Sliding Mode Control and Applications
Dr. Shyam Kamal
Department of Electrical Engineering
IIT(BHU) Varanasi
Week - 12
Lecture-57

Welcome back. In the previous class, I talked about the new notion of stability that is called arbitrary time stability. And what was our main goal? How can one control the time of convergence? And what was our objective? We want the time of convergence to be independent of the initial condition as well as the system parameters. Now, in this lecture, I am going to extend that kind of concept to designing terminal sliding mode control. So, we have already seen that in terminal sliding mode control, like in classical sliding mode control, we have two phases. One is actually reaching a phase and another is sliding into a phase.

And both phases actually occur in finite time, and due to that reason, overall convergence actually occurs in finite time, but the time of convergence is actually sensitive to the initial time as well as the parameter. Now, in this lecture, we are trying to control the time of convergence from the point where the trajectory is going to hit the sliding surface to the origin. So, during sliding mode, we are trying to control the time of convergence, and it is possible to show that in this way, somehow, overall, we can make it independent of the initial condition as well as the parameter. So, the objective of the lecture is to present arbitrary time terminal sliding mode control, and we have already seen the motivation because I want to design some kind of sliding surface such that we can reach here in finite time, and after that finite time, I can move from here to here.

We have already seen that classical terminal sliding mode control somehow suffers from singularity, and for that reason, non-singular sliding mode control comes into the picture. But the concept of non-singular sliding mode control is only limited to second order. So, it is possible to show that using this particular methodology, we can achieve the exact same kind of objective, and we can extend the design up to any order, and singularity problems can also be solved. This work is actually done in collaboration with Professor Xing Hao and Professor Bijnan Bandhapadhyay, along with my student Anil Pal. So, basically, we are trying to analyze the sliding phase control.

So, what is our main objective? Suppose that if I have some kind of initial condition, I will reach a certain point in some time t_r. So, this is reaching time; this time is actually a function of the initial condition as well as the parameter, but from here to here, whatever time we are going to select is independent of the initial condition, and due to that reason, overall time from here to here can be shown to somehow become independent of the parameter as well as the initial condition, because I can make this phase as quick as possible. Just one thing that we have to make sure of is that whatever time of convergence I select is greater than t_r. After that, we are also going to show you another algorithm that demonstrates how, even if this reaching time is unknown, one can actually retune the algorithm   to   achieve   this   particular   objective.   I   am   going   to   perform

a stability analysis based on the Lyapunov function that we discussed in the previous lecture.

And we are going to get some kind of form for an nth order integrator chain of integrator systems. So, obviously, what is the comparative benefit? So, I have already explained that this methodology somehow solves one of the very important classical problems associated with the terminal sliding mode control, which is the singularity problem. Over all time of convergence, it is possible to show that it is independent of the initial condition as well as the parameter. And we are also trying to show that there is some kind of explicit controller, a sliding mode kind of controller, and that the controller is also insensitive with respect to matched uncertainty. And we are also going to give the higher-order extension of this means beyond second order.

So, let us start with the system dynamics. So, this will represent some kind of non-linear system.

Consider the nonlinear system:

$$\dot{x} = f(t, x, \vartheta), x(t_0) = x_0$$

where: $x \in \mathbb{R}^n$: system states, $\vartheta \in \mathbb{R}^p$: system parameters, $f: \mathbb{R}_+ \cup \{0\} \times \mathbb{R}^n \to \mathbb{R}^n$: nonlinear function and origin $x = 0$ is an equilibrium point.

In this particular non-linear system, this ϑ, which is called var theta, is actually a design parameter or some kind of system parameter you can identify. x is the initial condition, and I am assuming that x belongs to $\mathbb{R}^n$, which means there are n state variables I have. Without loss of generality, I am assuming that $x = 0$ is my equilibrium point.

And obviously, this kind of notion we have already discussed in the previous class is fixed time stability. So, this definition I have taken from Polyakov's work.

Fixed-Time Stability (Polyakov 2012):
The origin is fixed-time stable if:

- Globally finite-time stable - Settling time function $T(t_0, x_0)$ is bounded $\exists T_{\max} > 0: \forall x_0 \in \mathbb{R}^n, t_0 \geq 0$
- $T(t_0, x_0) \leq T_{\max}$

So, $x = 0$, which is globally finite time stable, means that if I start anywhere in the state space, our trajectory remains bounded, and after that, we will converge in some kind of time that is upper bounded by some maximum time $t_{\max}$. So, this is nothing but the settling time that is a function of the initial time as well as x_0. It is possible to show that this is also a function of this parameter that is ϑ.

Now, with the help of the previous definition, we are going to actually define a new definition, and this is actually copied from our previous paper. So, the first paper on arbitrary sliding mode control. So, this time of convergence T_a, that is independent of any system parameter also. So, basically, this is independent of the initial condition. Why?

Because we are talking about fixed time stability, whenever we impose this kind of condition, it is also independent of parameters.

If T_a is greater than T_s, then T_s is basically some kind of time that is coming from the system. If the selected time T_a is greater than this, then we are saying that this is weak arbitrary time stability and T_s is strong arbitrary time stability; obviously, if T_a is not tuned based on the parameter, then we are referring to that kind of stability as free-wheel stability. So, the term "free will" I am not going to actually explicitly use here, because now, since obviously T_a is independent of v, some kind of arbitrary time stability comes into the picture. So, either you can interpret that as free will, or it is just a terminology. Particularly for a control engineer, control is very, very important, and the objective is important.

So, the objective is to control the total time of convergence, and T is the exact settling time. So, obviously, free will arbitrary time stability is a stronger notion than prescribed time. In the prescribed time, basically, regarding the time of convergence, we are trying to approach the end of this time of convergence, and that is only applicable up to $T_{\max}$. If you see the definition of prescribed time stability, this definition we are stating is extended up to T equal to infinity. It means that I can converge to an equilibrium point, and after that, I am also going to maintain fixed-time stability and finite-time stability, and in this way, this is a very strong notion.

In the Lyapunov characterization, we have already seen that if this Lyapunov function is bounded by a class $\mathcal{K}$ function. So, β_1 and β_2 are class $\mathcal{K}$ functions, and after that, $\dot{V} = 0$ for all t greater than or equal to t_f. Before that, they are going to satisfy some kind of expression, and this particular equality, if you see, we have already seen in the previous class that this is actually arbitrarily time stable, where t_f is nothing but $t_0 + T_a$, the time of convergence. So, we want to maintain the time of convergence T_a. That is defined like this.

Theorem (Anil et al. 2018):
For system $\dot{x} = f(t, x)$, assume:

- $V : [t_0, \infty) \times D \to \mathbb{R}_+ \cup \{0\}$ exists - $\eta > 1$ constant - Conditions hold:

$$\beta_1(x) \leq V(t, x) \leq \beta_2(x)$$
$$V(t, 0) = 0$$
$$\dot{V} = 0, \forall t \geq t_f$$
$$\dot{V} \leq \frac{-\eta(1 - e^{-V})}{(t_f - t)}; t_0 \leq t < t_f$$

So, in this way, basically this is a differential inequality, and after that, if I apply the comparison principle, then it is possible to show that within this particular time, if I start our system at t_0, then finally, we will converge to 0, and after that, due to this condition, since $V = 0$, already from this particular design, and $\dot{V} = 0$. So, if you apply Newton's first law, we are going to stay here. Now, one of the structures we suggested in the previous class, if I have a first-order integrator, then it is possible to show that if you design this v_1

like this, then within time t_f, and t_fI have already told you that t_f is nothing but $t_0 + T_a$. So, if you maintain $t = 0$, then basically t_f is equal to T_a. So, without loss of generality in this whole lecture, I am going to talk about the convergence time, t_f.

Scalar System Control:

Consider: $\dot{x}_1 = v_1$. Control law:

$$v_1 = \begin{cases} -\dfrac{\eta_1(1 - e^{-x_1})}{t_f - t}, & t_0 \le t < t_f \\ 0, & \text{otherwise} \end{cases}$$

where $\eta_1 > 1, t_f$: convergence time. Stability Analysis: Using $V = x_1^2$: Free-will weak arbitrary time stability - Convergence guaranteed by t_f

What does it mean? That I am going to start this system at $t_0 = 0$. And now, using the Lyapunov function again, I will get this kind of inequality, and one can show that $x_1 = 0$ is actually asymptotically stable. Now, one can able to extend this result for second order integrated system also. How can I extend? One is able to apply the back stepping control. So, what can you do? You can design some kind of change of the coordinates, and what is the beauty of this coordinate? I am assuming that $x_2 = -\phi_1$, and ϕ_1 is basically how we are going to design it.

System Dynamics:
Consider the second-order chain of integrators:

$$\dot{x}_1 = x_2, \dot{x}_2 = v_2$$

For arbitrary-time stabilization: - Desired $x_2 : x_{2\,\text{des}} = -\phi_1$ - Change of variables: $g_2 = x_2 + \phi_1$ - Transformed dynamics:

$$\dot{x}_1 = g_2 - \phi_1$$
$$\dot{g}_2 = v_2 + \dot{\phi}_1$$

So, if you are going to design ϕ_1 like this, then it is possible to show that x_1 is equal to 0 in a predefined time, and that predefined time t_f can be easily selected. So, if you start from $t = 0$, then at that t_f time, $x_1 = 0$. Now, one more important point that I have already discussed in the previous lecture. This is actually locally Lipschitz in x_1 and piecewise continuous in t up to t_f if you just exclude t_f. So, somehow discontinuity is going to lie in time.

So, since discontinuity is not lying on the x_1 state and for that reason, you are able to take differentiation up to t_f, and for that reason, I am able to take the derivative of this; that is the reason. Obviously, some kind of control is applied if you design ϕ_1 like $x_1^{1/2}\text{sgn}(x_1)$, something like this, with some gain k_1. Obviously, this is not differentiable at $x_1 = 0$.

Why? Because discontinuity is going to occur at $x_1 = 0$. But what is the beauty of this? Discontinuity is only a lie in time.

And due to that reason, the differential process is possible, and one can apply the backstepping method. Now, what am I going to do? Here, I am going to substitute x_2. So, $g_2 - \phi_1$ comes into the picture, and after that, I am going to define $\dot{g}_2$. So, $\dot{g}_2$ is nothing but $\dot{x}_2$. So, here v_2 comes into the picture and ϕ_1 comes into the picture.

So, $\dot{\phi}_1$ is differentiable, and it is also possible to show that it is bounded. And due to that reason, now I can able to design the control action with the help of Lyapunov function. So, this is actually totally inspired by the back stepping design. So, in backstepping design, first I will stabilize the first-order system, and after that, we will move on to the overall second-order system by making the coordinate change. So, the same principle applies here, and you can see that if I select ϕ_2 again, it is because I have to stabilize g_2.

Lyapunov $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Analysis:
Lyapunov function: $V = x_1^2 + g_2^2$

- $g_2 \to 0$ as $t \to t_f$

Control law:

$$v_2 = \begin{cases} -x_1 - \dot{\phi}_1 - \phi_2, & t_0 \leq t < t_f \\ 0, & \text{otherwise} \end{cases}$$

where:

$$\phi_2 = \frac{\eta_2(1 - e^{-g_2})}{(t_f - t)}$$

$$\dot{\phi}_1 = \frac{\eta_1\left[e^{x_1} - 1 + (t_f - t)x_2\right]}{e^{x_1}(t_f - t)^2}$$

So, I have to select ϕ_2, which is nothing but the result after the cancellation of all other terms, that ϕ_2 basically comes into the picture. So, the ϕ_2 structure should look like this. And if I do this, then by Lyapunov stability, you can see that I have this kind of term. So, $\dot{V}$ is either governed by this term or that term, depending on their value. So, in that particular way, since V is given like this, $\dot{V}$ is either the maximum of $|x_1|$ or g_2.

Time $\qquad\qquad\qquad\qquad\qquad\qquad$ Derivative $\qquad\qquad\qquad\qquad\qquad\qquad$ Analysis:
Time derivative of V :

$$\dot{V} = -2x_1\phi_1 - 2g_2\phi_2$$

$$\leq -\frac{2\eta_1|x_1|(1 - e^{-|x_1|})}{(t_f - t)} - \frac{2\eta_2|g_2|(1 - e^{-|g_2|})}{(t_f - t)}$$

Which implies:

$$\dot{V} \leq -\frac{2\eta_1 |x_1|(1 - e^{-|x_1|})}{(t_f - t)} \quad \text{or} \quad \dot{V} \leq -\frac{2\eta_2 |g_2|(1 - e^{-|g_2|})}{(t_f - t)}$$

Maximum   **Norm**   **Analysis:**

From $V = x_1^2 + g_2^2$:

$$V \leq 2(\max\{|x_1|, |g_2|\})^2$$

$$\sqrt{\frac{V}{2}} \leq \max\{|x_1|, |g_2|\}$$

For $\max\{|x_1|, |g_2|\} = |x_1|$:

$$\dot{V} \leq -\frac{2\eta_1 \sqrt{\frac{V}{2}}\left(e^{\sqrt{\frac{V}{2}}} - 1\right)}{e^{\sqrt{\frac{V}{2}}}(t_f - t)}$$

And in this way, if I substitute, I will again return to the exactly same equation. In the next slide, you can see that if I define the ρ variable as ρ, then I will get exactly the same kind of differential inequality. Like the arbitrary time stability, and due to that reason, $\rho = 0$ at arbitrary time t_f, and if $\rho = 0$, then V is also 0 or tending towards 0 at t_f, and now we are going to maintain $\dot{V} = 0$ in the next subsequent interval and due to that reason, it is possible to show that one can able to converge in this particular time. So, both variables are going to converge to 0 at that particular time.

Convergence   **Proof:**

Let $\rho = \sqrt{\frac{V}{2}}$:

$$\dot{\rho} = \frac{\dot{V}}{2\sqrt{2}\sqrt{V}}$$

$$\leq -\frac{\eta'(1 - e^{-\rho})}{(t_f - t)}, \eta' = \frac{\eta_1}{2} > 1$$

- ρ stabilizes within t_f

- $V \to 0$ in t_f

- $x_1, g_2 \to 0$ in t_f

- x_2 also converges via $g_2 = x_2 + \phi_1$

So, in that way, I can stabilize x_1 and x_2 within t_f time. Now, another important thing I have to maintain is that. So, how do we maintain it? So, after a while, you will get $x_1 = 0$ and $x_2 = 0$. So, just switch your dynamic to 0 , and due to that, in this type of control, switching is very, very important. Now, I am going to talk about how to extend this kind

of idea, whatever we have previously discussed, to any arbitrary order system, and here you can see that I have some kind of disturbance that is going to enter through the control channel.

nth-Order System:

$$\dot{x}_i = x_{i+1}, i = 1,2, \ldots n - 1$$
$$\dot{x}_n = u_n + d(t), |d(t)| \leq d_0$$

Objective: Design free-will arbitrary time convergent control for sliding phase - Case (i): Known reaching time t_r - Case (ii): Unknown t_r (algorithmbased design)

So, this is matched uncertainty, and we are assuming that this is bounded. So, for all supremum t, this is actually bounded by d_0. So, what is our objective? Now, our objective is to design free-will arbitrary time convergence. Free will means time of convergence I can select independent of the system parameter as well as the initial time, and due to that reason, this word has actually come into the picture. So, I have taken two cases: one where I know the reaching time t_r and another where the reaching time t_r is unknown. Most of the time, what happens is that whenever I have a practical system, the practical system lies in a compact set. So, by doing worst-case analysis, it is possible to show that one can estimate the time of convergence to the sliding surface. So, there is always some kind of t_r information, and it is possible to show that if you are going to keep t_f, the total time of convergence, greater than t_r.

Then you can always be able to move within that t_f. So, the only requirement is that t_f should be greater than or equal to t_r, but most of the time in practical situations this is unknown, and due to that reason, you have to slowly tune t_f, because I have to maintain t_r. So, we have to provide some kind of algorithm such that the controller will always check whether t_f is greater than t_r or not, and if that is not the case, they will improve their time. So, now I am going to start with this system, and I am going to assume here that whatever is actually bounded, it is possible to show that it is bounded. We have already shown in the literature that u is bounded.

Assumption   1:
For the system:

$$\dot{x}_i = x_{i+1}, i = 1,2, \ldots n - 1$$
$$\dot{x}_n = u_n + d(t), |d(t)| \leq d_0$$

There exists some bounded control u_n such that the states $x \in D \subset \mathbb{R}^n$. Now, what I am going to do is design when we assume t_r is known. If t_r is known, then what is the benefit that I can easily tune t_f without applying any algorithm? And second design, obviously, I am going to assume that t_r is unknown. So, here, obviously, in classical or conventional sliding mode control, coined by Professor Utkin or Emiliano, what are we basically doing? We are reaching from any initial point to here in finite time, and after that, we are maintaining asymptotically to the equilibrium point. Here we are trying to maintain this phase arbitrarily; that is our main goal.

We are not going to change this phase. Now, the key assumption here is that t_f is greater than t_r, but it is possible that in the next lecture I will give you some kind of algorithm where you will also be able to control this time of convergence. So, in arbitrary way you can able to converge from here to here. So, this time you can also control it, but I think that is not required, and for that reason, this design step is also good enough for the implementation purpose. Now, what is our main theorem? So, the main theorem is that we have finite time reaching phase dynamics, and after that, we have a free-wheel weak arbitrary time stable sliding phase. We are actually saying "weak" because most of the time, whatever t we are selecting.

Theorem (Free-Will Arbitrary Time TSMC): The system attains:

- Finite time reaching phase dynamics - Free-will weak arbitrary time stable sliding phase with control:

$$u_n = \begin{cases} -k\text{sgn}(s_n) - \dot{w}_{n-1} & t_0 \leq t < t_f \\ -k\text{sgn}(s_n) - k_1 x_2 - \cdots - k_{n-1} x_n & t \geq t_f \end{cases}$$

where: $k \geq d_0 \geq |d(t)|$ and k_1 to k_{n-1} make $s^{n-1} + k_{n-1}s^{n-2} + \cdots + k_1$ Hurwitz. Sliding manifold:

$$s_n = \begin{cases} x_n + w_{n-1} & t_0 \leq t < t_f \\ k_1 x_1 + \cdots + k_{n-1} x_{n-1} + x_n & t \geq t_f \end{cases}$$

So, our trajectory will go very close to t, but most of the time that is not exact. And due to that weak kind of stability, at least we expect, and after that, now you can see that within this time our control is given like this, and after $t = t_f$, again that will switch to the classical sliding mode control, because uncertainty is always present, and due to that, you have to always activate your control. So, once you are at the equilibrium point. Now, another aim is to determine how to maintain this equilibrium point, and for that reason, you have to switch to some kind of robust control. So, any robust control is okay, but I have to maintain insensitivity in the presence of any time-varying uncertainty, and for that reason, it is better to switch to some kind of classical sliding mode or higher sliding mode control algorithm.

You can see that here again control is 0 because x_2, x_n, and everything up to that point are 0. And what is the role of this sgn function that is actually going to switch with a very high switching frequency such that you can maintain it here? Okay, so here basically whenever we are selecting gain, based on hardware parameters, I am going to select the gain because somehow, once I am at the equilibrium point, I have to maintain it here. So this is nothing but some kind of sliding mode control design for an nth order system with $d(t)$. Okay, so classical sliding mode, I am going to switch. So, basically, our contribution here is about how to reach this given time t_f if $t_0 = 0$.

And obviously, if t_0 is T_a at that time, t_f is nothing but $t_0 + T_a$. So, the time of convergence is T_a. Before t_f, I have this kind of dynamics, w_n, and we are going to select based on the arbitrary time sliding mode and arbitrary time convergence principle. And after that t_f, I

basically have a classical sliding surface. So, here I am going to switch the sliding mode control as well as the sliding surface.

So, two different switchings are actually involved here. So, let us try to look at the second-order system. So, if I have a second-order system. So, what is the first step? First step is using back stepping.

You can try to design it. The first control is w_1 because if you have a second-order system, then obviously during the sliding phase you can see that the system dynamics is nothing but $s = 0$, and after that, the dynamics is given by $\dot{x}$ and whatever reduced-order dynamics come into the picture; due to that reason, first-order arbitrary time sliding mode is only enough. Okay, now what am I going to do? I am going to design this s_2 sliding surface that is $x_2 + w_1$. Why are we selecting like this? Because if w_1 is equal to $-v_1$, then what is v_1 ? From here, you can easily see. That whenever I have a first-order system that I have already designed, I am going to be consistent with the same terminology here. So, if I have v_1 and $\dot{x}_1 = v_1$, where this is negative, I can show that v_1 is tending towards 0 as t is tending towards some time t_f, if the initial time is 0 .

And due to that reason, the same kind of control I am going to maintain here, and somehow this control is at once while we are sliding. So, at that time, $s_2 = 0$ and x_2 is exactly replaced by $-v_1$. Now, what is our goal? Our goal is to converge to this sliding surface. So, for that, I am going to design based on the classical sliding mode principle, and I will select a disturbance that is greater than d_0. And after that, I finally have, if you see, if you apply this kind of control, then our overall dynamics, sliding dynamics, is $\dot{s}_2 = -k\text{sgn}(s_2) + d(t)$.

Second-Order System Analysis:

- Consider: $\dot{x}_1 = x_2, \dot{x}_2 = u_2 + d(t)$.
- Sliding surface design $\left(t_0 \le t < t_f\right)$:

$$s_2 = x_2 + w_1, w_1 = \phi_1 = -v_1$$

- Control law: $u_2 = -\dot{w}_1 - k\text{sgn}(s_2), k \ge d_0$.
- Resulting dynamics: $\dot{s}_2 = -k\text{sgn}(s_2) + d(t)$.

Sliding   Phase   Dynamics:
When $s_2 = 0$:

$$x_2 = -\frac{\eta_1(1 - e^{-x_1})}{(t_f - t)} \Rightarrow \dot{x}_1 = -\frac{\eta_1(1 - e^{-x_1})}{(t_f - t)}$$

By   Theorem, $x_1 \to 0$ in t_f,   implying $x_2 \to 0$.
So, in this way, I can show $\dot{s}_2 = 0$ in finite time. And that finite time depends on the initial time as well as the initial parameter. Now, if our final time is greater than t_r, you can see that $t_f - t_r$ comes into the picture. Because it is somehow giving the time, every time this is updated. So, this will convert x to 0 between t_f and t_r, for that amount of time that will

become 0 . So, if this time is short, obviously what happens is that this variable ω and $\dot{\omega}$ may be larger, and for that reason, you have to select a sufficiently large time.

Most of the time in a practical system, one can easily do it. So, once s_2 is equal to this, what happens after? Before time $t \leq t_f$, I have to apply or design the sliding surface based on the arbitrary time principle, but once I achieve the goal, I have to actually shift to classical sliding mode control, because as we move towards the equilibrium point, you can see that this term is actually tending towards 0 . Obviously, this is also tending towards 0 , and the 0 by 0 form comes into the picture. The beauty of this 0 and 0 by form is that it always gives you a finite value, and for that reason, this control is always finite. Okay, now I am going to shift to the classical sliding mode control.

Linear   Sliding   Surface:
For $t \geq t_f$:

$$s_2 = k_1 x_1 + x_2$$

Control law:

$$u_2 = -k\mathrm{sgn}(s_2) - k_1 x_2$$

Resulting dynamics:

$$\dot{s}_2 = -k\mathrm{sgn}(s_2) + d(t)$$

So, I hope that you are already well aware that if I have some kind of system $u + d(t)$, then I can design a sliding mode control. You can define the sliding surface like this, and after that, you can take the derivative of it. So, that is nothing but k_1 and $\dot{x}_1 = x_2$ and $\dot{x}_2 = u + d(t)$. and now you can design u. So, u if you design that is equal to $-k_1 x_1 - k_2 \mathrm{sgn}(x_2)$.

So, obviously what happens is that this term will cancel out, and after that, I have this kind of dynamic. So, within a finite time, it is possible to show that $\dot{s}_2 = 0$. So, we are already at 0 . So, if there is some kind of mismatch, classical sliding mode is also going to take care of this. So, this kind of behavior we are analyzing when $t \geq t_f$.

Now similarly, you can be able to extend exactly the same idea for the thirdorder system. But in a third-order system, obviously reduced-order dynamics, if you look carefully, are represented by a second order. And at that time, you need the second-order arbitrary time stability concept. So you have to go through the back stepping. So using backstepping, now I can design to select x_3 such that x_1 and x_2 will become stable in arbitrary time, and we have already seen during the backstepping how to design that kind of control.

Extension   to   Third-Order:
System dynamics:

$$\dot{x}_1 = x_2, \dot{x}_2 = x_3$$
$$\dot{x}_3 = u_3 + d(t)$$

Sliding surface:

$$s_3 = x_3 + w_2, w_2 = -v_2 = x_1 + \dot{\phi}_1 + \phi_2$$

Control law:

$$u_3 = -\dot{w}_2 - k\mathrm{sgn}(s_3)$$

Reaching dynamics:

$$\dot{s}_3 = -k\mathrm{sgn}(s_3) + d(t)$$

Once I have done that job, the next job is to maintain whatever transform variable s_3 that equals 0 for infinite time. For that, if I select control like this, it is possible to show that $s_3 = 0$, and once $s_3 = 0$, then you can see here that it equals 0 , then $x_3 = -w_2$, and what is w_2 ? w_2 is v_2, and by backstepping, it is possible to show that x_1 and x_2 both equal 0 at arbitrary times, and if x_1 and x_2 equal 0 , then x_3 is already 0 . So, there is no other choice, and x_3 is also equal to 0 in arbitrary time. In this way, you can extend this kind of concept for any order. So, every time you have to perform the back-stepping, based on back-stepping, you can select sliding mode control based on the last step, and after that, if you employ the control using the Lyapunov method, it is possible to show that you can achieve arbitrary time stability.

Induction   Hypothesis:
Assume true for $n = r$:

$(r - 1)$ th-order system stabilized by v_{r-1}

Sliding surface $s_r = x_r + w_{r-1}$

Control $u_r = -k\mathrm{sgn}(s_r) - \dot{w}_{r-1}$

Lyapunov Analysis:

For $(r + 1)$th-order system:

$$V_{r+1} = V_r + g_{r+1}^2 = x_1^2 + \sum_{i=2}^{r+1} g_i^2$$

Time derivative:

$$\dot{V}_{r+1} = -2\sum_{i=1}^{r+1} g_i\phi_i, \phi_i = \frac{\eta_i(1 - e^{-g_i})}{(t_f - t)}.$$

So, using induction, one can easily show that this kind of inequality comes into the picture, and finally, we will get this last inequality such that all $\rho + R = 0$ in finite time. So, whenever you are applying induction, you have to check for $n = 1$. So, for $n = 1$, that is arbitrary time stability. So, by induction it is possible to show mathematical induction that

for this result is true for all n belongs to $\mathbb{N}$. In this way, one can able to extend the proof for any order. Now, you have to become little more careful whenever you are selecting the parameter. So, parameter selection should satisfy this kind of condition in order to maintain the Lyapunov stability. So, it means that η_i, if you have a first-order system, then that is greater than 1 , but if you have a second-order system, then that is greater than 2 .

First-Order System Verification:

- For $n = 1$:

$$V_1 = x_1^2 \Rightarrow \sqrt{V_1} = |x_1|$$

- Time derivative:

$$\dot{V}_1 \leq \frac{-2\eta_1\sqrt{V_1}\left(1 - e^{-\sqrt{V_1}}\right)}{\left(t_f - t\right)}$$

- With

$$\rho_1 = \sqrt{V_1} : \dot{\rho}_1 \leq \frac{-\eta_1(1 - e^{-\rho_1})}{\left(t_f - t\right)}$$

- For $\eta_1 > 1, \rho_1 \to 0$ in t_f.

If it is a third-order system, then that is greater than 3 . So, these are the kinds of things you have to take care of whenever you are designing. So, if you have an nth order system, then n is whatever gain you are going to select, so all gains should be greater than n; that kind of thing we have to take care of, and obviously here that control design involves the conventional sliding mode control, so chattering must be present, and one can minimize the chattering by designing the first phase, the reaching phase, using the super twisting algorithm. So, at that time, instead of discontinuous control, you can involve this kind of control, $\text{sgn}(s)$ and plus x_2, and here $\dot{x}_2$ some fictitious variable. So, $\dot{x}_2 = -k_2\text{sgn}(s)$.

Stability   Proof:
Key inequality:

$$\dot{V}_{r+1} \leq \frac{-2\eta\sqrt{\dfrac{V_{r+1}}{r+1}}\left(1 - e^{-\sqrt{\frac{V_{r+1}}{r+1}}}\right)}{\left(t_f - t\right)}$$

Variable transformation:

$$\rho_{r+1} = \sqrt{\frac{V_{r+1}}{r+1}} \Rightarrow \dot{\rho}_{r+1} \leq -\frac{\eta(1 - e^{-\rho_{r+1}})}{(r+1)\left(t_f - t\right)}$$

For $\eta > r + 1, \rho_{r+1} \to 0$ in t_f (from   Theorem).
In that particular way, you can also minimize the chattering. So, chatter minimization is

always possible. So this kind of sliding mode control is called freewill arbitrary time sliding mode control because you can select the time of convergence. Obviously, every system has its own dynamics. So whatever time of convergence you select.

So, based on that, you have to select the time of convergence. Time of convergence will never be close to zero. Okay, some finite value will come because every system takes some time to respond to the control action, and obviously, this will handle some kind of uncertainty because I am involving the sliding mode control. The main intention of sliding mode control is to actually compensate for the matched uncertainty. Okay, now suppose that if t_f. So, I have to realize t_f, but t_r means that, suppose that this time t_r is not known to us.

Algorithm 1: Finding Realizable t_f

> 1.Select t_f = Arbitrary time 2. if ($s == 0$) then , Evaluate t_c = current time , Go to step 3 3. else repeat step 2 4. if $(t_f > t_c)$ then , t_f is realizable (Any time $> t_c$ is realizable) 5. else , Select new t_f , Go to step 1

So, for that, I have to actually write this kind of algorithm. So, if I can arbitrarily select some value initially, I will check whether s is equal to 0 or not, and whether the sliding surface is equal to 0 or not. So, now if $s = 0$. So, whatever current value I have selected is okay. Now, I will go to the third step, and after that, I will repeat the second step.

So, I will wait for the time when $s = 0$. If t_f is greater than t_c, then t_f is realizable; if not, then you have to select a new t_f. So, in this way, I can particularly modify the t_f slowly, and I will reach here; after that, I will be able to maintain the equilibrium point within some arbitrary time. So, now what I have done is taken a second-order system with this kind of matched uncertainty; we have selected the times of convergence to be 15 seconds and 20 seconds, and here I have selected the gains like this. And after that, what is our observation that stabilizes within t_f time? Control, obviously, actually chattering comes into the picture because I have to apply discontinuous control, classical sliding mode control, and I am applying it, but if I switch to super twisting control, then obviously this will minimize. And large control at t_f due to the forced convergence because what happens is that sometimes, suppose you are selecting $t_f = 5$ seconds, and this convergence will take place within 4.5 seconds. So, you have just 0.5 seconds to move from here to here. And, due to that reason, your control should be large enough to handle this convergence from here to here in 0.5 seconds. And, due to that reason, some kind of variation can be felt whenever we are converging from here to here, because this control is actually responsible for arbitrary time convergence from here to here. So, those kinds of things you can also see from the simulation, but whenever you are implementing control, first we are doing several trials, then we are implementing, and due to that reason you can easily select the time of convergence. So, this kind of phenomenon comes into the picture if your time of convergence t_f is not very large compared to t_r.

Parameters:

- Initial conditions: $x_1(0) = 1, x_2(0) = 2$, Disturbance: $d(t) = 0.1\sin t$ Control gain: $k = 0.2$, Convergence parameters: , For $t_f = 15s$: $\eta_1 = 4$, For $t_f = 20s$: $\eta_1 = 3.5$

And obviously, you can see that control not being so high means that it is actually less than 20 , and we have similar kinds of things here. So, this phenomenon is basically occurring because we do not know how much time I have to converge from here to here, but I have to make it arbitrary. So, if this time is short, then the denominator of the control, so you can see any control, then you can easily understand this $t_f - t_r$. So, this will very, very high and due to that reason, this kind of action basically comes into picture. Similarly, you are able to do an analysis for a third-order system, and it is now possible to show that in the case of a third-order system, control is actually chattering because we are implementing some on-off control.

But here, if the time of convergence I have selected is sufficient enough, you can see that that kind of spec will not come. So, now it is time to conclude this lecture. So, what have we seen? We have seen that one of the classical problems of singular terminal sliding mode control is that you cannot start from anywhere, because we feel that some kind of singularity will actually occur if you start anywhere in the state space, and due to that, a non-singular terminal sliding mode comes into the picture. However, this presents another solution to avoid the singularity of terminal sliding mode control: you can use the concept of arbitrary sliding mode control. One more thing, I am going to give you the hint that in the next class, I am going to show you that from here to here, one can also be able to reach in arbitrary time.

And due to that reason, you can now easily control from here to here in arbitrary time and here to here in arbitrary time. So, whatever problem you have regarding the time of convergence, you can easily solve it with the help of the next lecture. But most of the time, what we feel is that the time of convergence for the practical system is sufficiently large, and due to that reason, this kind of problem will not come into the picture. particularly those systems which are a little bit slow, particularly for the mechanical system.

And we have demonstrated the stabilization for the nth order pattern chain of integrators. And after that, we have done the comprehensive simulation for both second-order and third-order systems. And adaptive algorithm we have also proposed whenever t_r is not known in advance. So, with this particular remark, I am going to end this lecture. Thank you very much.