Sliding Mode Control and Applications
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Lecture-56

So, welcome back. In the previous lecture, I talked about discrete sliding mode control, and we discussed state feedback, output feedback, and how to extend it using sliding mode control. In this lecture, I am going to talk about the notion of stability and particularly some variants of finite-time stability, and this is a very important area of research. Recently, if you check the literature, both nonlinear control and sliding mode control are progressing in this particular direction, and for that reason, in the last module, I am going to incorporate these particular lectures, which are actually focusing on arbitrary time stability. So, for the purpose of discussion, we have one of the important stability concepts that is called finite time stability. So, now whenever we are talking about stability, we are all already well aware that I suppose I have some kind of system; for simplicity, you can take an autonomous system, and here x belongs to Rn.

Basically, we understand this system as having some kind of block where it is actually located, and after that, I provide the initial condition, and we generate the trajectory xt. And for this particular system, we are assuming that x equals 0 is the equilibrium point without loss of generality. If x equals 0 , it is the equilibrium point. So, if you apply a 0 initial condition here, then you will get a 0 output here.

And after that, we will be talking about stability in terms of epsilon and delta. So, we are selecting the output ball, which is basically related to xt. So, x-0. So, for every epsilon, there exists a delta that we are writing. For every epsilon, there exists a delta that is a function of epsilon greater than 0 , and after that, we are constructing a ball around $\mathrm{x(t)}$ minus $\mathrm{x(t_0)}$, wherex (t_0) istheinitialconditionminus 0 .

If that lies inside the delta, then it implies that $\mathrm{x(t)}$ minus 0 is going to lie in the epsilon ball. In this way, basically for all t , we are talking about the stability. And if, apart from the stability, all system trajectories from some initial condition converge to 0 , then we are talking about attractivity. So, basically, if in finite time all trajectories are attracted towards the equilibrium point, then we are talking about finite time stability. It is possible to show that the notion of finite time stability is somehow a function of the initial condition's time of convergence.

So, if you start from a large distance, then you have to wait a long time to converge to the equilibrium point. So, now people are trying to control that time. So, there are several notions or equivalent notions that we are going to clarify in this particular lecture, which basically address the meaning of fixed-time stability, predefined-time stability, arbitrary-time stability, and one stability concept that is prescribed time stability. So, I am going to talk about a kind of comparative analysis, and that kind of stability falls into this category that is rated convergence. And in several applications, convergence time is particularly very, very important for both offensive and defensive applications.

And obviously, we also have to talk about the convergence as well as the control boundedness for the arbitrary time stability. So, let us just talk for one minute about the arbitrary time stability. So, the notion of arbitrary time stability was actually coined by our group, and my PhD student, Anil, who is working with me, is one of the co-authors of this new notion of stability that is called arbitrary time stability. And finally, that is extended with the help of Professor Leonid Friedman, Professor Xing Yu, Professor Bijnan Bandopadhyay, and after that, Professor Menzik, Professor Thak, and Professor Zhiyong, so several people from several countries. After that, we all work together to establish this concept of arbitrary time stability.

So, I am going to discuss what the benefits are and what kind of extra things one can do with the help of this particular notion of stability. And we are trying to see the fine gap between finite time, fixed time, arbitrary time, and everything I am going to actually discuss. So, whatever theoretical difference between all kind of stability that I am going to discuss. So, when I started this course, I took this model. So, the main motive is that I have some system and what we have to do is design some kind of control.

So, this is nothing but some kind of force that is a thruster force, and what we have to do is solve two problems. Either that is a tracking problem or a stabilization problem. So, whenever we are talking about the tracking problem in this particular example, I am trying to control the velocity. So, we wish to maintain the velocity with some reference value v_d_t. This v_d_t may or may not be time dependent.

And after that second one, we are trying to maintain v_d equal to 0 in spite of disturbances, and due to that reason, we are talking about robust stabilization. In sliding mode control, particularly, we are talking about insensitivity with respect to matched uncertainty. So, whenever we have a linear system, we can see the linear system both in the time domain as well as the frequency domain if I have a linear time-invariant system. And at that time, performance measure one can talk both in the time domain as well as the frequency domain, meaning one can discuss the overshoot, undershoot, and afterwards the frequency domain response, gain margin, and phase margin. So, these kinds of notions somehow quantify the performance criteria.

But if we are talking about a non-linear system, most of the time it is very difficult to discuss a quantitative way to evaluate the performance criteria. And due to that reason, we are discussing performance criteria in a qualitative way. So, one of the ways I discussed when I started this lecture is stability. Somehow, the notion of stability is inspired by continuity, and using this notion, we are discussing the well-behavedness of the system, and after that, we are talking about convergence. So basically, how fast we are going to converge is also one of the parameters nowadays; this is called the speed of convergence.

Obviously, we have discussed accuracy whenever I was talking about higher or sliding mode control, homogeneous control, and you have already seen that in the presence of delay, in the presence of noise, and in the presence of some kind of uncertainty, how closely we are able to move towards our goal. So, somehow we are talking about accuracy, and if the accuracy of our control is high, then that is well acceptable for practical implementation. Robustness is with respect to internal or external uncertainty, and obviously, the fourth factor that is very, very important is cost. So, in this whole development, I am not going to talk about the cost. So, let us start with the notion of finite time stability.

So, you can easily understand this finite time stability from the natural phenomena. So, this is the cart, and this cart is moving, and we are assuming that this road has friction. So, this cart is just moving in one direction. You can see that if you remove the force, then what happens? This cart is going to stop automatically. So, this friction force is responsible for that.

So, here v is equal to 0 in some finite time. Why am I telling finite time? Because m is finite, k d is finite, k d is nothing but some kind of friction coefficient, and v 0 is finite. So, this time of convergence is a function of the parameter as well as the initial condition. So, if the initial condition v0 is high, then you have to wait a long time. And if you look carefully, in the whole sliding mode control, we are using exactly the same kind of differential equation with a discontinuous right-hand side to discuss the insensitivity with respect to uncertainty and finite time.

For example, consider the finite-time convergence of a mass with dry friction:

$$m\dot{v} = -k_d \text{sign}(v); \ \dot{x} := v$$

where m is mass, v is velocity, and k_d is the coefficient of dry friction. We have:

$$v = 0; \ \forall t \geq \frac{m}{k_d}|v(0)|$$

I am going to set this particular sliding variable equal to 0 . Now, another kind of stability. So, that is motivated by the homogeneous approximation. So, in 2008, this paper is very good, so those who are interested in output feedback or recursive observer development, please go through it. Okay, so what is actually the main intention of this paper? This paper discusses that when linearization fails, how to proceed further.

That is the main intention of this particular paper. One of the main points whenever people discuss non-linear control or linear control is that they claim there is no need for non-linear control because I can linearize the entire trajectory and then apply the control based on gain scheduling. However, there are several instances where linearization will not provide any information. So, for example, if you take the example of this particular first-order system, it means that x(t) basically belongs to R . We have just a one-dimensional system.

You can see here that this particular system, if you linearize it, then x dot is equal to 0 . So, what is the meaning of x dot equal to 0 ? It means that x is equal to a constant, but that is not the case for this particular system. But if you linearize around the equilibrium point or somewhere, it is possible to show that their local behavior is governed by this equation. Okay. So, this is nothing but locally asymptotically stable.

Take the system:

$$\dot{x} = -x^3 + x^5$$

This system is locally uniformly asymptotically stable. Linearization at the origin gives $\dot{x} = 0$, which provides no information about stability. The homogeneous approximation $\dot{x} = -x^3$ correctly concludes local asymptotic stability.

You can easily check based on Newton's philosophy that x 0 is the initial condition; then x dot is negative. So, we can move in this direction. But obviously, I cannot move from anywhere because this term is somehow restricting us from doing so. Okay. So, at least I can talk about local stability.

So, now, one example is constructed by Polyakov. So, Polyakov has actually given the notion of fixed time stability based on this particular paper. If you see this paper, then that kind of form is available in this paper. So, basically, I am going to take exactly the same system. This system, now I am going to keep inside running water.

Okay, so what happens now is two different forces that come into the picture: one is drag force and obviously the other is dry friction. Okay, and it means that the force is given like this, and the whole system is now given like this. If you solve this equation, the force solution can be obtained using several standard software programs to solve it, so it is possible to show that v_t equals 0 at this particular time. And now, it is also possible to show that the maximum deceleration time given by this is because arc 10 v 0 is always bounded by some kind of upper value that is pi. So, what is the observation here? You can see that the maximum time of convergence is independent of the initial condition.

Consider a rigid body moving laterally on a contact surface:

$$\begin{cases} \dot{x}(t) = v(t) \\ m\ddot{v}(t) = F(t), t > 0 \end{cases}$$

External forces: - Drag force: $F_{\text{drag}}(t) = -k_{\text{drag}} v^2(t) \sin(v(t))$ - Dry friction force: $F_{\text{dry}}(t) = -k_{\text{dry}} \sin(v(t))$

Total force acting on the system:

$$F(t) = -\left(k_{\text{dry}} + k_{\text{drag}} v^2(t)\right) \text{sign}(v(t))$$

Velocity evolution:

$$m\dot{v}(t) = -\left(k_{\text{dry}} + k_{\text{drag}} v^2(t)\right) \text{sign}(v(t))$$

Velocity solution:

$$v(t) = \tan\left(\arctan(|v(0)|) - \frac{\sqrt{k_{\text{dry}} k_{\text{drag}}}}{m} t\right) \text{sign}(v(0))$$

Convergence properties: - Velocity reaches zero at:

$$t \geq \frac{m \arctan (|v(0)|)}{\sqrt{k_{\text{dry}} k_{\text{drag}}}}$$

- Maximum deceleration time:

$$T_{\max} = \frac{m\pi}{2\sqrt{k_{\text{dry}} k_{\text{drag}}}}$$

Homogeneity analysis: - Near origin: $F \approx -k_{\text{dry}} \operatorname{sign}(v)$ (degree 0) - At infinity: $F \approx -k_{\text{drag}} v^2 \operatorname{sign}(v)$ (degree 2)

So, you can start anywhere in this maximum time you are going to converge, even if you start at infinity. And due to that reason, a new notion called the fixed time convergence concept comes into the picture. And if you are also including Lyapunov stability with fixed-time convergence, then that is called fixed-time stability. So, if you see this controller, it is bi-homogeneous in nature. So, near the origin, the degree of homogeneity is 0 .

Now, you are already well aware of how to calculate the degree of homogeneity; you can just scale v, and after that, the degree of homogeneity of the second term is 2 . So, this is called a bi-homogeneous system. So, bi-homogeneity comes into the picture, and due to that combination, it is possible to show that this is finite-time stability. So, just for the notion, in place of dy, I have written kd, and here I have written kv. So, now you can see that if I have this kind of system, then it is possible to show that you can start anywhere in the state space within this much time, which is independent of the initial condition, and you can converge to 0 .

Fixed-time convergence example:

$$m\dot{v} = -(k_d + k_v v^2)\operatorname{sign}(v)$$

where k_d, k_v are coefficients of dry and viscous friction. Then:

$$v = 0;\ \forall t \geq T_{\max} = \frac{m\pi}{\sqrt{k_d k_v}}, \forall [x(0), v(0)] \in \mathbb{R}^2$$

So, what is the difference between finite time and fixed time? So, the convergence time is uniform with respect to the initial condition in the case of fixed time stability. And in finite time stability, convergence time actually depends on the initial condition. But obviously, in fixed time stability, convergence time also depends on the parameters k d and k v . So now, a new question comes to mind. Is it possible that the time of convergence is uniform with respect to the initial condition or system parameters? Obviously, some gains are required, but if we give some kind of algorithm such that the time of convergence is independent of system parameters or initial conditions and the time of convergence is just a function of the gain we are going to provide, then that notion leads to arbitrary time stability.

And using this notion, you can explicitly control the time of convergence. Now I have done some kind of MATLAB simulation. So, in order to show the comparative study, So, I have taken a first-order system here, and after that, I am going to design this control. So, this is nothing but some kind of asymptotic control or exponential control that you can easily see, and why exponential. So, the reason you already understand is that as we move closer, the rate of change of x is going to decrease, and a similar kind of behavior can be seen in the time domain.

Consider the system:

$$\dot{x}(t) = u(t);\ x(0) = x_0;\ x, u \in \mathbb{R}$$

For asymptotic stabilization:

$$u(t) := -x(t) \text{ and behavior: } x(t) \to 0 \text{ as } t \to \infty$$

Now, if you update the control by one of the controls, you can see the behavior that if I start from 5 , then within 5 seconds I will converge. If I start with 2 , then within 2 seconds I will converge. So, somehow, the time of convergence is finite here. Now, you can just change algorithm like this. And it is possible to show that this somehow satisfies the property of bi-homogeneity.

Please check it. So, that is homogeneous near the equilibrium point with some weight, and if you are far from the equilibrium point, there is another weight. And it is possible to show now that the time of convergence is greater than or equal to pi. So, you can start anywhere Now, within this fixed time, that is 3.14 , you can able to converge. So, this is the maximum bound of the convergence time.

But if you look carefully at the simulation, before 3: 15, I will converge within 2 seconds. It means that you cannot control the time of convergence. But what is our main goal? Whenever I have a critical application, offensive or defensive, I also need to know at what time I can converge. So, for that, we have actually coined a new notion with the help of Professor Anil and Bhavana.

So basically, if you define it like this. So, here, this tf is somehow the design parameter. So, tf is you can define tf equal to t0 plus some extra time, so the time of convergence tc. So, within tc time, it is possible to show that I can converge to the equilibrium point. So, here tf I have set equal to 1 , t0 I am going to start from 0 . So, you can see that within one second, you can converge.

Arbitrary-time stabilization:

$$u := \begin{cases} \dfrac{-1.1\left(e^{|x|} - 1\right)}{e^{|x|}\left(t_f - t\right)} \operatorname{sign}(x), & \text{if } t_0 \leq t < t_f \\ 0, & \text{otherwise} \end{cases}$$

So, you can start anywhere within one second; you can converge. But obviously, if you are going to control the time of convergence, if you see that equation, if you look before that here, they are just using the information of the state. But here, we also have to use the explicit time information. So, automatically your system, closed loop system is non-autonomous. You have to pay for it because you are controlling the time, and if your time of convergence is very, very large, then the control magnitude is huge.

Now, why this kind of concept is required, I am going to give you some practical examples. So, when I started this course, the main idea of this course is based on on-off control and why we are basically doing on-off control. So, due to the flip-off convexification method, what happens? As we are at v equal to 0 , it is possible to show that the average value of the disturbance is exactly equal to the k signum x and the average value of the k signum v. And what is the physical interpretation of this? We are stating that the average velocity is equal to 0 , even if the instantaneous velocity is non-zero.

So, that is somehow the physical interpretation. And using the low pass filter, I can recover this disturbance. So, now, a day load of system is running on network. And whenever some disturbance comes into the picture, it is possible to show that one can estimate the time of convergence or estimate the disturbance as well as the time of convergence based on this particular algorithm. And here is the diagram we have already discussed. So, you can see that the time of convergence is finite, control is discontinuous, but what is beauty? If you pass this kind of k -signum v from the low pass filter, I can exactly reconstruct the disturbance just by designing the low pass filter.

Mathematical formulation:

$$\dot{v} = u + d(t) = -k\text{sign}(v) + d(t), k > |d(t)|$$

Sign function definition:

$$\text{sign}(v) = \begin{cases} 1 & v > 0 \\ \varepsilon[-1,1] & v = 0 \\ -1 & v < 0 \end{cases}$$

So, now a new question comes into the picture: is it possible to make some kind of estimator that will solve this problem in arbitrary time? So, for that, I need some kind of concept of an arbitrary estimator. So, how one can able to modify this algorithm such that notion of arbitrary estimator comes into picture. So, for that, I need some kind of sliding mode control that has arbitrary time convergence. So, in this particular module, I will talk about how to modify this algorithm so that I am able to discuss arbitrary sliding mode control.

That is one of our ideas. Another classical problem that we have observed in this particular course is something I have already given several classes on. Whenever we are trying to implement some kind of output feedback-based control. So, if we try to design some kind of homogeneous control, particularly based on the Laurent formulation, then I need a differentiator. So, the next question that comes into the picture is, is it possible to design some kind of differentiator that will differentiate the signal in arbitrary time, because of whatever differentiator has been proposed by Professor Laurent? So, the time when differentiation starts actually depends on the initial condition as well as the system parameters. So, is it possible to coin some kind of new differentiator such that at any arbitrary time I can start differentiation? And this problem can also be solved with the help of arbitrary time stability.

Robust observers aim to reconstruct unmeasured states within finite time:

$$\dot{x}_1 = x_2, \dot{x}_2 = f(t,x) + d(t,x);\ y = x_1$$

Only $y = x_1$ is measurable, x_2 must be estimated. Estimation of x_2 is required within a predefined time t_f.

Now, here you can see that some notion of predefined time stability comes into the picture. So, actually, what happens if you see the literature? So, in literature before arbitrary time stability, one notion of predefined time stability is there. And they are just talking about the upper bound of the settling time of a fixed-time stable system. So, somehow what we have observed in one of the examples is that theoretically, if you calculate this, it is the time of convergence. The time of convergence suggests that if you have calculated tp, what actually happens is that the trajectory sometimes converges here and sometimes converges there, but obviously it will not cross this time.

So, in this way, you cannot be able to control the time of convergence. So, somehow predefined time stability gives you some kind of close values such that you are able to move close to this tp, which means time of convergence. Obviously, you cannot exactly be able to do your job in this particular time, but that is only applicable for autonomous systems; whatever arbitrary time stability we are talking about is actually developed based on the notion of non-autonomous systems. It means that time information is also required. So, in that particular way, the notions of predefined time stability and arbitrary time stability are different.

Another important problem is the observation problem. And now, the same question again: is it possible to observe some signal? Suppose that if I have information about x 1 , then from x 1 , is it possible to observe x 2 in some kind of predefined time or in some kind of arbitrary time? So, that kind of problem is also solved by the notion of arbitrary time stability. Now, let us try to give the definition of arbitrary time stability. So, once we formulate problems, one of the main important things that we have to do is give the definition. You can see that several papers after 2020 onward, so in 2016 , We have actually developed algorithm. So, for four years we have documented, and after that, it takes two years to publish, and after that, we will get this paper.

So, the design of the controller with arbitrary convergence time is one of the very highly cited papers, and obviously, we have collaborated with Professor Leonid Fridman and Professor Bijnan Bandopadhyay, who is my supervisor. So, after that, this notion is established. So, what is the meaning of this? I have a non-autonomous system. So, time information is also here, and delta is a known disturbance.

System description:

$$\dot{x} = f(t, x, u, \delta), x(t_0) = x_0;\ x \in \mathbb{R}^n$$
$$u := u(t, x, \tau_a, \eta) \in \mathbb{R}^m;\ \tau_a, \eta \in \mathbb{R}$$
$$\delta \in \mathbb{R}^p;\ f(t, 0, 0, \delta) = 0$$

This is some kind of parameter. I have already told you that whenever I am going to develop some kind of algorithm, I need some parameters at that time. So, here delta, sorry, this is the parameter; tau is the parameter, control parameter; delta is the disturbance. So, what we are basically assuming is that in the presence of a disturbance, the equilibrium point is also 0 , and here, for arbitrary time stability, tau a is somehow a design parameter. So, now, here I have tau, as well as eta. So, I have two design parameters; one will provide the explicit information of time.

So, within what time I have to converge, that is actually denoted by tau a. So, suppose that t_f I will define t_0 plus tau a. So, within tau a time I can converge from any initial condition to x equal to 0 , that is the main interpretation. And we are assuming that u is a function of time x tau a, tau a the time of convergence or settling time. We are going to specify that eta is one design parameter that I am going to specify and that basically belongs to the real numbers.

Global finite-time stability: $x = 0$ is globally asymptotically stable $x(t, t_0, x_0, u, \delta) = 0$ for all $t \geq t_0 + \tau(t_0, x_0, u, \delta), \tau$ is the settling time function:

$$\tau: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}_{\geq 0}$$

Fixed-time stability: $x = 0$ is globally finite-time stable $\exists \tau_{\max} > 0$ such that: $\tau(t_0, x_0, u, \delta) \leq \tau_{\max}$ for all x_0, δ, u, t_0

And actually, after that, several clarification papers have been written in this direction because some kind of contradiction occurs in the community that arbitrary time stability and prescribed time stability concepts are both the same, but that is not the case. The prescribed time stability concept is only applicable up to 0 to tau a. Beyond that, they are not discussing anything, but in order to talk about stability, we have to discuss it for all time. So, basically, our notion is applicable throughout the time interval. So, somehow we have proposed arbitrary time stability on top of finite time and fixed time stability.

It is possible that without fixed time stability, you can also talk about arbitrary time stability. But in order to ensure consistency in the literature, we first stated that if x equals 0 , it is globally asymptotically stable. What is the meaning of globally asymptotically stable? Suppose that you are going to start your system at t_0, and this is the extra time that depends on t_0, x_0, u, and δ. Now, whenever we are talking about fixed time stability, this settling time function is upper bounded by some kind of tau max.

Arbitrary time stability: 1. System is fixed-time stable 2. $\exists \tau_a(\delta), \eta(\delta) > 0$ 3. Arbitrary assignment of τ_a via variation in δ, η4. For any δ, η : $-\tau_a \geq \tau_{tf}$ (Weak arbitrary stability) $\tau_a = \tau_{tf}$ (Strong arbitrary stability)

Free-will arbitrary time stability: 1. System is fixed-time stable 2. $\exists \tau_a > 0, \eta > n$, independent of system parameters 3. Either: $\tau_a \geq \tau_{tf}$ (Free-will weak) $\tau_a = \tau_{tf}$ (Free-will strong)

So, all trajectory You can start anywhere in a state space. If you construct some kind of compact set and are also able to feel these two things, then one can also talk about finite time stability in a compact set and fixed time stability in a compact set. Most of time practical system is behaving like that only. So, x equal to 0 is globally finite-time stable. It means that I can start anywhere from the state space and then, within a fixed time, I can converge to the equilibrium point.

What arbitrary time stability suggests is that there exists a tau a. And that is somehow a function of some kind of design parameter, delta and eta equal to delta. So, based on this, I can able to control the time of convergence. And if this time of convergence is independent of the parameter, then delta is somehow a parameter or disturbance of the system. So, now, if I make the parameter or disturbance a function of eta. So, most of the time since the disturbance, I do not know, and due to that reason, I am going to interpret some kind of parameter that is involved in the system.

So, if your time of convergence is determined based on the tuning of the parameter, then we are talking about arbitrary time stability. And what is the meaning of free will arbitrary time stability, if that is independent of the parameter? So, now tau and eta are both the design parameters of an arbitrary time-stable algorithm; if that is independent of the delta, which is a system parameter, then we are talking about this as free will arbitrary time stability. So, in order to ensure consistency, I am just going to utilize the notion of arbitrary time stability. So, arbitrary time stability means either you can basically tune parameters based on delta or independently of delta. Just say for the clarity of the notion because the notion of free will has actually been given in the literature.

So, some contradictions occur in the literature, and for that reason, after consensus, we are actually trying to preserve the name as arbitrary time stability. So, let us try to understand the notion of arbitrary time stability. So, basically, this is nothing but some kind of arbitrary time-stable system. So, how and why are we saying that this is an arbitrary time stable system? You can see here that the denominator is well defined.

Arbitrary time stable system:

$$\dot{x} = \begin{cases} -\eta \dfrac{(e^x - 1)}{e^x(t_f - t)} & \text{if } t_0 \leq t < t_f \\ 0, & \text{otherwise} \end{cases}$$

Solution:

$$x = \ln\left(1 + \kappa(t_f - t)^\eta\right), \kappa = \frac{e^{x(t_0)} - 1}{(t_f - t_0)^\eta}$$

So, somehow, if you see carefully, then this algorithm is locally Lipschitz in X. And piecewise continuous in t only piecewise continuity will lost when t equal to tf. So, in the neighborhood of tf, this is piecewise continuous; beyond that, it is also piecewise continuous because I am keeping this equal to 0 on the right-hand side. So, somehow we are patching two kinds of solutions where time is piecewise continuous; it means that in this algorithm this continuity only occurs in time, not in the state. So now, you can see that if you solve it, then it is possible to show that as t tends towards tf , then obviously x equals 0 .

System behavior: It is easy to see that $\exists \eta > 1$ s.t. $x \to 0$ as $t \to t_f$. It is also clear that $\dot{x} = 0, \forall t \geq t_f$. Therefore, it implies that x remains stationary at the origin for any time $t \geq t_f$.

Arbitrary time stabilizable control $\dot{x}$:

$$u = \begin{cases} \dfrac{-\eta\left(\kappa\left(t_f - t\right)^{\eta-1}\right)}{\left(1 + \kappa\left(t_f - t\right)^{\eta}\right)}, & \text{if } t_0 \leq t < t_f \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

From (1), it is concluded that for $\kappa \geq 1, |u| < \eta\kappa, \forall t \geq t_0$ and for $\kappa < 1, |u| < \eta, \forall t \geq t_0$. So, before tf, this algorithm is going to converge. So, the same kind of things I have actually written here. Now, it is possible to show that this particular right-hand side of the differential equation can be bounded. This is not unbounded. Obviously, if you start from infinity and bring something from infinity to 0 , an infinite amount of control is required.

That is true for almost all controls. Even if you can talk about state feedback control, if x is infinity, then obviously the right-hand side is infinite. But if your domain x is finite, control should not go to infinity at that time. So, that kind of thing is actually proved here. It means that you can easily implement an arbitrary time stable philosophy in the practical system. Now, once you develop the notion, the second important thing here during the development of the notion is that it looks like this is only true for the scalar system.

And here I have taken a scalar positive system; it means that x is greater than 0 . And why is it a scalar positive system? You can easily understand the philosophy because we know that if I develop some theory for a scalar system. So, now I can develop the notion of energy-like function, that is, Lyapunov function, and using the compression lemma, I can map an n -dimensional system, where x belongs to R n̂, to some equivalent system, where V comes into the picture in R+̂. Here, we will develop some kind of theory that is applicable in the Rn domain. So, somehow we are going to map to the one-dimensional real line, because V x is going to belong to it finally. And why we are doing this is because I have to do completion, and completion is only possible in one-dimensional space.

Theorem: Consider the system $\dot{x} = f(t, x; \delta)$ and let $D \subset \mathbb{R}^n$ contains the origin $x = 0$. Assume that there exists a real-valued smooth function $V: [t_0, \infty) \times D \to \mathbb{R}_{\geq 0}$ and there exists a real number $\eta > 1$ s.t. $k_1\|x\|^a \leq V(t, x) \leq k_2\|x\|^a, \forall t \geq t_0, \forall x \in D \setminus \{0\}$

$$V(t, 0) = 0, \forall t \geq t_0$$

$\dot{V} = 0, \forall t \geq t_f$ -

$$\dot{V} \leq \frac{-\eta\left(e^{k_3\|x\|^a} - 1\right)}{e^{k_4\|x\|^a}\left(t_f - t\right)} \ \forall V \neq 0, \forall t \in \left[t_0, t_f\right),$$

where k_1, k_2, k_3, k_4 and a are positive constants, then the origin is weak ATS and $t_f \geq \tau_{ac}$. Strong ATS condition: Finally, if

$$\dot{V} = \frac{-\eta\left(e^{k_3\|x\|^a} - 1\right)}{e^{k_4\|x\|^a}\left(t_f - t\right)} \ \forall V \neq 0, \forall t \in \left[t_0, t_f\right),$$

then the **origin** is strong ATS and $t_f = \tau_{ac}$. So, now, if you have developed some kind of theory for one-dimensional space, and whenever you have an n-dimensional system, then it is possible to show that if you go via the Lyapunov analysis, one can show that the same kind of theory is applicable for higher order systems. One more important thing I am going to highlight here is that whenever you are talking about the convergence behavior, whatever this parameter is, I am talking about two parameters, and this is somehow tuned by the user, so eta should be greater than 1 for convergence. And tf, tf is somehow a parameter that is actually tuned by the user. So, two parameters, tau and eta, I have to tune in order to talk about arbitrary time convergence, because t0 is the starting time. So, during Lyapunov function analysis, I am going to consider some kind of Lyapunov function again, and here a time-dependent Lyapunov function is also allowed, because we are talking about the stability of non-autonomous systems.

This is V equal to 0 , which should be 0 for all t greater than or equal to t 0 , and after that, V should be greater than or equal to tf; this should be 0 because it can automatically be shown due to this relation that for t greater than or equal to tf, V is 0. V is 0 and V dot is 0 , which means that it is going to maintain V0 equal to 0 . So, I can maintain the equilibrium point. And this expression we have designed in such a way that it is possible to show the exact same kind of expression that comes into picture. So, in the next lecture, I will again talk about input-to-state stability; I will come back to the proof of this.

So, basically, if this is less than or equal to 0 , then it is possible to show that here x equal to 0 is weakly arbitrary time stable, and if that is actually equal to 0 , then that is strongly arbitrary time stable. Here I am going to take one system and try to show you how this system will basically satisfy the property of this particular Lyapunov function. It means that the arbitrary time stability concept is not limited to the scalar system that I am trying to show you. So, I have constructed one example, and here I am assuming that this parameter q depends on time but is bounded by some gain k ; k is positive, and the rate of change of q is always less than qt ; that is the kind of assumption I have. And this function you can see that this is just in arbitrary time stable system I have this kind of function.

System dynamics:

$$\dot{x}_1 = \begin{cases} -\eta_1\theta(x_1,t) - x_1 - q(t)x_2, & \text{if } t_0 \le t < t_f \\ -x_1 - q(t)x_2, & \text{otherwise} \end{cases}$$

$$\dot{x}_2 = \begin{cases} -\eta_2\theta(x_2,t) + x_1 - x_2, & \text{if } t_0 \le t < t_f \\ x_1 - x_2, & \text{otherwise} \end{cases}$$

where $\eta_1, \eta_2 > 1$ and $q(t)$ satisfies: $0 \le q(t) \le k$

$$\dot{q}(t) \le q(t), \forall t \ge 0$$

$$\theta(s,t) := \frac{(e^s - 1)}{e^s\left(t_f - t\right)}$$

So, I have defined that function as es -1 ext f − t. So, basically these two terms are responsible for the arbitrary time stability. And once tf greater than equal to 0 , at that time system is going to switch to this value and this value. Now, I have taken some kind of Lyapunov function; you can see that this is a time-dependent Lyapunov function, and whenever you are going to take the time-dependent Lyapunov function, at that time you have to show some property that is positive definite. So, obviously this is positive definite, because it is bounded by two positive definite functions. It is also possible to show that it is always upper bounded by, or actually upper bounded by, x 1 squared plus b and lower bounded by x squared plus x 2 squared, and radially unbounded.

This means that as x 1 and x 2 tend towards infinity, this whole function tends towards infinity. And after that, what have we done? We have taken the derivative of this particular Lyapunov function, and then I have actually substituted this particular inequality. And once you substitute this kind of inequality, where phi 1 and phi 2 are defined like this. and where theta is nothing but what we have already defined. So, in place of x, I have just substituted x 1 and x 2 .

So, you can see that you will get some kind of form where this value also comes into the picture. So, from the inequality finally, you will be able to replace it with 2 , and this whole thing becomes some kind of quadratic form. You can easily see that this is a quadratic function and that q is positive definite. So, this whole thing is negative definite, and for that reason, I am able to write like this. Now, from here V is less than or equal to minus 2 × 1 phi 1 minus 2 × 2 phi 2 , I have to show that I will get the arbitrary time stability form. So, we know that V is a combination of x 1 squared and x 2 squared, and for that reason, this relation is true.

Obviously, this relation is true, or either this or that is true, and based on that, I can develop this kind of differential inequality if I do this kind of coordinate transformation. So, you can see that again I will get exactly the same form as arbitrary time stability, and in this way, I am able to show that zeta is tending towards 0 as t tends towards some predefined time t f . So, Zeta is 0 . So, V is 0 . So, in this way, I can prove that this second-order system is asymptotically stable, which means I can achieve the equilibrium point in this time t_f.

After that, obviously, the system is going to switch. So, I have to maintain it. After this, I have to prove that if I start with x 1 equal to 0 and x 2 equal to 0 , you can easily see that if x 1 equals 0 and x 2 equals 0 , then x 1 dot also equals 0 and x 2 dot also equals 0 . And this system is also globally exponentially stable. So, what happens is that even if some fluctuation in x 1 and x 2 comes into the picture, I can maintain x equal to 0 . So, in this way one can construct some kind of arbitrary time-stable systems.

So, now it is time to conclude this lecture. So, what have we seen in this lecture? We have discussed the finite time stability that is actually inspired by the nature of fixed time stability, after that arbitrary time stability. In between, I have also talked about the predefined and prescribed time stability concepts and how arbitrary time stability is different from all the others. And using this particular method, I can able to control the time of convergence. And due to that reason, you can see that in current literature on several practical problems, including missiles, people are using this kind of concept. I have also shown that the control is bonded, and in practical implementation, I have done practical implementation on a two-tank system, followed by a helicopter system, in order to show people that the control we are proposing is not unbounded.

And, after that we have also given some kind of practical insight how to implement these kind of system for the real world problem. I have discussed some key problems, and in the next subsequent lectures, I am going to show you how to solve one of those key problems. So, with this remark, I am going to end this lecture. Thank you very much.