Sliding Mode Control for Nonlinear Systems

So, welcome back. In the previous class, I talked about the sliding mode control for the non-linear system. So, originally, sliding mode control was developed to handle non-linear systems because, in nature, most physical systems actually fall into the category of non-linear systems. In the previous class, I was discussing how to design on-off control for first-order highly uncertain systems. But originally, most of the system is either second order or higher order. So, in this lecture, I am going to talk about sliding mode control.

for the second order nonlinear systems. So, the purpose of the discussion and outcomes. So, we are going to demonstrate robustness. I have already discussed that for any nonlinear system, we cannot claim that I know the model accurately.

There are several uncertainties that are associated with the parameters, as well as disturbances from the control channel or through external sources. So, how basically sliding mode control works, and I have already told you that sliding mode control is nothing but a kind of on-off control, and we have to do on-off with very high speed; that is our goal. In this lecture, we are going to see how non-linear systems are converted into linear systems after some finite time. After that, we are also going to talk about insensitivity with respect to some classes of perturbations. What is the meaning of insensitivity? It means that after finite time, system becomes insensitive.

They do not feel any kind of disturbance. So, that is the meaning and what is the outcome? I have already told you that sliding mode control is very easy to design. So, we are also going to talk about the design aspects. So, let us start today's lecture with a very simple system, a second-order system, and this is basically a second-order uncertain system. So, if you have seen the previous lecture.

So, at that time I was talking about exactly the same system, but here this dynamics is not present and our coordinate system is σ. So, $\dot{\sigma} = h(\sigma) + g(\sigma)u$, and again we are going to assume exactly in the same way that $h$and $g$are unknown, but we have some kind of assumption associated with $h$and g. Then, only I can be able to design the control action, on–off control action. And here, this assumption is required. What is the physical interpretation of this assumption? We are assuming that the control matrix is here; the reason I am mentioning the matrix is that it might be possible in this class I am talking about second-order systems, but in the next class, I am going to talk about multi-input multi-output systems.

At that time this becomes somehow a matrix. So, the control we are going to apply is through this matrix. So, I assume that this is not going to change the sign. So, how can we ensure that? You can see that I am always assuming that $g$is bounded by $\bar{g}$, and the objective is exactly the same as in the previous class. Now, we have to design a state-feedback control.

It means that I am assuming that I know $x_1$and x_2, and then by designing the control action $u$based on the on–off control with high switching frequency, I have to make sure that $x_1$and $x_2$both tend towards 0as $t \to \infty$. So, here you can see that I have relaxed the assumption. In the previous class, I was trying to force the trajectory of a first-order system in finite time, but here I am talking about the asymptotic behavior after a very, very long time. And what are we basically going to do? We are going to design control actions such that the system's motion actually converges to a pre-selected manifold. Manifold means some kind of lower-dimensional space

So, in two-dimensional space, I am going to create one-dimensional space because I only have one control action. And what am I going to do? Now, I am going to make sure that $\sigma = 0$in finite time. So, here I have made this choice, but you can take any non-linear combination of $x_1$and x_2, which is also possible. I will also justify why I am claiming $c_1 > 0$. So, similar to the linear case, whenever I first discussed on–off control in the first module and after that what we have done, whenever we have a second-order system, we are defining the linear combination and after that we are transforming the system.

With the help of that linear combination. So, similarly, I am going to do that here. So, here you can see that this manifold $\sigma(t)$is $c_1 x_1 + x_2$. You can easily plot for $\sigma > 0$. So, this is $x_1$and this is x_2.

Then, this is the line that is going to represent $\sigma = c_1 x_1 + x_2 = 0$. So, this is nothing but $\sigma(t)$, which is a lower-dimensional space sliding manifold, and what is our job? I have infinite time; I have to converge here, and asymptotically or exponentially, I am willing to converge to the equilibrium point. By designing an on–off control with a high switching frequency, we are essentially trying to maintain along this surface; that is our objective. So, here I have $\dot{x}_1 = x_2$. So, I have already defined $\sigma = x_2 + c_1 x_1$.

So, for that reason, I removed x_2, and in place of x_2, I made this kind of substitution. After that, I am going to take the derivative of $\sigma(t)$. So, here you can see that I am going to use Newton's philosophy. What is Newton suggesting? Suppose that if you want to control some quantity, just control its rate of change. For that reason, I am going to calculate the rate of change.

Here, I calculated the rate of change. I have substituted $x_1$and $\dot{x}_2$ from the dynamics. And then, now by designing this control action, I am going to maintain $\sigma(t) = 0$for infinite time. Please see carefully; here I am maintaining $\sigma = 0$for an infinite amount of time. If you see the second subsystem, it is exactly equivalent to the previous system, like the previous class, but here I have just one extra term.

So, what am I going to do now? I am going to assume that these two terms are part of the uncertainty. And now, I am going to assume that the absolute value of this term divided by this term is bounded. So, that kind of assumption comes into the picture. So, what is an observation? Motion on $\sigma(t)$should be independent of $h$and g. So, we have to design control such that I can force $\sigma(t) = 0$in finite time, and after that, I have to maintain it.

It means that in finite time I am going to converge here, and after that I am going to maintain exactly along the sliding surface; practically, you will be able to see it like this. So, now the question is how we can drive the trajectory to the manifold in finite time and keep it there. So, again, if you see this problem carefully, you can see that by using some kind of backstepping, this is a method of backstepping. So, what are we basically doing? We are trying to partially decouple this system. So, now, without worrying about $x_1$dynamics, I can design control u, and I am forcing $\sigma = 0$.

And, once $\sigma = 0$, this is actually nothing but the first-order system. And, using Newton's law, now you can design gain. So, you have already learned that in the first module that if $x_1 > 0$, then $\dot{x}_1 < 0$, and if $x_1 < 0$, then $\dot{x}_1 > 0$. And similarly, we have to do this here; then we will converge to the origin, and for that reason, $c_1 > 0$. So, one more important thing I have already told you is that you can select any kind of manifold, but here, once you design a manifold like this, your subsystem will look like $-\alpha x_1(t)$.

Now, you have to design some kind of control mechanism. And it might be possible that some more terms come into the picture in the second dynamics, but reduced order dynamics will come like this. So, using Lyapunov method, you have to stabilize this system. Now, we have somehow partially decoupled the system. So, I am just going to look into this dynamic, and this dynamic is exactly the same as the previous class.

So, I am going to adopt a similar kind of methodology. What am I going to do? I am going to design some kind of gain that depends on the state. So, this is a state-dependent gain. Because our target is to start anywhere in two-dimensional space, and after that, I have to converge to the equilibrium point asymptotically; that is our aim. And since this is uncertain, I do not know the exact value, and for that reason, I have to actually define some kind of state-dependent gain.

And this is β_0, which is the constant gain, because once you converge very near to it, then at that time, the value of this is very, very small. So, this value will maintain the sliding mode control; that is our main target. And $h$and g, so in the previous class since I have just this term, so now I have these two terms, and due to that reason, this kind of inequality comes into the picture. So, this is our assumption. So, this assumption you have to check at each and every time.

So, now you can see that, like in the previous class, since I have to stabilize $\sigma = 0$in finite time, I am going to apply Newton's method of Lyapunov stability for finite-time stability. I am going to select the same $\frac{1}{2}\sigma^2$; this is positive definite. After that, $\dot{v}$, if you do the calculation like in the previous class, and due to that reason, I have actually removed that kind of calculation. Now, you can see that I will get this kind of inequality. So, what is our claim? $\dot{v} \leq 0$.

And in the previous class, we have already proved that if this kind of dynamics comes into the picture, I will do a substitution of $|\sigma|$in terms of V, and somehow some term associated with $V^{1/2}$comes into the picture, and using the Comparison Lemma, I can prove that $V = 0$for some finite $t \geq t_f$, and that will be maintained. And due to that reason, now I am not going to repeat the same kind of exercise; please do it by yourself. So, once $\sigma = 0$, now you can see that magic happens; there is no uncertainty in this system. And after that, what happens? The second-order system is actually converted into a first-order differential–algebraic system. What is the meaning of a differential–algebraic system? The first equation is a differential equation, and the second equation is an algebraic equation. So, now this is actually fully decoupled.

So, you can apply Newton's law and maintain $c_1 > 0$. By the solution, you can also see that for any initial condition, $x_1$tends towards 0as $t \to \infty$. And since $\sigma = 0$, the linear combination is 0, $c_1$is positive, and here $x_1$is actually tending to $\mathbb{R}_0$. So, I can conclude that $x_2$is also tending to $\mathbb{R}_0$. Since the convergence of $x_1$is exponential, the convergence of $x_2$is also exponential; otherwise, they will not maintain that equal to 0.

So, in this way I can achieve sliding mode behavior for any non-linear second order system. So, now if you are working with any practical system, you can take that practical system, do mathematical modeling, check the condition, and after that, design the control action. actually most of time what happens that wherever we are applying control. So, during the application of control, actuator bandwidth is fixed, and for that reason, for simplicity's purpose, what people actually try to establish is that they want to use some kind of constant switch gain. So, the gain of the controller should be finite, and if the gain is finite, or some kind of value that does not depend on the state, it is possible to show that I cannot steer the trajectory from everywhere, and then we will converge to the equilibrium point.

So, suppose this is a two-dimensional space. So, in two-dimensional space, it means that now I cannot start from anywhere and converge to the equilibrium point. Why? Because we do not have sufficient control, we have to estimate this region. Most of the time, non-linear systems are actually evolving in some kind of limited space, and due to that reason, this methodology is completely effective in practical situations. So, now, what am I going to assume? I am assuming that I have a simple relay.

What is the meaning of relay? Some kind of gain and switch, and then I am going to force $\sigma = 0$ an infinite number of times. That is our objective. So, again you can see that in the previous place, if you see the assumption. So, basically, when this $k$becomes constant, the assumption is going to change. Here, this is a function of x, but once you maintain just a constant gain, then this becomes constant.

So, now, I have to calculate the region of attraction. What is the meaning of the region of attraction? I am able to guarantee that if I start somewhere in two-dimensional space, then I can converge to the equilibrium point. So, that kind of guarantee, guaranteed region I have to select. How do I select? So, again, the methodology is very, very simple. What can you do? You can start with the Lyapunov function.

And then, you have to maintain $\sigma\dot{\sigma} \leq 0$. Since $k$is finite, you are going to get this kind of criteria. Now, what can you do? You can apply this kind of criterion to the reduced-order dynamics. So, this is not completely reduced because here $\sigma(t)$, once $\sigma(t)$ becomes exactly 0, then only I can say that the second-order system is converted into a first-order system, but here this is still a second-order system. In order to find the region of attraction, what am I going to do? I will just define the Lyapunov function with respect to the first variable.

I will actually substitute this kind of condition, and using this condition, I can tell that if you start from this particular ball, then you are able to converge to the equilibrium point, okay. This set is called a positively invariant set. What is the meaning of invariant set? The meaning of an invariant set is that if you start from that set, you will remain there forever. You cannot go leave. What is the meaning of positive? So, in the engineering system, we are assuming that it only evolves in a positive direction.

So, here $t = 0$, then this is t^2. So, we are moving in a positive direction. But for mathematicians, direction is not a matter. Time may also be negative. At that time, they were just using invariance. So, now you can be able to calculate it; it means this condition, it means that you can start anywhere in this space, I am going to give you the guarantee that you are going to converge to the equilibrium point.

Now, this depends on a, and the choice of $a$fully depends on you. So, this kind of stabilization is called semi-global stabilization. It means that if I enlarge a, then my region of attraction is going to increase. And in this way, you can see here that both dynamics $x_1$and x_2 are involved.

Why? $x_1 \leq \frac{a}{c_1}$. And what is c_1? That is the gain from the sliding surface. $\sigma$is nothing but $x_2 + c_1x_1$. So, in this way, we have a restriction on $x_1$and x_2, and if you plot it, then you are able to find some kind of ball. Here, ball does not mean some kind of spherical thing.

Ball may be a line, point, or anything. That depends on the situation. Since in the previous problem, you have seen that I was talking about semi-global stabilization. Most of the time, students are very confused with this kind of terminology, and for that reason, what I am going to do here is tell you about all sorts of stabilization that people are particularly using in the course of non-linear control. So, from Khalil's book, you can see it. Up to this time, what we have seen is that if I have a second-order system, then I have two different methodologies: I will convert the second-order system into the framework of the sliding surface using backstepping, and then the second-order problem is converted into exactly what resembles a first-order problem.

I will solve the first order problem, I will apply Newton's law, and then I will be able to solve the overall problem. So, everywhere you can just be able to apply some idea of the Lyapunov function as well as some idea of Newton's. So, now in this section, I am going to talk about the types of stabilization. So, I am going to quickly introduce everything.

o, suppose I have a forced system. This system is called a forced system. Here, I am assuming that $x \in D$, where $D \subset \mathbb{R}^n$. What does it mean? Because it is a non-linear system, the mathematical model of the non-linear system is not valid everywhere. So, I am assuming that it is only valid in D.

I am not able to apply anything. So, I am assuming that control will lie in this region that is a subset of M. Now, I am going to design some kind of state-feedback control. So, most of the time you might have actually learned during the linear control course that people suggest that whenever you have a non-linear system, you can linearize it about the operating point and then apply the control. So, if you do and if you get $x(t)$tending towards 0as $t \to \infty$, that kind of stabilization is called local stabilization. It means that your control only gives you certainty that if you start in the neighborhood of the equilibrium point, you will be able to tend towards 0as $t \to \infty$.

So, now the second kind of stabilization is called regional stabilization. So, in regional stabilization, in some way, we are first able to understand the physical meaning of the system where the mathematical model is valid. And after that, using the Lyapunov method, we are trying to establish some kind of region such that if I start from this region, I will converge to the equilibrium point. Here, the region does not depend on some kind of parameter. So, regional stabilization means using some kind of Lyapunov method or some other technique; I am going to guarantee first that if you start anywhere in this space, then you will converge to the equilibrium point.

And after that, we are designing control and asymptotic control accordingly. So, that kind of stabilization is called regional stabilization. What is the meaning of global stabilization? So, global stabilization is nothing but a guarantee that if you start anywhere from the state space. In the first problem, we have already seen that I was talking about global stabilization. It means that irrespective of our initial conditions, I will converge to the equilibrium point.

What is the meaning of semi-global stabilization? Now, this word I have already used, and what you have seen is that our region of attraction is a function of the parameter. So, $\sigma$is lying between $\pm a$, and I can decrease or increase it. So, similarly, you can see that whatever control I am going to design, that is depending on the parameter, some choice of parameter. And if you tend this parameter towards infinity, then you will achieve global stabilization again.

So, somehow semi-global control always depends on the parameter. So, what is the difference between global and semi-global? In semi-global stabilization, I am first forming some kind of compact set. What is the meaning of a compact set? Some kind of set that is closed and bounded, okay. So, sometimes a student will confuse closed and bounded. They think that closeness is bounded, but that is not the case. Bonded means I have some kind of upper bound; closed means there exists some kind of sequence such that you will converge to the equilibrium point.

So, if you are confused, please actually go through the real analysis book; you will be able to understand it easily. Now, what I am going to do is explain that I have taken this non-linear force system, x^2; due to x^2, this system is non-linear. Now, I am going to linearize this system. So, if you linearize this system, then I have just $\dot{x} = u$.

So, you can just reject the non-linear term. Now, I will design controls like this. So, you can see that if system like this, then I can able to talk about global stabilization, but if you substitute this control due to this term, I cannot able to achieve the global stabilization. So, at that time, I will talk about regional stabilization. Using the Lyapunov method, you can confirm that this is the region of attraction. Okay, and how original stabilization and semi-global stabilization are different, I have already told you that in semi-global stabilization our control depends on some kind of parameter, and that parameter I can increase and decrease, and based on that I can achieve the outcome.

Semi-global stabilization. In the previous problem, you can see that I am able to design some kind of control $kx - x^2$, and then our system is fully linear, and you are able to get global stabilization, but now you can see that the control magnitude is very, very high. If your initial condition is high, then you can see that control is very, very high. So, most of the time, this is the practical situation. In this practical situation, what happens is that we are talking about practical stabilization. Okay, so most of the time you can see here that this term is always bounded by some kind of δ, and this is not equal to 0 at the origin.

Due to that reason, the origin is no longer an equilibrium point. So, at that time, we are basically not hoping for some kind of uniform asymptotic stability. Why the uniform? Because this dynamic depends on time. So, we have to give guarantee that for all time this result will true. So, due to that reason, the uniform term comes into the picture. Now, in this particular situation when the equilibrium point or origin is not an equilibrium point, we are talking about a uniform ultimate bond.

It means that I want to maintain a trajectory around the equilibrium point. The ultimate or uniform bond concept I have already discussed during the first lecture. So, what are we basically going to do now? We are going to design a control such that our futuristic trajectory will lie in this region for all finite time. So, you have to calculate the finite time, and after that, you have to maintain it. So, if you can maintain it locally, then this is practical.

and where control gain does not depend on the parameter. If the control gain also depends on the parameter, at that time we are going to use some kind of term that is a semi-global kind of terminology. Okay, so let us see it. I have taken this system; I have designed this kind of control action. Now I will select a non-linear system. Most of the time, if you have a non-linear system, it is good to treat it using the Lyapunov function.

Okay, and I am going to calculate $\dot{v}$. Here, you can see that this term is actually negative; all other terms are positive. So, I am going to divide this term into three parts: $\frac{k}{3}, \frac{k}{3}, \frac{k}{3}$, and after that, this part is going to maintain the convergence. Another part is going to take care of one term, one term is going to take care of x^3, another one is $x$and d. And then, it is very easy to check that you are able to maintain this. So, if you select $k \geq \frac{3\delta}{\varepsilon}$, and $k$you can select very, very large, then you will achieve semi-global practical stabilization.

Why semi-global? Because now, by selecting gain k, you can cover a larger region. Now, how can we achieve global practical stabilization? So, in global practical stabilization, I am saying that you can start anywhere. So, if your gain is tending towards infinity, then I can able to select any value. So, in semi-global stabilization, it might be possible that, even if the gain tends towards infinity, I cannot cover the whole region. But here, in global practical stabilization, I am able to cover all regions and the trajectory is going to maintain about the equilibrium point.

So, this is a very easy calculation. You can apply this control. You can design this Lyapunov function and maintain this kind of stability. Please check for yourself. Now, I have taken a pendulum system. So, this is a very, very practical system.

Several problems are actually driven by this practical system. So, this is a second-order system, and I have just designed the relay-based control here. What is the meaning of relay? Gain is a constant sliding surface of $a_1 x_1 + x_2$. So, in place of c_1, I have just written a_1. You can apply this kind of parameter, and please do the simulation.

So, after doing the simulation, you can be able to see the response like this. So, I am also going to provide MATLAB code. During the tutorial session, I am also going to give the MATLAB code so you can define the initial condition. This is the time span, which means up to what time you are willing to do the simulation. After that, you can apply ODE45, and then you can define $x_1$and x_2. In the dynamics part, you can define this, which will give the plotting command, and after that, you can define the dynamics and then simulate it.

Okay, and you will get. this kind of result. So, here theta is actually going to converge to some value, the theta difference that is 1.5, and the sliding surface you can see is converging towards the equilibrium point. So, now it is time to conclude this lecture. So, we have discussed the sliding mode control for a second-order system.

We have also established semi-global stabilization. And after that, we try to understand different kinds of notions, which are actually used in the nonlinear control courses. And this notion is very practical. And I tell you again, for higher-order systems, please try to understand these notions carefully. So, thank you very much.