Sliding Mode Control and Applications
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Welcome back. In the previous class, I talked about the difference equation with minima. And what we have seen is that in the case of a continuous-time system, whenever we are realizing the sliding mode control, at that time a differential equation with a discontinuous right-hand side comes into the picture. And in a similar way, whenever we are trying to achieve sliding mode control in the case of the discrete-time system, we have already seen the counterpart of the differential equation with a discontinuous right-hand side; now we have a difference equation with minima. In this lecture, we are trying to explore the same kind of algorithm that we have explored for the continuous-time system, which is nothing but the higher-order sliding mode control algorithm. And what was our observation in the case of a continuous time system? In a continuous time system, whenever we talk about second order sliding mode control, differential inclusion with a discontinuous right-hand side is going to appear on the second derivative, and similarly for the third derivative.

Here, we also have a similar kind of observation: whenever we are moving for the twisting, super twisting. So, twisting is relative degree to algorithm. So, here since the relative degree, we have to define it in a different way. So, we have to define, suppose that $y = z_1(k)$. So, $y_1(k+1) = z_1(k+1)$, then control will not appear; then we will go for the higher iteration, that is $y(k+2)$. And it is possible to show that when control explicitly appears, one can again be able to design some kind of algorithm based on a difference equation with minima. So, we have already seen that in the case of classical discrete-time sliding mode control, if I have to implement some kind of discontinuous algorithm. So, suppose that I have to implement $\dot{x} = -k\text{sign}(x(t))$, and here obviously we have a disturbance. Then one can think that, in the absence of disturbance, I will apply some kind of Euler discretization, and finally, we will adjust the gain; in that way, basically, Gao's reaching law comes into the picture.

So, we have already explored this. Now, what is actually the difficulty with Gao reaching the law? once I try to implement this particular algorithm by discretizing it, and here again I am assuming that I have just first order system $x(t) \in \mathbb{R}$, here you can easily able to replace x by s. So, what happens? In the absence of disturbance, we are also unable to maintain this sliding surface exactly. So, what happens is that the trajectory will converge here, and after that, they are going to do a zigzag motion. So, due to discretization, inherent chattering comes into the picture.

Due to that reason, Euler discretization is not a very perfect or correct way to implement this kind of algorithm. So, what is an alternative approach? So, one approach we have already seen is that without using the discontinuous term, I can land on the sliding surface, and after that, I am going to maintain it at all subsequent time instants, and that is called Utkin's law, which suggests $x_{k+1} = 0$. Now we are trying to find some kind of equivalent to this. So, what we have seen is that some kind of difference equation with minima comes into the picture. And what is the beauty of the difference equation with minima that, in the absence of disturbance, at least I can exactly slide along this line? The number of steps I can take is also compatible with the actuator bandwidth.

Obviously, in the presence of disturbance, I cannot exactly maintain my position along the sliding surface. Why? Because control is only available at discrete instants. So, if you come here and control is not available, we will obviously cross and recross. So, that kind of thing happens. So, what is our main intention for this particular lecture now? We have seen in the case of continuoustime systems that we have a very fascinating algorithm that is twisting and super twisting; the super twisting algorithm, as well as the twisting algorithm, has several applications in observer design, differentiated design, and significant applications in parameter estimation, and for that reason.

If we are able to find some kind of discrete counterpart to this, one of the ways to find a discrete counterpart is by using Euler discretization. So, there are several equivalent researches actually existing in the literature, which somehow discuss a kind of Euler discretization approach. But here, in this particular lecture, I am going to suggest another approach that is based on the difference equation with minima. And it is possible to show how one can implement it in the absence of disturbance. And that is the beauty; obviously, once we establish a disturbance-free case, we will talk about finite time input to state stability whenever perturbed factors come into the picture.

And, one of the obvious applications of the super twisting algorithm is in designing some kind of super twisting-based observer. So, I am also going to discuss that. In this lecture, I am going to discuss a type of design that is based on Lyapunov theory. So, suppose that I have some kind of first-order system and I know the minimum mean difference equation with a minima-based algorithm. So, how do we extend it for the second, third, or higher order? So, it is possible to show using backstepping that one can do it.

Here, in this particular lecture, I am not going to prove the twisting and super twisting using the Lyapunov method. So, those who are interested can see our recent paper in the IEEE Transactions on Automatic Control, which is basically based on the minimum operator. Whenever we are talking about a discrete-time system, the dynamics are given by this:

$$z(k+1) = \psi(z(k), \nu(k))$$

where $z \in D \subseteq \mathbb{R}^n, k \in \mathbb{Z}_{\geq 0}, \nu(k) \in \mathbb{R}^m$. The mapping

$$\psi: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

is continuous with $\psi(0,0) = 0$. The solution is:

$$z(k) := z(k, z_0, \nu(k)) \text{ for any initial condition } z_0.$$

So basically, the interpretation is exactly the same. Here, I have a step k, here I have the initial condition z_0, and obviously, the control I am going to apply is $\nu(k)$. Then I am going to generate $z(k)$. So, that is the interpretation of the solution. And in the previous class, we have already revised what the meaning of a class $\mathcal{K}$ function is.

So, the class $\mathcal{K}$ function is actually continuous and strictly increasing. And whenever we are talking about $\mathcal{K}_\infty$, then obviously, this a is actually mapped to infinity; then the whole function is mapped to infinity. And after that, we have defined a generalized $\mathcal{K}$-class function. So, this function is continuous and strictly everywhere except this particular time whenever that will come, then at that time that is going to show the non-decreasing behavior. And after that, we have defined a generalized $\mathcal{KL}$ class function because whenever we talk about twisting and super twisting higher-order sliding mode control, the notion of finite time stability is required.

So, for that, I have two arguments whenever we are talking about generalized $\mathcal{KL}$ class functions. So, with respect to the first argument, that is s, this represents some kind of generalized $\mathcal{K}$-class function. And with respect to the second argument, if I fix the first argument, similarly here we have fixed the second argument, then it is possible to show that in finite time this becomes 0 . And using this particular generalized $\mathcal{KL}$ class function, one can define the finite time input to state stability.

A mapping $\Phi: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a generalized KCL type (GKLfunction) if: (i) for fixed $t \geq 0, s \mapsto \Phi(s,t)$ is generalized K-function (ii) for fixed $s \geq 0, t \mapsto \Phi(s,t)$ is continuous and

$$\lim_{t \to T} \Phi(s,t) = 0 \text{ for some } T \leq \infty$$

The system is Finite-Time Input-to-State Stable (FTISS) if $\exists \beta(\cdot,\cdot) \in \mathcal{GKL}$ and $\gamma(\cdot) \in \mathcal{K}_\infty$ such that $\forall z_0, \nu(k)$:

$$\|z(k)\| \leq \beta(\|z_0\|, k - k_0) + \gamma(\|\nu\|)$$

with $\beta(\zeta, k) = 0$ for $k \geq K(\zeta)$, where $K(\zeta)$ is continuous with $K(0) = 0$. Now, I am going to take a very, very simple example. So, this is an example of a first-order system, where $z(k) \in \mathbb{R}$, and now I am going to propose some kind of algorithm like this, which is based on the minimum operator.

$$z(k+1) = u(k) + \varphi(k), z(0) = z_0$$

where $z \in D \subset \mathbb{R}, k \in \mathbb{Z}_{\geq 0}, u \in \mathbb{R}$ and φ is bounded with $|\varphi| \leq \varphi_0$. Control law:

$$u(k) = z(k) - \text{sign}(z(k)) \min(|z(k)|, l)$$

where $l > \varphi_0$.
So, if this term is not present, and if $\varphi(k)$ is not present, then it is possible to show that an infinite number of steps can maintain $z(k) = 0$. And what is the property of this minimum

operator? So, suppose that if I select some kind of gain L and $z(k)$, if the absolute value of $z(k)$ is greater than L, then every step I am going to do will decrement by L, because the signum of $z(k)$ will give us the sign. So, suppose I assume $z(k)$ is positive; then, at every step, I am going to decrease it by L, and at some point, $z(k)$ becomes minimum.

At that time, this value and this value are exactly the same, and in this way, one can show that in the absence of disturbance, we will achieve finite time stability in the case of the discrete time system. And this is the theorem which suggests that if you apply this kind of control to this first-order system, then there exists some kind of Lyapunov function.

Consider the system with control law $u(k)$. Let there exist continuous $V : D \to \mathbb{R}_+$ satisfying:
$\alpha_1(z(k)) \leq V(z) \leq \alpha_2(z(k)), \forall z \in D \setminus \{0\}$

$V(z = 0) = 0$

$\Delta V(z) \leq -\min(V(z), l) + \alpha_3(|\varphi(k)|)$
where $\alpha_1, \alpha_2, \alpha_3 \in$ class $\mathcal{K}$.
Then the system is FTISS with:

$$|z(k)| \leq \varphi_0 \text{ for } k \geq K(\varphi_0, z_0) \leq \left\lceil \frac{V(z_0) - \varphi_0}{l - \varphi_0} \right\rceil$$

where $z_0 \in D \setminus B_{\varphi_0}(0), l > \varphi_0$.
And after that, if they satisfy this kind of condition, it is possible to show that $z(k)$ is always bounded by the maximum value of φ_0, and φ_0 is nothing but the maximum bound of $\varphi(k)$. With this finite number of steps, I can reach this particular set, and after that, we cannot leave that set.

So, suppose that if $|z| < l$, then within one step it is possible to show that I can remain inside this. So, this case is obvious. Now, whenever $|z| > l$, I have to show how we are going to decrease. So, here one inequality is very important. Suppose that the absolute value of the first term is greater than the absolute value of the second term always; then it is possible to show that $|a - b| \leq |a| - |b|$.

So, based on that, it is possible to show that this expression comes into the picture and φ_0 is the maximum bound of φ_0. So now, if this condition is satisfied, this many steps we are going to show, and this is nothing but the settling time, and this function is called the ceiling function. The ceiling function is required because this k is always some kind of integer, and for that reason, the ceiling function is necessary. It is not difficult to show because in each step I am going to decrease. So, in this way, if I decrease, it is possible to show that finally, once this condition is satisfied, our Lyapunov function is going to be bounded by some kind of φ_0.

And if $\varphi = 0$, then finally I can show that I can converge to the origin, and this much of a step is required to converge here. There is no need to become confused because $V(k)$ is

$V(k+1) \leq V(k) + \varphi(k)$ I have retained. So, $V(k+1) - V(k)$, that is $\Delta V(k)$, that kind of expression here comes in $\Delta V(k)$. So, basically, $V(k+1) - V(k)$ will come here. So, finally, $V(k+1)$ is bounded by φ_0.

So, what is the $\mathcal{K}$ observation perturbation bound that φ_0 control must be? So, this kind of condition should required such that monotonically one can able to decrease and Lyapunov function provide the explicit convergence. So, this is the case whenever we have a single order system. Suppose that now I have a second-order system. So, how can we talk about the finite time stability of $z_1(k) = z_2(k) = 0$ exactly if there is no disturbance? So, one of the ways one can design the deadbeat control. Now, in deadbeat control within one step, it is possible to show that z_1 and z_2 both equal 0 .

Another way one can design based on the minimum operator is that the beauty of the minimum operator allows you to control the step size. It means that it is very comfortable with respect to actuator bandwidth. And here we are going to show that if we have some kind of disturbance come into the picture, then our trajectory always remains in some kind of bound, and that bound is nothing but the ultimate bound.

Second-order system:

$$z_1(k+1) = z_2(k), z_2(k+1) = u(k) + \varphi(k)$$

where $z = [z_1, z_2]^\top \in D \subset \mathbb{R}^2, k \in \mathbb{Z}_{\geq 0}, u \in \mathbb{R}$ and φ satisfies $|\varphi| \leq \varphi_0$. Switching function:

$$s(k) = cz_1(k) + z_2(k)$$

where $-1 < c < 1$ is a positive scalar. The dynamics of $s(k)$ are:

$$s(k+1) = cz_1(k+1) + z_2(k+1)$$

Control law:

$$u(k) = -cz_2(k) + \mu_1 s(k) - \operatorname{sign}(s(k))\min(\mu_1|s(k)|, l_1)$$

where $l_1 > \varphi_0, \mu_1 \in (0,1)$. So, again I have defined the sliding surface as $cz_1 + z_2$. So now one of the ways whenever you have to design some kind of controller is that you have to maintain $z_1(k)$ and $z_2(k)$ exactly equal to 0 .

So obviously, I cannot use this. So, in the next subsequent slide, I will show you how one can achieve this based on the backstepping. One of the ways that I suppose I have a second-order system is that I can design the sliding surface, and after that, in finite time, I can force all trajectories in the vicinity of the sliding surface, and after that, I can maintain them along that. So, that is nothing but a kind of sliding mode control design. So, how do we achieve that? What can you do? You can take an increment, and after that, once you take an increment, this is just a first-order system, and you can apply exactly the same algorithm as here. Whatever algorithm is applicable for the first order, because in the coordinate of $s(k)$ this is a first order system, and in this way, I am able to design control.

Now, here you can see that whenever we are designing some kind of sliding manifold. So, what our aim means is that I have infinite time to reach here in several sub-states, and after that, I have to maintain this. So, for that, I need two different kinds of stability. So, from here to here, at least s, variable s should converge in finite time, at least in some band, and after that, we can be able to slide along this particular band. So, here we have completed the transformation.

So, the whole system is represented in terms of z_1 and s, and after that, we have defined the Lyapunov function, which contains two parts. So, one part V that contains both state and another is V_1, because I have to show that the overall system remains stable, but this subsystem, the second subsystem $s(k+1) = \mu s(k) - \text{sign}(s(k))\min(\mu|s(k)|, l) + \varphi(k)$, is actually finite-time input-to-state stable in the presence of $\varphi(k)$, this kind of disturbance. So, it is possible to show that if this condition is satisfied, then in finite time $s(k)$ actually remains bounded. With some kind of bound that is decided by the variable $\varphi(k)$, it is possible to show that within this number of steps one can converge. Further, if that will remains in that bound, it is possible to show that $\Delta V(k) \leq 0$.

Consider the system $z_1(k+1) = s(k) - cz_1(k), s(k+1) = \mu_1 s(k) - \text{sign}\big(s(k)\big)\min(\mu_1|s(k)|, l_1) + \varphi(k)$. Let there exist continuous functions $V: D \to \mathbb{R}_+$and $V_1: D \to \mathbb{R}_+$satisfying:

$$\alpha_1\big(s(k)\big) \leq V_1(s) \leq \alpha_2\big(s(k)\big), \forall s \in D \setminus \{0\}$$

$$V_1(s=0) = 0$$

$$\Delta V_1(s) \leq -\min(V_1(s), l_1) + \alpha_3(|\varphi(k)|)$$

where $\alpha_1, \alpha_2, \alpha_3 \in$ class $\mathcal{K}$. Then $s(k)$ is FTISS and ultimately bounded by φ_0 with settling time: $K(\varphi_0, s_0) \leq \left\lfloor \frac{V(s_0) - \varphi_0}{l_1 - \varphi_0} \right\rfloor$ for $s_0 \in D \setminus B_{\varphi_0}(0), l_1 > \varphi_0$.

Moreover, if $V(z_1, s) > 0 \forall (z_1, s) \in D \setminus \{0\}, V(0,0) = 0$ and $\Delta V(z_1, s) \leq 0 \forall |z_1| \geq \frac{\mu_1\gamma - l_1 + \varphi_0}{1 - |c|}$ and $|s| \leq \gamma$, then the system is Lyapunov stable.

So, asymptotically I can able to remains inside that particular bound. It means that we are always not asymptotic stability $\Delta V \leq 0$. So, we are remains stable whenever we will converge to this bound. So, that is called Lyapunov stability. So, now what I am going to do is first show finite time input to the state stability of $s(k)$.

Next, I am going to pick this equation, the second equation, and after that, I am going to define the Lyapunov function. Finally, I will show that $s(k)$ is ultimately bounded. So, $s(k)$ is ultimately bounded. So, that is the same proof I can apply to whatever we have developed for the first-order system, because in the domain of $s(k)$. So, this dynamics $s(k)$, if you replace $s(k)$ with $Z(k)$ and multiply by the gain, then the exact same kind of algorithm basically comes into the picture.

And, for that reason, there is no need to re-establish it again. Now, it means that s always remains in some kind of band, and after that, using this Lyapunov function, I can proceed further, and then several sub-conditions come into the picture. So, we have actually decomposed the proof into several sub-parts, and it is possible to show that this condition will be satisfied, provided this condition is satisfied. So, what this condition is basically telling us is that I cannot start anywhere in the state space and after that I will converge to some band. So, always, by designing this, I can design μ, and it is possible to show that I can cover anywhere in this particular space, because as you can see, c is very, very small; c is very close to 1 , then this band is actually tending towards infinity.

So, in this way, we can cover and $s(k)$ always remains inside bound γ in this particular case, and it is possible to show that if $s(k)$ remains on γ. So, finally, that will converge to this particular band infinite time we have already seen. Now, the second approach, which is based on backstepping and backstepping design, is one of the most beautiful designs. Suppose that in a continuous time system, those who are comfortable with continuous time systems have already seen that if I have to design control for this system, but if you only know how to design control for a first order system, then what can you do? You can apply that control.

So, I will apply $x_2 = kx_1$. So, if x_2 is equal to kx_1, then x_1 tends towards 0 as t tends towards infinity. Now, what can you do? Since in place of x_2, you have substituted kx_1. So, you can define some kind of composite variable with the help of these two variables k and x. So, now you can see that if I force s to equal 0 , then obviously $x_2 = -kx_1$, and in this way, now $\dot{s} = \dot{x}_2 + k\dot{x}_1$, and $k\dot{x}_1$ I know if you substitute and if you design u such that $s = 0$. So, in a composite way, I can design the Lyapunov function for the whole system, and based on that, we can also select the control.

Similar kinds of things we can do for the discrete time system. In a continuous time system, I cannot do this whenever finite time stability comes into the picture, because if you put finite time stable control here, then you cannot take the derivative of s, because suppose that if I take $s^{1/2}\text{sign}(s)$. So, the power half derivative will not appear, but at least for a discrete time system, that kind of problem will not arise. So, what am I going to do? I am going to define the sliding surface like this. Why? Because we already know that if $z_1(k + 1) = \phi(z_1(k))$ and $z_1(k)$ if I select based on the minimum operator, then z_1, if I design it like this, can show that $z_1 = 0$ infinitely many times, and for that reason, we have selected ϕ like this.

Nonlinear switching function:

$$s(k) = z_2(k) - \Phi(z_1(k))$$

where

$$\Phi(z_1(k)) := \mu_1 z_1(k) - \text{sign}(z_1(k))\min(\mu_1|z_1(k)|, l_1)$$

$\mu_1 \in (0,1), l_1 \in \mathbb{R}_+$.
Control law:

$$u(k) = \Phi(z_1(k+1)) + \mu_2 s(k) - \text{sign}(s(k))\min(\mu_2|s(k)|, l_2),$$

where $\mu_2 \in (0,1), l_2 \in \mathbb{R}_+$.
Closed-loop system:

$$z_1(k+1) = s(k) + \Phi(z_1(k))$$
$$s(k+1) = \mu_2 s(k) - \text{sign}(s(k))\min(\mu_2|s(k)|, l_2) + \varphi(k)$$

So, based on the first-order difference equation, you can select the $\phi(k)$, and after that, you can define the composite variable; and once the composite variable is defined, then you can proceed further by taking an increment. Now, you can push $s(k+1) = 0$ infinitely and then overall control comes into the picture. So, overall system can be represented by this. Now, whenever you have this kind of system, so what is your aim? You have to show first that $x(k) = 0$. So, once $x(k) = 0$, so all subsequent step that is going to maintain it.

So, $x(k) = 0$, so that is again a first-order system, but the proof of this is not so easy. It is possible to show that there exists again a Lyapunov function, and here α_1 and α_2 are class $\mathcal{K}$ functions, and they will satisfy this kind of difference equation, and at that time the settling function is given like this. So, you can start with some kind of Lyapunov function that is the absolute value of both states. Here, the modified state is nothing but you can see z_1 and s, and due to that reason, we have defined the Lyapunov function like this. Again, I will do the increment; I will substitute the control because we have already calculated the control based on the backstepping logic.

Consider the closed-loop system. Let there exist continuous $V: D \to \mathbb{R}_+$ satisfying: $\alpha_1(z(k)) \le V(z) \le \alpha_2(z(k)), \forall z \in D \setminus \{0\}$

$V(z = 0) = 0$

$\Delta V(z) \le -\min(V(z), l) + \alpha_3(|\varphi(k)|)$
where $\alpha_1, \alpha_2, \alpha_3 \in$ class $\mathcal{K}$.
Then the system is FTISS and $s(k)$ is ultimately bounded by φ_0 with settling time:

$$K(\varphi_0, z_0) \le \left\lceil \frac{V(z_0) - 2\varphi_0}{l - 2\varphi_0} \right\rceil$$

for $z_0 \in D \setminus B_{2\varphi_0}(0), l = \min(l_1, l_2) > 2\varphi_0$.
Now, I have to prove whether the control I have calculated is correct or not. So, how do we prove that? So, for that, we have to make several cases because proving stability in the discrete case is a little bit difficult and not as obvious as in the continuous case, and in every case, we have to show that $\Delta V(k)$ is somehow bounded by some kind of $-l_2 + \varphi_0 l_1/\varphi_0$ if this condition is satisfied. Again in the third case, you can see that it is bounded by $l_1 + l_2$, and when this condition comes into the picture, it is finally bounded by this. Okay, so in this way I can break this whole big expression into several small expressions, and in each and every case I am going to show that I always have a decrement. So,

whenever at least one case comes at a time, one can show that $\Delta V(k)$ is always decreasing in such a way that finite time stability comes into the picture.

Proof. Lyapunov candidate: $V(z(k)) = |z_1(k)| + |s(k)|$

$$\Delta V(z) = |s(k) + \Phi(z_1(k))| + |\mu_2 s(k) - \text{sign}(s(k))\min(\mu_2|s(k)|, l_2) + \varphi(k)| - |z_1(k)| - |s(k)|$$

Using $\mu_1|z_1(k)| \geq \min(\mu_1|z_1(k)|, l_1)$ and $\mu_2|s(k)| \geq \min(\mu_2|s(k)|, l_2)$:

$$\Delta V(z) \leq (\mu_1 - 1)|z_1(k)| + \mu_2|s(k)| - \min(\mu_1|z_1(k)|, l_1) - \min(\mu_2|s(k)|, l_2) + \varphi_0$$

** Scenario I:** $\mu_1|z_1(k)| \leq l_1$ and $\mu_2|s(k)| > l_2$

$$\Delta V(z) \leq -(1 - \mu_2)|s(k)| - l_2 + \varphi_0 \leq -l_2 + \varphi_0$$

** Scenario II: **$\mu_1|z_1(k)| > l_1$ and $\mu_2|s(k)| \leq l_2$

$$\Delta V(z) \leq -l_1 + \varphi_0$$

** Scenario III: **$\mu_1|z_1(k)| > l_1$ and $\mu_2|s(k)| > l_2$

$$\Delta V(z) \leq -(l_1 + l_2) + 2\varphi_0$$

** Scenario IV:** $\mu_1|z_1(k)| \leq l_1$ and $\mu_2|s(k)| \leq l_2$

$$\Delta V(z) \leq -V(z(k)) + 2\varphi_0$$

Combining all scenarios:

$$\Delta V(z) \leq -\min(V(z(k)), l) + 2\varphi_0$$

Settling Time:

$$K(\varphi_0, z_0) \leq \left\lceil \frac{V(z_0) - 2\varphi_0}{l - 2\varphi_0} \right\rceil$$

where $z_0 \in D \setminus B_{2\varphi_0}(0), l > 2\varphi_0$

System converges to invariant set in finite time. Obviously, that will not converge to 0 because we have a disturbance. So, within this time, we are able to be bounded by some kind of bound that is given by the $2\varphi_0$, and the same kind of things, basically, you can also observe in simulation. So, here $z(k)$, you can see that in the presence of disturbance, what happens? If I have sinusoidal disturbance, then this kind of response, and control is given like this. Now, for the second-order system, we have again taken a sinusoidal disturbance and the maximum value of φ_m.

Second-order system:

$$z_1(k+1) = z_2(k)$$
$$z_2(k+1) = 2z_1(k) - 0.3z_2(k) + u(k) + \varphi(k)$$

where $\varphi(k) = \sin(0.6k), \varphi_m = 1$

Initial conditions: $z_1(0) = 100, z_2(0) = 50$

Design parameters: $c = 0.5, l_1 = 8, \mu_1 = 0.8$
I have taken one. These kind of initial condition we have taken. Here, we have compared this kind of approach with several existing approaches. And you can see from this particular two graphs that whatever this means, this is the proposed approach, and it is also beneficial in terms of the control. All other approach control is very very huge. In the case of Bartoszewicz, the control is obviously small, but the band is basically higher.

You can see that this is the band for the Bartoszewicz case. So, what we have achieved is minimal band, low inertial control effort, and complete chattering elimination, which we have compared with these two papers. Now, we are going to extend the twisting algorithm. So, we have already seen the twisting algorithm in the case of the continuous time system that is given by $k_1\text{sign}(x_1) - k_2\text{sign}(x_2)$. So, if you discretize it without any uncertainty using the Euler method, it is possible to show that x_1 and x_2 remain bounded, but they are not
going to converge to the equilibrium point. It means that you cannot achieve higher-order sliding mode control even in the absence of the disturbance.

But if you write algorithms in this way, obviously whenever you write in this manner, that is going to create some kind of restriction. What kind of restriction comes into the picture? You can see that the twisting algorithm was initially developed to remove chattering, and the philosophy is that if I have a first-order system, in a first-order system the control is actually discontinuous, meaning classical sliding mode control. So, I will actually increase the relative degree by 2 , and after that, I will design the twisting algorithm. So, a similar kind of interpretation actually comes into the picture here. So, this algorithm is not directly applicable to a relative degree 2 system.

Although twisting is directly applicable to a relative degree 2 system, it will cause chattering due to the discontinuous term. So, here this is applicable for relative degree 1 , but control is continuous; however, the difficulty arises in proving that z_1 and z_2 are equal to 0 in finite time at least when there is no disturbance. And it is possible to show using this again if this condition satisfies whatever gains μ_1 and μ_3. So, this Lyapunov function is constructed by one of my PhD students, Parijat, and after that, $V(z) \geq 0; V(z) = 0$ is 0 , and ΔV is the minimum of this. So, it is possible to show that if you select l, then the minimum number of steps for both z_1 and z_2 to equal 0 is finite.

Unperturbed Case:

$$z_1(k+1) = \mu_1 z_2(k)$$
$$z_2(k+1) = \mu_2 z_1(k) - \text{sign}(z_1(k))\min(\mu_2|z_1(k)|, l_1) + \mu_3 z_2(k) - \text{sign}(z_2(k))\min(\mu_3|z_2(k)|, l_2)$$

where $z = [z_1, z_2]^\top \in D \subset \mathbb{R}^2, k \in \mathbb{Z}_{\geq 0}, l_1 > 0, l_2 > 0, \mu_1, \mu_2, \mu_3 \in (0,1)$.
And, one can easily extend this whole result by adding $\varphi(k)$. Basically, whatever Lyapunov function is proposed here, this Lyapunov function is actually a strictly positive function. So, obviously, that will not provide finite time convergence, but will always provide finite time input to state stability. So, that kind of analysis I have not done here, because more time is required to show each and every step, but for those who are interested, one can see our recent paper in the IEEE Transactions on Automatic Control. And here you

can see that an infinite number of steps one can be able to converge. So, exactly we have same behavior like the, like the means classical twisting algorithm which is actually in case of the continuous time system in absence of disturbance.

One can be able to extend the super twisting algorithm also exactly based on the same idea. So, here you can see that initially, whenever we talk about the super twisting algorithm in the continuous time case, I have this kind of structure: $\text{sign}(x_1) + x_2$ and $\dot{x}_2 = -k\text{sign}(x_1)$. Again, philosophy is that if you try to implement this for some kind of computer control system or using a micro processor, then you have to do some kind of discretization. So, if you do discretization, obviously x_1 and x_2 cannot be maintained exactly equal to 0 in the absence of disturbance as well. So, what is it, and how do we exactly implement it? So, for exact implementation, you can go through the difference equation with the minima.

So, the minimum function we have defined is like this, and actually, we have added this term. In place of the whole term, we have similarly added this term in place of the second term. It is possible to show that in the absence of disturbance, now in finite time, z_1 and z_2 both equal 0 . And, if you see carefully then this function is again continuous. So, on higher-order derivatives, exactly the minimum kind of things is going to appear.

In that particular sense, you can be able to feel that this is nothing but the higher-order version of the discrete super twisting algorithm. Here, one restriction comes into the picture due to the selection of the Lyapunov function. If you check the gain condition, the gain condition requirement is very, very high. Practically, this is suggesting to us that it is better not to avoid starting our initial condition on this axis, because even if you can start, at that time I cannot prove stability, but still what happens is that somewhere you are going to fall here, and again basically our system becomes stable. So, this kind of restriction has come about due to the restriction of the construction of the Lyapunov function, but if you simulate this algorithm for any value of ρ_1 and ρ_2, if ρ_1 and ρ_2 satisfy this kind of condition, you will see that they will always give us the finite time stability of z_1 and z_2, and this is the better version of that particular case, and again the gain condition remains the same.

And this particular function, a $\mathcal{K}_\infty$ class function, that $\mathcal{K}$ class function ρ_3 comes into the picture, where ρ_1 and ρ_2 are $\mathcal{K}_\infty$ functions. So, again I can prove that this is finite time stable. So, the Lyapunov function is bounded by a $\mathcal{K}$-class function. Now, this is the response in absence of the disturbance unperturbed case and exactly you can able to see that for we have taken several initial condition and after that we have shown that for all initial condition in finite time I can able to converge. Whenever you have disturbance this kind of disturbance you can still able to see that behavior is quite satisfactory.

And this is the phase portrait because I have a second-order system. So, I can easily draw a phase portrait. So, this is unperturbed case and this is the perturbed case. One of the very useful applications of the super twisting algorithm is observer design. So, what have we done? We have taken the pendulum system, and after that, we have designed the observer for this particular system. So, how are we basically designing the observer? So, in order to design an observer, first you can copy the dynamics.

So, how do you copy the dynamics? $\bar{z}_1(k), \bar{z}_2(k)$, and after that, you have to add two correction terms l_1 and l_2. Now, l_1 and l_2 terms can easily be designed based on this particular algorithm. And, in this way, you can show that z_1 and z_2 both equal 0 , meaning the estimate of z_1 and z_2 and the actual z_1 and z_2, if you take their disturbance, their actual increment.

So, that is called an error. So, the error is always 0 . So, the error is 0 . So, finally, one can be able to estimate the state. So, we have simulated this again. And, after that, we have actually this is the parameter we have used, and it is possible to show that in the presence of the disturbance, whatever observer is performing that is extremely good, means z_1 is actually able to estimate $\hat{z}_1$, and similarly, z_2 is able to estimate $\hat{z}_2$. And, obviously in the presence of disturbance, the disturbance means that whatever error exists between $z_1 - \hat{z}_1$ and $z_2 - \hat{z}_2$ is not exactly equal to 0 .
So, that always remains in some vicinity. So now it is time to conclude this lecture. So, we have discussed the novel discrete time algorithm such that one can realize higher sliding mode control in the discrete case; we can also discuss the backstepping kind of design, and one can also talk about the Lyapunov-based design. And we have also seen one application of how to design a discretetime super-twisting observer such that I can achieve finite-time estimation. in disturbance free system and if I have disturbance then I will come up with the finite time input to state stability. We have taken two cases, the unperturbed case and the perturbed case, and then we have shown the finite time stability, which is very useful whenever we are talking about practical systems because nowadays almost all practical systems are actually computer-controlled, and at that time discrete algorithms have several significances.
and obviously root of these algorithm is not based on the Euler discretization. So, we have developed separate kinds of algorithms for both the twisting and super twisting. So, with this remark, I am going to end this class. Thank you very much.