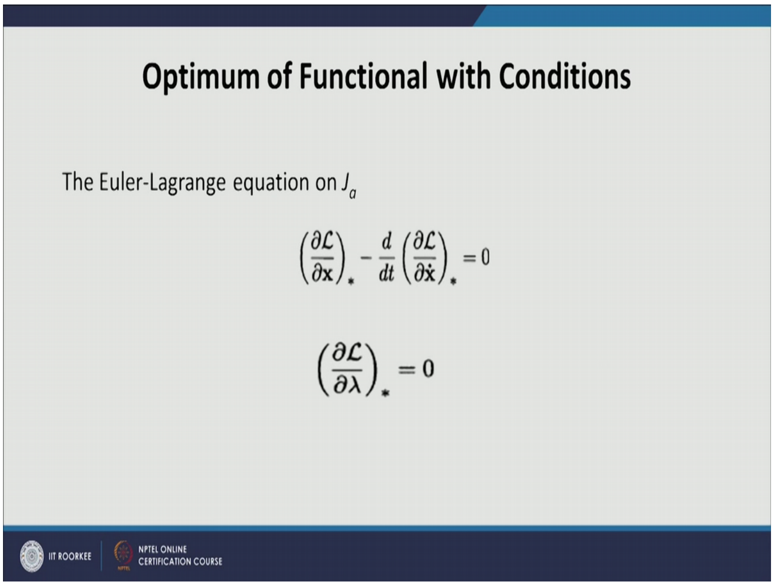

We will lead to the m number of the Euler equations which is del x by del x minus d by d t del L by del x dot equal to 0, and the m number of my condition equation which is del L by del lambda equal to 0; as del L by del lambda giving me the g. So, if we will see we have the total n plus m number of the equations. So, what the first equation giving me del L by del x minus d by d t del L by del x dot, because all the n number of the state.

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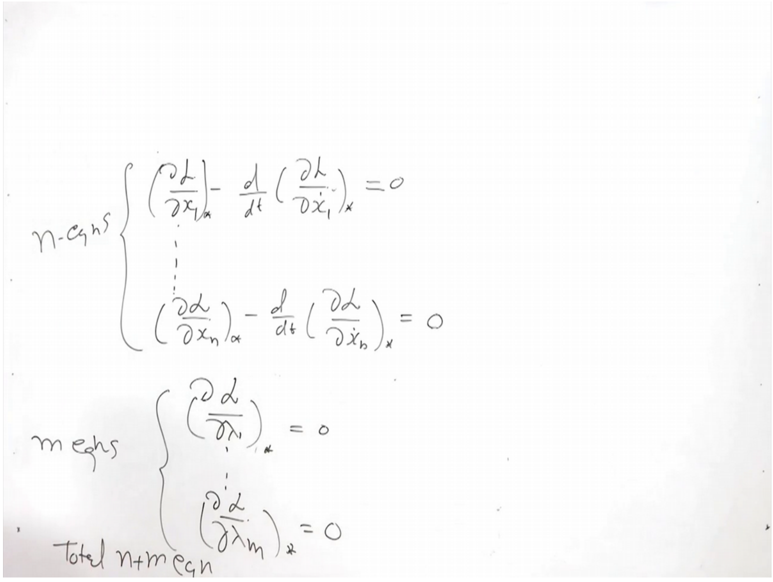

So, this is nothing, but del L by del x minus del L by del x 1, d by d t, del L by del x 1 dot, and these we are finding for optimal condition, this must be equal to 0 and similarly my nth equation is del L by del x, n minus d by d t, del L by d x n dot that must be equal to 0 and the conditional equations are del L by del lambda this means this is nothing, but my del L by del lambda 1 is 0 up to del L by del lambda m equals to 0.

So, these are n equations and these are m equations. So, my total n plus m equation I have. So, total n plus m equations we have, and number of variables are x 1 x n and lambda 1 to lambda m; so total n plus m variable, n plus m equations which can be solved to determine the all my x 1 and lambda. So, as we have seen that we have n number of the equations given by Euler equation here, and the m number of my condition equation.

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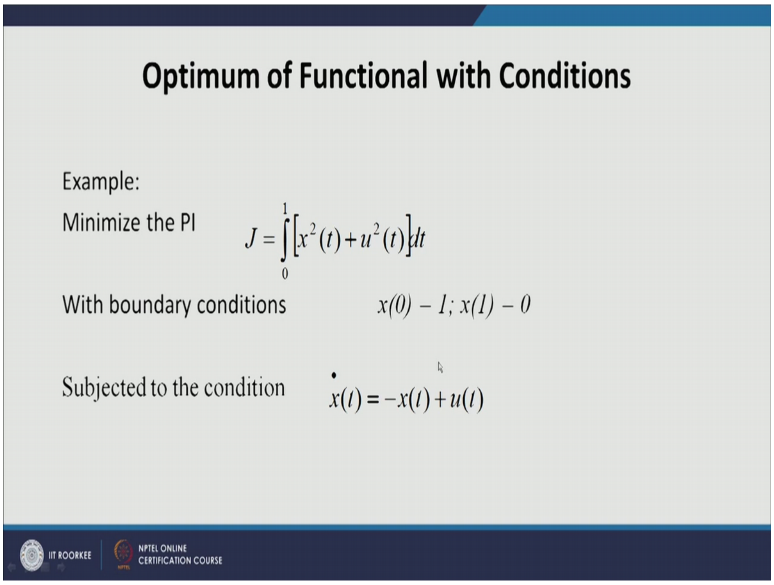

So, next we will take one example and see how this knowledge can be implemented to determine the different variables. My problem here is I have to minimize the J which is 0 to 1 x square t plus u square t d t, subjected to the boundary condition x 0 equal to 1 x 1 equal to 0, and my condition given here is the x dot as minus x t plus u t.

So, means we have to minimize the J subjected to this condition. So, what I have in the integrant x square plus u square this is nothing but my V, and last equation nothing, but giving me the condition.

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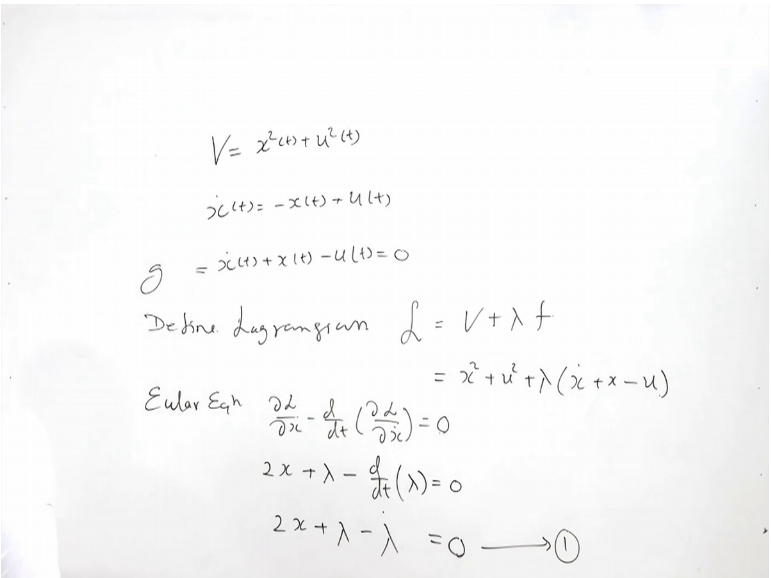

So, if I will write all variables on ne side I can write this as my g. This means we have the V as given here x square t plus u square t, V is x square t plus u square t, and the condition given here is as the x dot t as minus x t plus u t, this can be written as x dot t plus x t minus u t equal to 0, and this I can take it as my g. So, my first step is to define the Lagrangian L as V plus lambda f, which is nothing but my x square t I am dropping out plus u square plus lambda, x dot plus x minus u. So, this is my Lagrangian and what we have to do, we have to now write my e L equation and the condition equation.

So, if I will write my Euler equation as del L by del x minus d by d t, del L by del x dot equals to 0. So, del L by del x will be what. So, this is 2 x plus lambda minus d by d t of del L by del x dot, del L by del x dot is I have nothing but my lambda equal to 0. So, this is 2 x plus lambda minus lambda dot this equals to 0, I say this is my note down equation 1.

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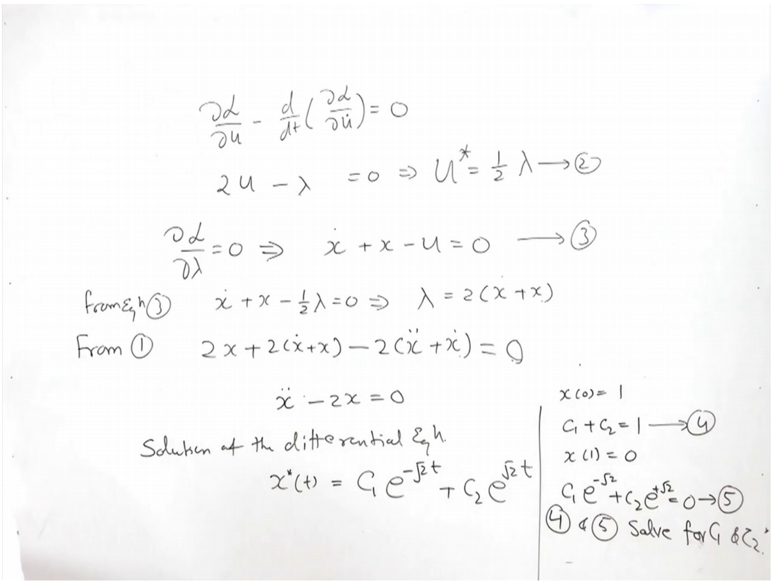

My next Euler equation will be my another variable is del L by del u minus d by d t del L by del u dot, that must be equal to 0. We know my; what is my L if I will write this. So, this gives me nothing but 2 u, minus lambda equal to 0, this means I can write my optimal u as 1 by 2 lambda. And my condition equation is del L by del lambda equal to 0 with given value of L my this equation comes out to be nothing, but what my condition is x dot plus x minus u equal to 0.

I say this is my equation 2 this is my equation 3. So, as a three equations equation number 1 2 x plus lambda minus lambda dot equal to 0, second equation as my u star as 1 by 2 lambda, and the third equation x dot plus x minus u equal to 0. So, these three equation I have to solve simultaneously to determine the value of the x star t, u star t, and lambda t. So, simple way to solve this equation I have u star as 1 by 2 lambda I place. So, from equation three what we can write x dot plus x minus 1 by 2 lambda equal to 0, this implies my lambda is nothing but 2 x dot plus x, as lambda is x plus sorry 2 x dot plus x and my equation number 1 is 2 x plus lambda minus lambda dot. So, I place into now from 1 what we can write? 2 x plus lambda in place of lambda I write 2 x plus x dot plus x, minus lambda dot means I am taking the derivative of this 2 x double dot plus x dot equals to 0.

So, if I will simplify this I will simply get x double dot minus 2 x equal to 0. So, the solution of this differential equation is we can write as x t, C 1 e to the power minus under root 2 t, plus C 2 e to the power under root 2 t. Now this C 1 and C 2 we can find out using the terminal conditions which is x 0 equal to 1 and x 1 equal to 0. So, I have x 0 equal to 1 this means the t s 0 x 0 is 1. So, what I will have? C 1 plus C 2 equal to 1 and my another condition is x 1 equal to 0 where t is 1. So, this is C 1 e to the power minus under root 2 plus 2 e to the power plus under root 2 that is 0.

So, 1 2 3 I can say this is my 4 this is my 5 equation 4 and f5 you solve this four solve for C 1 and C 2. So, if we will solve this for C 1 and C 2 we get.

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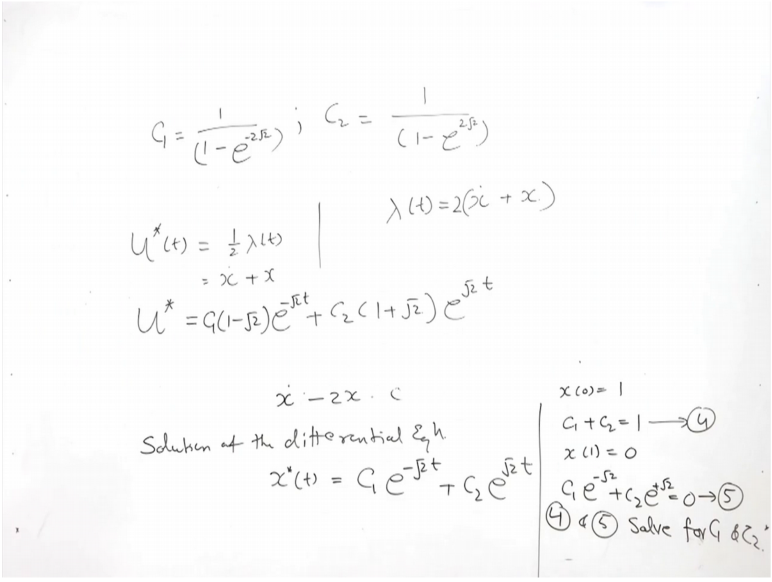

C 1 as 1 upon 1 minus e to the power minus 2 is square root of this, and C 2 is 1 upon 1 minus e to the power 2 root 2. So, if you know C 1 and C 2 we can also write what is my u star t, which is nothing but 1 by 2 lambda t, sorry first we have to find out lambda t which is nothing, but my x dot plus x.

So, this is sorry lambda t is twice of x dot plus x. So, if I am writing u t this is nothing, but x dot plus x, I know; what is my x t? So, I have x dot and x if I will write. So, this will give me C 1, 1 minus under root 2, e to the power minus root 2 t, plus C 2 1 plus under root 2, e to the power under root 2 t. So, this will be my u star, and lambda star can directly be written by this. So, all 3, x 1, x star and lambda star all the values can be determined utilizing my necessary condition del L by del x minus this, sorry del L by del x minus d by d t del L by del x dot equal to 0 which is nothing, but my Euler equation, my 2 variable are x and u which is given here x and u are my 2 variable. So, has the 2 equation as del L by del x, del L by del u, and the third will be del L by del lambda.

So, these can be solved to determine my all the variables; if everything is known them directly with the value of the x t optimal value of x and u, I can find out what will be my minimum value of the J.

So, in this class we have discussed about the optimal value of a functional with given condition, in the next class we will apply our apply this concept to a optimal control problem. So, today I stop here.

Thank you very much.