

Sliding Mode Control and Applications

Dr. Shyam Kamal

Department of Electrical Engineering

IIT(BHU) Varanasi

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Lecture-58

So, welcome back. In the previous class, I talked about the terminal sliding mode control that is based on the concept of arbitrary time stability. So, if you see the literature on arbitrary time stability, parallel to arbitrary time stability, there are several notions, and the aim is exactly the same that evolve within the literature. Okay, so after the publication of this paper, some new names have actually come into the picture, which were coined by another group. They suggested a name that I will discuss in the next part of this particular lecture, and based on that particular name, I will explain why I am going to maintain that name, as we have the same concept but several different names. So, you should be aware of different names so that you will not be confused whenever you are reading the literature.

And due to that reason, we are equally respecting all names like predefined time stability, prescribed time stability, some kind of predefined time stability of the upper bound of settling time, arbitrary time stability, free will arbitrary time stability, and almost every. The stability concept which I have just discussed, M , is exactly the same. We have to somehow try to control the time of convergence, and based on that, we have to develop a new theory such that the theory is applicable to the system whenever some uncertainty comes into the picture. So, this is the main goal of the lecture.

So, in this lecture, I am going to extend the theory of arbitrary time stability in the framework of input-to-state stability. That is a very, very important notion whenever we are basically dealing with non-linear systems. So, the main objective of this lecture is to understand basically one of the variants of arbitrary time stability that is called predefined upper bounded settling time. So, for stabilization, we are trying to solve the stabilization issue with the predefined upper bound of the settling time. It means that whatever settling time function τ is, we have seen that in the case of finite time stability, this settling time is a function of t_0, x_0 , and the design parameter.

Time of convergence is positive, obviously. So, either you can put the absolute value or not put the absolute value; that does not matter, and this is T_a . So, somehow we want to achieve this. So, either you can use the word "arbitrary time" or a stabilization with a predefined upper bound of settling time; both are exactly the same. So, the notion of predefined time stability is initially developed for the autonomous system.

What is the meaning of an autonomous system? So, where time is not explicitly involved. And whenever we actually look into the arbitrary time stability, that is actually available

for the non-autonomous system. We are trying to fill the gap. So, this notion we have already predefined time stability. So, how can we extend this kind of notion for the time-varying system, meaning a predefined time stability notion for the non-autonomous system? So, basically, if you look closely, that is nothing but arbitrary time stability.

But since the predefined name is already coined, and for that reason, in order to maintain some kind of transparency within the literature. Later, we decided to give the name of arbitrary time sliding mode control or arbitrary time stability as a predefined upper bound of settling time-based design. So, in this lecture, I am going to be consistent with this name that defines the predefined upper bound of settling time. It means that we are talking about the predefined time stability of a non-autonomous system. And obviously, we are going to extend this notion of input to state stability.

And you are already well aware that for a linear system, we know that if the system is stable, it is possible to show that a bounded input will give us a bounded output. But this is not always actually valid in the case of the non-linear system. Even if there is no control or uncertainty, if our nonlinear system is stable or asymptotically stable, it is possible to show that some class of disturbance is going to destroy that stability. It means that a bounded disturbance will not give a bounded output for each and every non-linear system. Due to that reason, we are trying to focus on that class of non-linear systems, which will somehow show behavior similar to that of linear systems.

It means that if non-linear system is stable without control, then we expect that bounded input will give us the bounded output. Obviously, we have to identify that class of the non-linear system, and it is possible to show that if that class comes into the picture, then somehow it will satisfy this kind of input to a state stability property. So, in another way, we are saying that if a non-linear system is input to a stable state, it will behave like a linear system, meaning a bounded input will give us a bounded output. Now, what is the main intention of this lecture? Non-linear systems that will satisfy the property of arbitrary time stability or a predefined upper bound of settling time stability; and if a disturbance comes into the picture, then what is their behavior? We are trying to understand state stability with the help of the input. And obviously, in order to understand the input to state stability, we have to develop some kind of Lyapunov function.

And obviously, whenever I am going to use this kind of notion, we have to talk about the predefined upper bound of settling time based on the Lyapunov function. Based on this particular notion, I am going to define the sliding mode control with matched disturbance. So, whatever sliding mode control we discussed in the previous lecture is somehow based on the classical sliding mode control, where I am reaching here in finite time using conventional control, and after that, we are actually converging to the equilibrium point. In this lecture, I am going to show you how it is possible to demonstrate that actually reaching a phase also allows you to tune arbitrarily for any higher or sliding mode control. It is possible to show that if you add some term in the Levant controller, then in an arbitrary amount of time you can reach the equilibrium point.

So, somehow, how to initialize the Levant controller such that you can get arbitrary time stability that we are going to look into, and obviously, we are going to take some academic examples as well as some implementations. So, obviously, whenever we are talking about the ISS notion, then the role of class \mathcal{K} and \mathcal{KL} functions comes into the picture. But whenever we talk about some notion or variant of finite time stability, I need a special class of \mathcal{KL} functions called the generalized \mathcal{KL} class function. So, the generalized \mathcal{K} function, which is also called the \mathcal{GK} function, we have already seen in several places. So, their behavior is exactly like the class \mathcal{K} function, but for some reason they are going to show this kind of behavior.

So, somehow, if you see the graph of this class of functions, it looks like this. So, that will 0 for some time interval. It is possible to show that if that \mathcal{GK} class function, the generalized \mathcal{KL} class function, we are basically going to tell. So, that generalized \mathcal{KL} class function, if it satisfies some kind of property, where τ . So, whenever we are talking about the generalized \mathcal{KL} class function, we are talking about the generalized \mathcal{K} function and \mathcal{L} function, and that \mathcal{L} function should, once it tends to some finite time τ , then that whole function should tend towards 0, if S is fixed.

That is our definition of the generalized \mathcal{KL} class function. Now, we have actually defined some new notion that is an arbitrarily generalized \mathcal{KL} class function. So, if you see this τ , this depends on the initial condition as well as the parameter. So, the exact value you cannot select based on the system, but somehow if you are able to select this τ arbitrarily, then that is called the arbitrarily generalized \mathcal{KL} class function. So, we have actually come up with a new notion of the generalized \mathcal{KL} class function that is called the arbitrarily generalized class \mathcal{KL} function whenever we talk about arbitrary time stability or predefined upper bounded settling time-based stability.

In this particular lecture, whenever I talk about the predefined upper bound of settling time-based stability, we will use this kind of notion of norm. Essentially, it means that at some point, it might be possible for the disturbance to take a very, very large value, say 200, at time t equal to 200, and before and after that time, if the disturbance is bounded by plus or minus 5. However, at that time, the supremum is just 5, not 200. So, that is the physical interpretation of this essential supremum, and after that, obviously, we are going to define L_∞ like this:

$$\|d\|_{[t_0, t_1]} = \text{ess sup}_{t \in [t_0, t_1]} \|d(t)\|, \|d\|_\infty = \|d\|_{[0, +\infty)}$$

$$L_D = \{d \in L_\infty : \|d\|_\infty \leq D\}$$

This is the definition of the ball $B_\delta = \{x \in \mathbb{R}^n : \|x - x_0\| < \delta\}$ and Lie derivative $L_f g$ we are going to utilize whenever we define the notion of the predefined upper bound of settling time-based stability or arbitrary time stability.

If you are confused, then you can assume that this is arbitrary time stability. And this class of functions is somehow essential whenever we are defining arbitrary time stability. So, I have written that we use the predefined upper bound of settling time in the controller to denote the predefined time, which is the predefined upper bound of the settling time-based convergent system. So, for simplicity, whenever this term comes into the picture, you can

imagine that I am talking about arbitrary time stability. Now, suppose that I have a system that looks like this; again, this is a non-autonomous system where this δ is some kind of system parameter we are assuming.

Consider the system:

$$\dot{x} = f(t, x, \delta), x(t_0) = x_0$$

where: $x \in \mathbb{R}^n$: System states, $\delta \in \mathbb{R}^p$: Tunable parameters, $f: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$: Nonlinear vector field, $f(t, 0, \delta) = 0$: Origin is equilibrium point, $t_0 \geq 0$: Initial time, x_0 : Initial states

So, it is some kind of tunable system parameter. Based on that, I can adjust the time of convergence. So, if we are going to adjust the time of convergence based on δ , then that is called arbitrary time stability, and we know that if that is independent of δ , then it is called free will arbitrary time stability. So, whatever time of convergence or upper bound of settling time we are talking about, that is somehow independent of this particular parameter. And obviously, we have discussed the finite time stability.

System $\dot{x} = f(t, x, \delta), x(t_0) = x_0$ is globally finite-time stable at origin if: - It is globally asymptotically stable - Any solution $x(t, t_0, x_0)$ satisfies:

$$x(t, t_0, x_0) = 0 \quad \forall t \geq t_0 + \tau(t_0, x_0) \quad \text{where } \tau: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$$

is the settling time function
Fixed-Time Stability: System $\dot{x} = f(t, x, \delta), x(t_0) = x_0$ is fixed-time stable at origin if: it is globally finite-time stable and settling time is bounded:

$$\exists \tau_{\max} > 0: \forall x_0 \in \mathbb{R}^n, \tau(t_0, x_0) \leq \tau_{\max} \quad \text{for all } t_0 \in \mathbb{R}_{\geq 0}$$

So, in finite time stability, whatever the time of convergence, this is a starting time t_0 . So, that is a function of t_0 and x_0 , and in some cases, that is also a function of the parameter. Whenever we talk about fixed time stability, we then discuss some maximum settling time. So, this is not the true settling time because we have already seen in several places that the time of convergence may be 2.5 seconds, but τ_{\max} will come out to be like 300 seconds.

So, theoretically, this particular time of convergence will not give us true information for that reason. So, you can see one paper that is written by Professor Morino, and they are trying to develop some kind of differentiator based on fixed-time stability, and there you can see this kind of phenomenon. Theoretically, the time of convergence is very, very high, but the actual time of convergence is very low. Due to that reason, this time of convergence is not providing us with true information. And due to that reason, we need another notion.

And I have already told you that either you can talk about the predefined upper bound of settling time-based control or arbitrary time-stable control; both are the same. Just in order to maintain consistency, we have actually used this kind of name, and I have already repeated that the predefined time stability is already developed for the autonomous system. So, I want to create a unified notion so that

no confusion arises whenever a new reader enters this particular field. So, if τ_a depends on a tunable parameter, then basically I can talk about the arbitrary time stability. And if that is arbitrarily chosen, then somehow we are talking about arbitrary time stability.

PUBSTC Definition: System $\dot{x} = f(t, x, \delta), x(t_0) = x_0$ is PUBSTC at origin if: - It is fixed-time stable - Exists $\tau_a > 0$ depending on tunable parameters δ - τ_a can be stated a priori for given δ - Arbitrary τ_a values can be assigned via δ changes - $\tau_a \geq \tau_{ac}$ for any given δ

So, our notion is that we are assuming this time of convergence does not depend on the initial condition as well as the parameters of the system. And obviously, for that, I need some kind of very, very specific form, and this form we are telling that here whatever I have, this particular \dot{x} , so the right-hand side of \dot{x} is given like some kind of η and θ , okay? And θ we have already defined whenever I am introducing the notion $e^s - 1, e^s(t_f - t)$, okay? So this form you have already seen in the previous two classes, so just in order to make. In short, we have used the $\theta(s, t)$. So, whenever $\theta(s, t)$ comes into the picture, you can always somehow correlate that with this particular kind of function that appears:

$$\theta(s, t) = \frac{(e^s - 1)}{e^s(t_f - t)}$$

PUBSTC Nonautonomous System: Dynamics:

$$\dot{x} = \begin{cases} -\eta\theta(x, t) & \text{if } t_0 \leq t < t_f \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$x = \ln(1 + \kappa(t_f - t)^\eta)$$

$$\text{where } \kappa = \frac{e^{x(t_0)} - 1}{(t_f - t_0)^\eta}$$

Properties: $x \rightarrow 0$ as $t \rightarrow t_f$ for $\eta > 1, \dot{x} = 0 \forall t \geq t_f$,

Now, if you solve this equation, since I have started the notion of arbitrary time, I have already talked about this; somehow, this parameter k is coming like this.

And it is possible to show that as t tends towards t_f , this value is somehow 0, and due to that, x is equal to 0. An origin is nothing but uniformly stable because whatever t_f I have selected is t_0 plus some time of convergence t_a . So, time of convergence t_f is somehow independent of t_0 ; why is this independent of t_0 ? So, it is possible to show that you can select the value of T_a that is just a little greater than t_0 , which means you can start your system, and after that, you can select any time, then you can adjust t_f . Particularly, whenever you are doing this kind of thing for a practical system, you have to be careful because I have already told you that every practical system has inertia. So, you cannot be able to make this time close to 0.

You have to give sufficient time so that the system will respond to the controller, and for that reason, in a practical system, τ_a is not close to 0 ; it is some finite value. Now, you can easily design the control. We have already seen that if our state space is in a compact set, then obviously the control is also bounded, and for that reason, it is practically implementable. Now, what we are saying is that x equal to 0 is a predefined upper bound of settling time control based on stability or predefined time, and there exists some kind of function, a new class of function. So, arbitrary generalized \mathcal{KL} class function β , and I have already told you that whenever we are talking about an arbitrary general class \mathcal{K} function, then we have two arguments.

So, with respect to the first argument x_0 , we now have a generalized \mathcal{KL} class function. With respect to the second argument as t tends towards t_f , and t_f now if we are able to tune, then that is nothing but an arbitrarily generalized \mathcal{KL} class function, because t_f does not depend on anything anywhere. So, now whenever I have a system like this, I have to show that somehow this kind of relationship comes into the picture. And here I am assuming that $x(t)$ will lie in any compact set because γ is a C^∞ function.

So, I can also cover the whole space. So, one can easily show that it is possible to show that t_f already contains the information of t_0 plus τ_a . So, without loss of generality, I can take t_0 equal to 0 . So, whenever I am going to start my system, I will set t_0 equal to 0 , and due to that t_0 , I will keep it here without any ambiguity, and after that, it is possible to show that this is uniformly stable, because for any t_0 , suppose that if a finite t_0 comes into the picture, then t_f should be t_0 plus τ_a . And obviously, Δ , I will select the minimum of these two. So, now, for every ϵ , there exists a δ such that if I start from a δ ball, I am going to converge to the ϵ ball.

And in this way, I can show that this particular arbitrary time-stable system or predefined upper bound settling time stable system is always stable since this is a time-varying system, and due to that, we are talking about uniform stability, which means that the concept of stability is independent of the initial selection of time. So, how do I make myself independent again? I am repeating that whatever you are going to select. So, you can add t_0 inside that then everywhere you can able to show that whatever system that is uniformly stable. Now, this is the Lyapunov characterization. So, here this function is nothing but, somehow, you can easily see that as x tends towards infinity.

Equivalent PUBSTC Definition (Lemma): The equilibrium $x = 0$ of system is PUBSTC if \exists class \mathcal{AGKL} function β and class \mathcal{K}_∞ function γ s.t.

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t_f - t), \forall t \in [t_0, t_f), \forall \|x(t_0)\| < \gamma(c)$$

Proof: $\|x(t)\| \leq \beta(\|x(t_0)\|, 0)$ shows uniform stability, For $\epsilon > 0$, choose $\delta = \min\{\gamma(c), \beta^{-1}(\epsilon, 0)\}$, $\|x(t_0)\| < \delta \implies \|x(t)\| < \epsilon$, $\|x(t)\| \leq \beta(\gamma(c), t_f - t) \rightarrow 0$ as $t \rightarrow t_f$

So, this is radially unbounded. And after that, if that satisfies this kind of property, obviously, \dot{V} is less than or equal to 0 and \dot{V} equals 0 as t is greater than or equal to t_f . So, all definitions of stability of the Lyapunov for the non-autonomous system are going to be satisfied. One can easily show the proof that if this condition is satisfied and \dot{V} is less than or equal to 0. Then x equals 0 or x equal to 0 is actually stable.

Lyapunov Theorem: For system $\dot{x} = f(t, x)$ with origin in $D \subset \mathbb{R}^n$, if \exists smooth $V(t, x)$ and $\eta > 1$ s.t.: $k_1 \|x\|^a \leq V(t, x) \leq k_2 \|x\|^a$, $V(t, 0) = 0$, $\dot{V} = 0$ for $t \geq t_f$,

$$\dot{V} \leq \frac{-\eta(e^{k_3 \|x\|^a} - 1)}{e^{k_4 \|x\|^a} (t_f - t)} \quad \forall t \in [t_0, t_f)$$

then origin is PUBSTC with $t_f \geq \tau_{ac}$. If equality holds in last condition, then $t_f = \tau_{ac}$.

Proof (Part 1): Using comparison lemma with $k_3 = k_2, k_4 = k_1$:

$$\dot{V} \leq \frac{-\eta(e^V - 1)}{e^V (t_f - t)}$$

Solution:

$$V(t, x(t)) \leq \ln(\kappa(t_f - t)^\eta + 1)$$

$$\text{where } \kappa = \frac{e^{V(t_0, x_0)} - 1}{(t_f - t_0)^\eta}.$$

State bound:

$$\|x(t)\| \leq \left[\frac{\ln(\kappa(t_f - t)^\eta + 1)}{k_1} \right]^{\frac{1}{a}}$$

Proof (Part 2): From solution, $V(t, x(t)) \rightarrow 0$ as $t \rightarrow t_f$, $\dot{V} \leq 0$ when $t \rightarrow t_f$, $x(t)$ remains at origin $\forall t \geq t_f$, Therefore $t_f \geq T_{ac}$, For equality case, similar analysis gives $t_f = T_{ac}$.

Class \mathcal{AGKL} function identified:

$$\beta(r, s) := \left[\ln \left(\frac{(e^{k_2 r^a} - 1)}{(t_f - t_0)^\eta} (t_f - s)^\eta + 1 \right) \right]^{\frac{1}{a}}$$

So, stability is evident. Now, I also have to show the attractiveness, and I have to show that in arbitrary time, I can achieve x equal to 0. So, only the attractive part is important, because we know this part from the classical non-linear literature. So, if you adjust the gain like this, k_3 equal to k_2 and k_4 equal to k_1 , then what are k_1, k_2, k_3 , and k_4 that are coming from here? Then you can easily see that I will get this kind of differential inequality, and in the previous two classes, we have seen that if this is this kind of differential inequality,

then the solution looks like this, and finally, this class of means arbitrarily generalized \mathcal{KL} class function comes into the picture. So, for arbitrary time stability, if you select control in this particular way, then it is possible to show that the solution always satisfies this kind of arbitrarily generalized \mathcal{KL} class function if there is no input. And obviously, whenever this \mathcal{KL} class function is called, it has two arguments, and we have already seen this.

So, using this particular arbitrarily generalized \mathcal{KL} class function, one can show that if t_f is τ_{ac} , then whatever time of convergence I have selected, within that time of convergence, this is equal to 0. Now, suppose that a disturbance also comes into the picture. So, I know that our x equals 0 in the absence of disturbance, which will satisfy the property of the predefined upper bound of settling time-based stability. Now, what do I have to do? I have to extend this kind of concept if the disturbance is bounded by L_∞ . We have already defined what the meaning of L_∞ is.

System Definition: Consider the nonautonomous system

$$\dot{x}(t) = f(t, x(t), d(t)), t \geq t_0$$

where: $x \in \mathbb{R}^n, d \in \mathbb{R}^m, d(t) \in L_\infty$ and $f(t, 0, 0) = 0$. $S(t, t_0, x_0, d)$ denotes all forward solutions.

Stability Definitions: Finite-time ISS: $\|x(t)\| \leq \beta(\|x(t_0)\|, t) + \gamma(\|d\|_\infty)$ where $\beta(r, t) = 0$ when $t \geq T(r)$, Fixed-time ISS: Finite-time ISS with $\sup_{r \geq 0} T(r) < +\infty$, PUBSTC-ISS: $\|x(t)\| \leq \beta(\|x(t_0)\|, t_r - t) + \gamma(\|d\|_\infty)$ where $\beta(r, t) = 0$ when $t \geq t_r$ (prescribed).

So, suppose that our origin when d is equal to 0 is 0, and this is the forward solution. So, we are just talking about the forward solution because we are discussing finite time stability, and I have already told you that whenever you are talking about finite time stability. So, in finite time, two trajectories will meet. So, you are not able to understand where this trajectory is coming from. So, in the backward direction, you are not able to see, and due to that reason, we are just talking about the uniqueness in the forward direction.

If you see the definition of finite time stability, then at that time, this β is basically nothing but a generalized \mathcal{KL} class function. Now, this γ is actually the \mathcal{K}_∞ function. Now, what we are going to do, whenever I talk about the fixed time stability at that time, the generalized \mathcal{KL} class function's time of convergence is upper bounded by some T_r , which is less than infinity. However, how this is different from fixed time stability is that the time of convergence I can select means the time of convergence is always prescribed, given by the user, and this is the definition of the prescribed upper predefined upper bound of settling time based on the input to a state stability concept. Suppose that whenever I have to check this, then obviously one of the ways I can solve the differential equation, and after that I will formulate this kind of arbitrary generalized \mathcal{KL} class function.

Lyapunov Conditions: For $V(t, x): \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}$: 1. $\exists \mathcal{K}_\infty$ functions $\alpha_1, \alpha_2: \alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$. 2. $\exists \mathcal{K}$ functions α_3, α_4 and $\mathcal{AGK}\beta$:

$$\|x(t)\| \geq \alpha_4(\|d(t)\|) \Rightarrow \dot{V}(t, x) \leq -\beta(\|x_0\|, t_f - t) - \alpha_3(\|x\|)$$

PUBSTC-ISS Lyapunov Definition: $V(t, x)$ is PUBSTC-ISS Lyapunov if: ISS Lyapunov conditions hold, $\beta(\|x\|, t_f - t) \mapsto \eta_1 \theta(V, t), \alpha_3(\|x\|) \mapsto \eta_2 V^\alpha$ where $\eta_1 > 1, \eta_2 > 0, \alpha > 0$

This step is very tedious, and for that reason, it is better if I provide some kind of Lyapunov characterization. So, even if I have an n th order system, I will keep that n th order system within the framework of Lyapunov. It means that I am going to move from an n th order system to one dimension, and after that, I will apply or try to find the same kind of arbitrary generalized \mathcal{KL} class function. So, that is our main motivation. So, it is possible to show that if a Lyapunov function is bounded by a $2\mathcal{K}_\infty$ class function, and after that, α_3 and α_4 are $2\mathcal{K}$ class functions, and suppose that if that will satisfy this kind of condition, once this comes into the picture.

Proof (Part 1): - Define invariant set: $\Xi = \{x: V(t, x) \leq \alpha_2(\alpha_4(\|d\|_\infty))\}$. - For $t > \tau$ (entering time of Ξ): $\|x(t)\| \leq \gamma(\|d\|_\infty), \gamma = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_4$. - For $t \leq \tau: \dot{V}(t, x) \leq -\eta_1 \theta(V, t) - \eta_2 V^\alpha$.

Proof (Part 2): - For $t \leq \tau: \|x(t)\| \leq \beta(\|x_0\|, t_f - t)$ - Combining both cases: $\|x(t)\| \leq \beta(\|x_0\|, t_f - t) + \gamma(\|d\|_\infty) \forall t \geq 0$ - Thus system is PUBSTCISS

So, for any \mathcal{K} -class function, if this relation satisfies, then I can tell that x equals 0 of the original system, which is nothing but the predefined upper bound of settling time based on input to a state stable. And proof is not so difficult, but you can construct the β function like this, and you can construct the α_3 function like this. Then it is possible to show that I can always get, so what I am saying here is that I am claiming that if some system has a predefined upper bound on settling time based on stability, then I always have some kind of Lyapunov function. So, I have to prove that. So, the Lyapunov function condition is given by conditions 1 and 2 .

So, if conditions 1 and 2 are satisfied, then one can easily see that this condition is also satisfied, because we are stating that this result is only valid when this condition is satisfied, and for that reason, I have substituted on the right-hand side of the Lyapunov function. Now, one can also see from this that $x(t)$ is actually given by some kind of γ , and this is the norm of d_∞ . Due to that reason, γ_1 can be written like this. So, I hope that this step becomes clear to everyone, and after that, t less than or equal to τ , I have this relation, and whenever I have this kind of relation, then obviously, t less than or equal to $2t$, some kind of arbitrarily generalized \mathcal{KL} class function comes into the picture. So, now in the presence of disturbance, basically I have the influence of disturbance as well, because when t is greater than or equal to τ , we are going to enter inside this particular ball, and if you combine these two things, then basically the combination will give you the input to a stable state.

So, by selecting t_f you can able to make system input to a state stable in presence of the disturbance. Obviously, disturbances should be bounded. Now I am going to construct a

very simple system. So here you can see that without this x^3 and u , we know that this system is somehow the arbitrary time stable or predefined upper bound settling time-based stable. Now, what we have to show is that in the presence of the input u , this will again satisfy some kind of property like ISS.

System Dynamics: First-order system

$$\dot{x} = \begin{cases} -\eta\theta(x, t) - x^3 + u, & \text{if } t_0 \leq t < t_f \\ -x^3 + u, & \text{otherwise} \end{cases}$$

where: $x \in \mathbb{R}$ is state and u is control input.
 Lyapunov Analysis: Lyapunov function: $V(x) = x^2 \Rightarrow |x| = \sqrt{V}$.
 Time derivative for $0 \leq t < t_f$:

$$\begin{aligned} \dot{V} &= 2x\dot{x} = -2\eta x\theta(x, t) - 2x^4 + 2xu \\ &\leq -2\eta|x|\theta(|x|, t) - 2(1 - \delta)x^4 - 2\delta x^4 + 2xu \\ &\leq -2\eta|x|\theta(|x|, t) - 2(1 - \delta)x^4, \forall |x| \geq (|u|/\delta)^{1/3} \\ &\leq -2\eta|x|\theta(|x|, t), \text{ where } 0 < \delta < 1 \end{aligned}$$

Stability Analysis: From Lyapunov derivative:

$$\dot{V} \leq -2\eta\sqrt{V}\theta(\sqrt{V}, t)$$

Using transformed variable $\zeta = \sqrt{V}$: - Similar analysis as previous sections shows:

$$V = 0, \forall t \geq t_f$$

- Which implies:

$$x = 0, \forall t \geq t_f$$

So, how do we prove it? I am going to start with this simple Lyapunov function. I will take the derivative, and you can see here that it is possible to show that if now, this condition comes into the picture. So, under this particular condition, I am able to show where δ is nothing but that which is going to lie between 0 and 1, that \dot{V} will satisfy this condition. And from this condition, if you again do the substitution, I will finally get the arbitrary time stability or predefined upper bound settling-based stability.

So, V equal to 0 and x equal to 0, I can maintain. So, obviously, this condition will give the information of some kind of bond, ultimate bond. Now, suppose that I have some system like this. So, whenever we are talking about sliding mode control, we just need to ensure that in the presence of disturbance, our trajectory remains bounded. What is our greater hope? That in the presence of matched uncertainty, our system becomes insensitive with respect to matched uncertainty. And due to that reason, the notion of sliding mode control comes into the picture.

And whenever we have a system and if we know the output, using inputoutput based linearization or normal form, I can convert this system into the form of relative degree.

And what is the definition of relative degree? You can take the y and calculate the Lie derivative, or you can simply take the derivative, and if control explicitly appears in the n th derivative, then the order of this particular system is basically n . So, just for simplicity, here we are assuming that these terms β and $a(t, x)$ are known, but you can easily design when these two terms are also not known. So, for simplicity, I am assuming that these two terms are known and that whatever disturbance we have actually absorbed here. So, here without loss of generality you can also be able to assume that this disturbance also contain the state that is not going to harm our analysis provided this is bounded by d_0 .

So, for the sake of simplicity or for a simple case, I am going to consider this kind of system. So, if you design v , if these two terms are known, then I will design the inverse of $b(x)$. So, this will cancel out and minus $a(x)$. So, finally, this system is converted like this system. Now, I want to design some kind of use such that at arbitrary times, I can maintain x_1 ; x_1 is nothing but the \dot{y} , and x_2 is the \ddot{y} , like that.

System Transformation: Original system:

$$\dot{z} = \mathcal{F}(z, \varrho(t)) + \mathcal{G}(z, \varrho(t))v, y = h(z)$$

Normal form (relative degree n):

$$\begin{aligned} \dot{x}_1 &= x_2, \dots, \dot{x}_{n-1} = x_n \\ \dot{x}_n &= a(x, t) + b(x, t)v + d(t) \end{aligned}$$

Chain of integrators form:

$$\dot{x}_1 = x_2, \dots, \dot{x}_n = u + d(t)$$

So, output and they are all higher derivative equal to 0. So, in this way, I can incorporate the higher order sliding mode control. So, first we are going to see how one can modify Utkin's algorithm for classical sliding mode control so that one can achieve arbitrary time stability or predefined upper bound type stability. So, if you look carefully, if I have just a relative degree one system, and if I design an algorithm like this: $x(t)$, whatever, if s is here,

then you can put s . If x is here, then I am going to put x , and whether t is right or not does not matter, dt . So, if I have this kind of algorithm, it is possible to show that $x(t)$ equals 0 in some finite time, t greater than or equal to t .

But this finite time is going to depend on the initial conditions. If I have some kind of parameter, it is possible to show that if some parameter is involved here, then some uncertainty is present, and this also depends on that. Now, what is our aim? Our aim is to modify this algorithm such that, at any arbitrary time, you are able to select it so that I can move to the same objective x at time t equal to 0. That is our goal. So, how do we solve that? So, for that, I am going to add some new terms. And again, you can see that after t_f , I am going to switch to the classical sliding mode, the classical first-order sliding mode.

So, if you take this kind of Lyapunov function, you can see that exactly same structure will appear again. So, in this way I am able to show that as t tends towards t_f . then x equal to

0, and after that our structure is switched to the classical first-order sliding mode control. And obviously, in the presence of any unknown disturbance, whether that is 0 or not 0, we can maintain x equal to 0 by the design of classical sliding mode control. Similarly, in exactly the same way, I can extend this idea for the relative degree 2 system.

Okay, so how to proceed towards a relative degree two system? You can apply the backstepping, and after that, by doing backstepping, you can design the sliding mode in the second step. So, how do you select this σ_2 ? So, in order to select σ_2 , what am I going to do? Here I am going to define σ_2 as x_2 plus ζ_1 . ζ_1 is nothing but an arbitrary time-stable control. So, what is our main objective? Using sliding mode control, I am going to observe this term $d(t)$, and I am going to cancel it out. So, once that is cancel it out, then this whole system is free from the uncertainty and our system is now governed by the arbitrary or predefined upper mode based controller such that x_1 and σ_2 , σ_2 is just acts like the sliding variable that equal to 0.

System Dynamics: Second-order system with disturbance:

$$\dot{x}_1 = x_2, \dot{x}_2 = u + d(t)$$

where

$$|d(t)| \leq d_0$$

Okay, and x_2 equals ζ_1 . ζ_1 is an arbitrary time controller, and due to that reason, x_1 and x_2 both equal 0. So if you modify the second or sliding mode control like this, here after t_f , you have to exactly switch to the Levant controller. So any second-order controller, twisting whatever you want, any second-order controller you have to switch. Why? Because once I get to the origin, at that time I have to turn off this arbitrary time controller, because in the denominator I have a term like $t_f - t$.

Okay, and due to that reason, I have to turn off that controller. So we have to switch to some suitable controller, so it is better that you can switch to the Levant controller. So, by using arbitrary time, we are initializing the Levant controller so that I will reach here in arbitrary time, and after that, I will maintain. And again, proof is not so difficult. You can take the Lyapunov function and prove that we have an arbitrary time kind of construction that comes into the picture. Okay, so after that, what we have done is implement this kind of result for first-order and second-order systems in the presence of the disturbance, and you can see the result from one initial condition.

I have given different times: four seconds, five seconds, six seconds, and eight seconds for the second order; four, five, six, seven, and eight seconds are given for the first order. Okay, and now...What is beauty? That one can be able to initialize the Levant controller in the predefined time.

Simulation Parameters: - First-order system: Initial condition: $x(t_0) = 5$, Parameters: $\eta = 1.5, k = 0.12$ - Second-order system: Initial conditions: $x_1(t_0) = 6, x_2(t_0) = 4$, Parameters: $k = 0.2, k_1 = k_2 = 0.1, \eta_1 = \eta_2 = 2$ - Disturbance: $d(t) = 0.1 \sin t$

And back stepping somehow gives us a way such that I am able to initialize the Levant control in a predefined time. And control is also remains finite and it is also possible to reject the disturbance. Now, when I have started the notion of arbitrary time stability, I

have told you that one of the main importances of this notion is how to design an exact differentiator that will initialize at arbitrary time. So, you can see that if you remove this term, then this is exactly like the Levant differentiator. So, what is our contribution, or my contribution? My contribution is how to initialize the robust exact differentiator at a given time or at an arbitrary time.

State-space Representation: - Let $m(t)$ be the input signal - If $r = m(t)$ and $s = \dot{m}(t)$,

$$\dot{r} = s, \dot{s} = \ddot{m}$$

Robust Exact Differentiator:

$$\dot{z}_1 = -\eta_t \phi(\zeta_1) - \alpha |\zeta_1|^{\frac{1}{2}} \text{sgn}(\zeta_1) + z_2$$

$$\dot{z}_2 = -\frac{\beta}{2} \text{sgn}(\zeta_1)$$

$$\zeta_1 := z_1 - r$$

Variables:

$$\phi(\rho) := \theta(\rho) \text{sgn}(\rho) = \frac{(e^{|\rho|} - 1)}{e^{|\rho|}} \text{sgn}(\rho)$$

$$\eta_t := \begin{cases} \frac{\eta}{t_f - t} & \text{if } t_0 \leq t < t_f \\ 0 & \text{else} \end{cases}$$

Error Dynamics:

$$\dot{\zeta}_1 = -\eta_t \phi(\zeta_1) - \alpha \psi_1(\zeta_1) + \zeta_2$$

$$\dot{\zeta}_2 = -\beta \psi_2(\zeta_1) - \ddot{m}(t)$$

$$\zeta_2 := z_2 - s, \psi_1(\zeta_1) = |\zeta_1|^{\frac{1}{2}} \text{sgn}(\zeta_1) \text{ and } \psi_2(\zeta_1) = \frac{1}{2} \text{sgn}(\zeta_1)$$

Lyapunov Analysis:

$$V(\zeta) = \sigma^T M \sigma, \zeta = [\zeta_1, \zeta_2]^T$$

where $\sigma^T = [\psi_1(\zeta_1), \zeta_2]$ and $M = M^T > 0$ is a solution of:

$$\begin{bmatrix} A^T M + M A + \epsilon I + 4L_1^2 C^T C & M B \\ B^T M & -1 \end{bmatrix} \leq 0$$

for

$$A = \begin{bmatrix} -\alpha & 1 \\ -\beta & 0 \end{bmatrix}, C = [1 \quad 0], B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

with $\epsilon > 0$ and (α, β) in set K . Convergence Theorem: Assume that $|\dot{m}(t)| \leq L_1$ and $|\zeta_2| \leq L_2$. Also, there exists a positive constant c such that $|m_{12}| \leq c \leq m_{11}$, where m_{ij} for $i, j \in \{1, 2\}$ is an element of M , then for a positive constant, $\eta > \frac{8\lambda_{\max}(M)}{(m_{11}-c)}$, the differentiator is robust exact differentiator with arbitrary convergence time, t_f for the gains α, β contained in the set K given as:

$$K = \left\{ \left((\alpha, \beta) \in \mathbb{R}^2 \mid \beta - \frac{\alpha^2}{2} \leq 0 \text{ and } \frac{1}{\beta} < \frac{1}{2L_1} \right) \right\} \\ \vee \left((\alpha, \beta) \in \mathbb{R}^2 \mid \beta - \frac{\alpha^2}{2} > 0 \text{ and } \frac{1}{\alpha^2 \left(\beta - \frac{\alpha^2}{2} \right)} < \frac{1}{4L_1^2} \right)$$

So, for that, we have added this term, and here ζ is $z_1 - r$. So, I have to differentiate this signal r , that is $m(t)$, and r equals $\dot{m}(t)$. So, where again is this term $\phi(\zeta_1)$ that is defined like this, and $\eta(t)$ I am going to select like this? So, a closed-loop system is given like this, and using the Lyapunov function construction where this LMI comes into the picture, it is possible to show that there exists a set of gains that will satisfy this kind of condition. Okay, and using that kind of condition, you can show that I can differentiate any signal in arbitrary time. Okay, so the same kind of things we have actually shown using this kind of simulation result. You can see that this graph is for the Levant differentiator, and one can initialize the differentiator at an arbitrary time of 0.5 seconds. If you want to initialize at 1 second, you can initialize at 1 second. So, what is beauty? By using the notion of arbitrary time stability, you can initialize the differentiator at any arbitrary time. But for the Levant differentiator, you cannot be able to do that. Everything depends on the system. Similarly, for the observer, you can redesign the extra term, and again the proof is exactly the same as this.

Simulation Parameters: - Input signals: $m_1(t) = 5t + \sin t$ and $m_2(t) = 5t + 0.5\sin t + 0.001\cos(20t)$ - Parameters: $t_f = 0.5s, \eta = 15, L_1 = 1, L = 2, c = 0.3, z_1(0) = 4$ and $z_2(0) = 5$ - Matrix: $M = \begin{bmatrix} 1.3896 & -0.2938 \\ -0.2938 & 0.4505 \end{bmatrix}, \epsilon = 0.2724$

Why? Because error dynamics looks exactly like this. So, proof and gain selection for the observer is exactly the same as the previous one. and this is the observer dynamics. Again, if you select the gain, it is possible to show that e_1 and e_2 are equal to 0 at infinite times or arbitrary times. At that time, you will be able to select, and in this way, you can construct some kind of arbitrary time observer for the secondary system. Now, we have actually implemented these algorithms on the two-tank system, which is available in lab of IIT, BHU .

Perturbed Double Integrator:

$$\dot{x}_1 = x_2 \\ \dot{x}_2 = f(t, x) + \psi(t, x); y = x_1$$

Observer Design:

$$\begin{aligned}\dot{\hat{x}}_1 &= \eta_t \phi(y - \hat{x}_1) + \alpha \sqrt{|y - \hat{x}_1|} \operatorname{sgn}(y - \hat{x}_1) + \hat{x}_2 \\ \dot{\hat{x}}_2 &= f(t, \hat{x}) + \beta \operatorname{sgn}(y - \hat{x}_1)\end{aligned}$$

Error Dynamics:

$$\begin{aligned}\dot{e}_1 &= -\eta_t \phi(e_1) - \alpha \sqrt{|e_1|} \operatorname{sgn}(e_1) + e_2 \\ \dot{e}_2 &= -\beta \operatorname{sgn}(e_1) + \Delta(t)\end{aligned}$$

where $\Delta(t) := \psi(t, x) + f(t, x) - f(t, \hat{x})$ represents the residual error. So, if you see the configuration, so we have configuration like this. And what is our main goal? By controlling the level of the first one, one can control the level of the second. So, we have actually two level-based controls. So, in configuration one, we just have to control the level in tank one, and in configuration two, I have to control the water level in tank two. Whenever we try to control level 2, we have just one control: what is the control input? That is the pump voltage. So, if you talk about configuration 2, then that is represented by a second-order system; configuration 1 is a first-order system, and for that reason, it is possible to show that any arbitrary time theory can be easily implemented.

System Configurations: - Configuration I: Control water level in tank 1 Configuration II: Control water level in tank 2 - Dynamics for Configuration I:

$$L_1 = \frac{-A_{o1} \sqrt{2gL_1}}{A_{t1}} + \frac{K_p}{A_{t1}} u$$

where u is the pump voltage v_p . - For the tank in configuration II, the diffeomorphic change of variables for the error dynamics leads to a second order chain of integrators.

Here, basically, we have provided some sufficient time such that the system will stabilize. So, you can see that I have given 30 seconds and 40 seconds, which are predefined times. This is the control; you can see that the control is also very nice, meaning finite, which is not infinite, and similar kinds of things one can see for configuration 2, and I have to maintain the level of both tanks equal to 15 centimeters. So, it is possible to show that using control, which is 22 again, one can achieve exact stabilization for the two-tank system. In this particular implementation, I am assuming that there is no disturbance because the setup is inside our lab.

So, we have just deployed the backstepping-based control. Now, it is time to conclude this lecture. So, basically we have developed or discussed several important notions, particularly the input to state stability. That is an extremely good notion because every system is actually somehow affected by disturbance. So, whenever you are designing some kind of non-linear theory-based control system or something, you have to show that the system will satisfy the ISS property. And we have extended the notion of arbitrary time stability; we have obviously given a name for consistency.

The predefined upper bound of settling time is just the extension of arbitrary time stability, and we have also demonstrated how to actually design first-order as well as higher sliding mode control based on the same concept. We have also shown the implementation results, so now if you see the literature, several practical applications are being deployed by people using these kinds of controllers, particularly for missile robots, for the two-tank system, for industrial plants, for missile guidance, and for quadcopters. So you can select your system and try to implement these kinds of controls. Thank you very much.