Finite - Time Stability Using a Lyapunov Function

Welcome back. So, in the first module, what we have observed is how to tune a PID controller, or proportional-integral-derivative controller, just using Newton's law. And after designing the feedforward control, what was our observation? For the underwater vehicle, that system is represented by a linear system. And then we have also designed the on-off control for that linear system using the same philosophy. Most of the systems are non-linear, and it is very difficult to control a non-linear system. So, how to design sliding mode control for a non-linear system, that is one of the challenging topics.

Due to that reason, in this module, we are going to explore sliding mode control for the non-linear system. So, what is the purpose of our discussion? I am going to start with some kind of a non-linear system. And now whenever we have a non-linear system, we have to treat a non-linear system in a little bit different way. So, we have to involve Lyapunov methodology here; we have already understood the meaning of the Lyapunov function, which applies to an nth order system.

So, we are trying to convert it into a one-dimensional system, and after that, we are going to apply the completion lemma, which is the basic philosophy of Lyapunov. So, in today's lecture, we are trying to first introduce the definition of finite time stability and after that how to check the stability without solving the differential equation. So, for that, we are going to involve the concept of the Lyapunov function. After that, we are also going to calculate the time to convergence at the equilibrium point. First, I will map the system in a Lyapunov way, and after that, I am going to solve the differential inequality, and then I am going to calculate the time of convergence.

So, what is the outcome? You are able to understand the role of the Lyapunov function in proving finite-time stability. You are also going to gain insight into the design of control input for finite-time convergence. And after that, we are also able to understand the behavior of the system in the presence of highly uncertain situations. If our system is in a highly uncertain situation, then how do we design the control action? So, for the simplicity of the presentation, I am again going to start this lecture by taking the example of the first-order system. In the next class, I will talk about the second-order and higher-order systems.

You can see here that this system, what is written here, that is $\dot{\sigma} = h(\sigma) + g(\sigma)$, that is highly uncertain. Why I am saying that is highly uncertain? Because I do not know this, I do not know the gain that is associated with the control. So, both are uncertain. Now, I have to design a control action such that I can force $\sigma(t) = 0$in some finite time t.

This is our objective. So, how can we achieve this objective since this part is uncertain? This part is also uncertain. I am not going to design any feedforward control. So, I am just going to rely on feedback, and after that, I am going to design feedback to achieve $\sigma = 0$. If you learn the first module, our system in the first module is very, very simple.

What kind of system do we have? So, we have a system like $\dot{\sigma} = u + d(t)$. So, you can see here that uncertainty is not associated with control. And then our task is very easy; I am able to design k sign(σ). You can interpret σas s, then $\dot{s} = -k$ sign$(s) + d(t)$. And you know that if $k > d(t)$, then I can be able to force $s = 0$.

In this particular representation, $\sigma = 0$. But here, what kind of challenge do I have? You can see that control, the gain of the control, is also uncertain. System is still first order. It means that $\sigma(t) \in \mathbb{R}$. So, we have just one-dimensional system.

That is the same as the dt; here I have generalized it, assuming that this part is also a function of the state. And what kind of assumption do I have? Assumption is the gain that is associated with control that always follows this kind of rule. This is our assumption. This will not change the sign. If this is going to change the sign, then I have to design the control in a different way.

Now, I am going to propose the control; again, this is somehow switch gain control, but here gain is not a constant; it contains some kind of constant part plus some kind of state-dependent part. Now, we are also assuming that $h$and $g$are going to satisfy this kind of condition. Here you can see that I have already assumed that $g(\sigma(t)) > 0$. So, this is not equal to 0. So, this is well defined and in this way I can able to cover the whole space even if mathematical model of this system is valid everywhere.

Most of the time, a non-linear system is only valid in some specific region. What does it mean? $\sigma(t) \in D$; $D$is a subset of $\mathbb{R}$. So, based on that, we have to tune this particular parameter. Now, what is our aim? So, the aim of this lecture is to understand the finite-time stability concept, and after that, we have to prove that once the loop is closed, if I substitute $u$here, then I can guarantee $\sigma = 0$ for all time $t \geq 0$. Why is this particular treatment required? Because in the previous class, we already understood that in sliding mode control, whenever we are trying to apply it to a higher-order system, we have to design the sliding surface.

Before coming to the sliding surface, I have to direct the trajectory to this sliding surface. And that problem is equivalent to the one-dimensional problem if I have a second-order system with only one control input. Similarly, a third-order system with one control input. It might be possible I have a multi-input, multi-output system. So, at that time, I had to generalize this for a multi-input, multi-output system.

So, finite-time stability is very natural in nature. You can see that I have taken the example of the cart; the cart is moving, and I have some kind of friction here. So, if you stop applying any kind of force, what happens? Due to the friction force, this cart will stop. Okay. So, this is the mathematical model.

If you solve this mathematical model, it is possible to show that in finite time, I can stop this cart. Okay. And due to that reason, you can see that if a truck is very, very heavy, you have to wait for more time to stop even if you can apply the brake. That is the physical interpretation that you always have to take care of. You should not stand in front of the truck because even if they apply the brakes, more time is required to stop it.

Now, what is our conclusion? If I have a differential equation like this, $m$is just a scaling factor. So, I can again write like $\frac{k_d}{m}$ sign(v). You can see that this structure is exactly the same as $-k$ sign(s). So, this is actually motivated by this particular example. So, finite-time control is very natural.

So, we have made this observation. Now, whenever I have a system like $\dot{x} = u$, then I will substitute u, and I will achieve $s = 0$in some finite time $t \geq t_0$. But in this lecture, what is our motivation? You can see that I am going to apply this kind of control for highly uncertain systems. Why am I telling about a highly uncertain system? Because $h(\sigma(t))$is also not known, the exact value of $h(\sigma(t))$or $g(\sigma(t))$is also not known. Just the bound is actually known to us.

And now I have to design the controls. So, control now contains some state dependent gain. So, this is the physical interpretation or some kind of physical phenomenon where finite time is occurring. So, now we have to give a definition. So, how do you give a definition? I am going to generalize this definition for an nth order system.

So, I have taken the autonomous systems. What is the meaning of an autonomous system? I have already told you that whenever I have an autonomous system, whenever I am going to solve this system, you can see that the solution of this system, I can easily write like this: $x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau))\, d\tau$like this.
So, if I apply the initial condition here and the clock is always in my hand, I know the time; then I can see the futuristic state because the right-hand side of the differential equation is known by the mathematical model. So, in this representation, you can see that I am just going to apply $x(t_0)$ and I will get $x(t)$.

No external input is required. So, that kind of system is called an autonomous system. And what is the equilibrium point? We are going to talk about those kinds of points. If I start, I will stay there. So, I am assuming here that $x(t) = 0$is my equilibrium point.

And this mapping is from $\mathbb{R}^n$to $\mathbb{R}^n$. So, in order to talk about finite-time stability, I again have to apply the concept of continuity. So, this definition is actually inspired by continuity. And how are we basically applying continuity? We already know that in order to apply continuity, I have to plot between the dependent variable. So, here the dependent variables are $t_0$and $x(t)$. This plot is not a better plot because this system lies in n-dimensional space.

So, this is just approximate. I have several trajectories here. After that, I am going to zoom in on this $x(t)$and correspondingly, we are trying to see whether $x(t_0)$is also going to lie on some ball or not. So, I have an origin. After that, I created a kind of ball where $x(t)$is going to lie, and we are trying to find a kind of ball where the initial condition will lie.

So, if system is Lyapunov stable. So, if I start from this ball, I am not going to leave the bigger ball. That is the idea. So, the same kind of things I have written in the form of the definition for every ε. So, you can select the $x(t)$ball based on your desiredness, and corresponding to every ε-ball, there exists a δ. So, once you fix ε, δdepends on εsuch that if you start with x_0, what is the meaning of x_0? $x_0 - 0$, 0is the origin.

So, less than δ, then $x(t)$and 0are going to map to 0. So, 0, because 0is the equilibrium point. So, $x(t) - 0 < \varepsilon$ for all t. This is the Lyapunov stability condition, and now we are also going to talk about finite-time attractivity. What does it mean? That all solutions, if you start from here, are at the equilibrium point, so you stay here, but if you start near the equilibrium point, I guarantee that you will reach here.

If this kind of phenomenon happens everywhere in the state space, then that is called the global finite time stability for any initial condition. You can see here that in observation, the example that the time of convergence is always a function of the initial condition. It means that if the initial velocity is very, very high, then you have to wait more time to stop. So, a similar kind of observation here can be seen in the case of time. It means that finite time convergence is sensitive with respect to the initial conditions.

This is the formal definition. So, I have an nth-order system. So, one way I will solve the differential equation is by obtaining the information of $x(t)$, and then I will apply the ε–δ definition. What is another way? You can map this system to one dimension. How do you do that kind of mapping? You can define some kind of Lyapunov function, which will map from $\mathbb{R}^n$ to $\mathbb{R}^+$, and I am going to maintain positive definiteness. What is the meaning of positive definiteness? For all non-zero x, $v(x) > 0$and $v = 0$ only at $x = 0$, and here radial unboundedness is required because I am talking about global finite-time stability.

whatever our initial condition, Lyapunov ball should capture that and due to that reason, this condition is required. And after that, I also want to ensure the finite-time attractivity, and you can see that using this condition, since $\dot{v}(x) \leq 0$, that condition is actually inbuilt inside this, and due to that reason, stability is ensured. But I have little more than that, and due to this term, I have finite-time stability. So, here αshould lie between 0and 1. Why? Because I want to enforce that once our trajectory is close to the equilibrium point.

At that time, they will converge with very high speed. How can I confirm that? You can just take the derivative of $v(x)^\alpha$with respect to x, and you will be able to see that since αlies between 0and 1. So, as you are very close to the equilibrium, the rate of change, or Jacobian, of this is tending towards infinity, and due to that reason, I will get the finite-time attractivity. So, that is a geometrical interpretation that you can see physically or mathematically by solving this. So, this is the mathematical way to show finite-time attractivity.

What have I done? I have taken the same equation, I have done the variable separation method, and after that, I have $v(x_0)$at $t = 0$; at $t = t_f$, this is 0. So, I have integrated this, and after that, I will get this kind of expression. So, here $v(x_0)$is finite, $c$is finite, everything is finite, and for that reason, t_f is also finite. So, in finite time, if I have this kind of condition, then $v = 0$in finite time also implies $x = 0$. So, at the equilibrium point, we will reach it in finite time; that is the conclusion.

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After that, I have to calculate the rate of change. So, this is a differentiable function. So, $\sigma\dot{\sigma}$, 2and 1/2 will cancel out, and after that, I have substituted the system dynamics, and then I have substituted the control. So, this is our system. In this particular development, since our system is highly uncertain, I am assuming that I do not need to worry about $\sigma = 0$. What kind of things occur to $\sigma = 0$? Practically, in finite time, we will go to the close vicinity of the equilibrium point, and I am going to stay there.

So, practically, $\sigma = 0$. And for that reason, we are not worrying about that. And due to that reason, I have defined $\text{sign}(\sigma) = \frac{\sigma}{|\sigma|}$. And then, what am I going to do? I am going to substitute the signum σlike this. And then, I am going to apply the gain of the controller; I am assuming that it contains the constant part. Why is a constant part required? Because I am going to make sure, the system is very, very uncertain.

So, even if this part is equal to 0, because this depends on the state, I have some kind of gain to handle the uncertainty, and due to that reason, the gain of the switching control I am maintaining is $\beta = \beta_0 + \rho(\sigma)$. After substituting this, I will get this expression. Now, I am assuming this kind of condition, and you have already seen that this should be greater than 0. So, this is the kind of condition I have.

So, here is something like $g_0$that I have. So, due to this condition, I am able to write like this, and I can also separate like this, because $\sigma\, g(\sigma)$is always positive. Now, what am I going to do? I am going to apply the norm inequality here, and using the norm inequality I will be able to get this, and after that, I am going to substitute, and after substitution, now you will get this kind of expression. So, you can see here that $\dot{v}(\sigma) \leq 0$. So, the system is stable, but finite-time stability is not concluded from here; for that, I have to do some more exercises.

So, first I am going to show you how to talk about ε–δ stability; even if I transform an n-dimensional system into one dimension, you will still be able to show ε–δ stability

So, I have selected an epsilon ball that is associated with $\sigma(t)$, and with respect to the initial condition, I have defined some kind of delta ball. So, this epsilon ball is associated with the trajectory, and the delta ball is associated with $\sigma(0)$, and I have to show that there is a relation between $\varepsilon$and δ. εI can change independently, then δwill fix. So, similarly, I have first defined ε, and then I will fix the δ, and $\sigma(0) < \delta$. So, I am able to write

$$\frac{1}{2}\,\delta^2 = \varepsilon.$$

And since I know from the previous expression that $\dot{v} < 0$, it means that $v$is positive. So, $v(\sigma)$is positive, and $\dot{v} < 0$. So, what happens? Whatever futuristic value is lesser than the initial value, and due to that reason, this condition comes into the picture. Using that relation, I can show that there is a relation between ε and δ.

So, in this way you can show the first part. Second part, now from here I have to derive the convergence time. So, again, this is just a first-order equation. So you can apply the Comparison Lemma. So, you can convert it into a differential equation. So, here I am directly applying the Comparison Lemma at $t = 0$; I have $v = v(x_0)$, and at $t = t_f$, I have $v = 0$.

And after integrating, this is a routine process you can do, and after that, you can apply $v(0) = \frac{1}{2}\sigma_0^2$, and then you will be able to get this kind of time of convergence. So, the initial condition σ_0, $g_0$is positive, and $\beta_0$is defined as the constant part of the gain. So, the time of convergence is always finite. So, what is our conclusion? Even if I have a very highly uncertain system, I can still achieve finite-time stability. And this kind of concept I am going to use in almost all futuristic slides.

For that reason, you have to understand this part carefully. So, what is our conclusion now? So, the conclusion is that I can employ the finite-time uncertainty concept and finite-time stability concept to any uncertain system. Now, at that time you have to design a gain that is based on the combination of two gains: one is a constant gain, and the other gain depends on the state of the system. After that, you can see that several systems, several first-order systems, are modeled using the differential equation discussed in today's lecture, like

$$\dot{\sigma} = h(\sigma(t)) + g(\sigma(t))\,u.$$

So, whenever you have an uncertain system, there is no need to worry.

Again, you can apply sliding mode control. What is sliding mode? Whenever you have one dimension, the meaning of sliding mode is nothing but the origin, because in infinite time you can reach some kind of manifold, and in that manifold, you are going to maintain, so due to that reason, $\sigma = 0$is also interpreted as sliding mode control. Now, I am going to employ this kind of concept for higher-order systems in the next class, and then again, what is our objective? Our objective is exactly the same. I will design either a linear or a non-linear sliding surface. I will employ this kind of concept for that sliding surface, and after that, I will maintain it. So, for maintaining purpose, I can able to design any kind of classical control, because once we are on the sliding manifold, our system is insensitive with respect to any kind of matched uncertainty.

So, any classical control technique that is not able to take care of the uncertainty or at least match the uncertainty, we can employ and design the sliding surface based on that. So, that is the topic for the next class. So, with this remark, I will end this lecture. Thank you very much.