

## **Sliding Mode Control and Applications**

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**Lecture-53**

So, welcome back. In the previous class, I was talking about discrete sliding mode control and what we have observed is that in discrete sliding mode control, we have two schools of thought. So in the first thought, basically what we are doing is discretizing the continuous control, and then we are bringing all trajectories towards the sliding manifold, and after that, we are maintaining. Second thought that is based on without any switching function using continuous control, we are actually reaching the sliding surface and after that, we are maintaining. And why the second approach is valid in the case of a discrete time system is that the discrete differential equation is inherently discontinuous in nature. In this lecture, we are going to solve one question that is very important.

So, we are already well aware that whenever we are talking about the continuous time sliding mode control. So, at that time, the differential equation with a discontinuous right-hand side comes into the picture. Either you can talk about classical sliding mode control or higher-order sliding mode control. So, what equivalent kinds of things will appear in the case of the discrete time system that we are actually going to explore? And it is possible to show that in order for the sliding mode control to occur in the case of a discrete time system, some difference equation with a minimum comes into the picture.

So, this particular concept was developed by one of my PhD students under my guidance, Parijat, and the same work I am going to present in this particular lecture. So, we are first going to highlight the limitations of Gao's as well as Utkin's approach for discrete sliding mode control. And after that, we are going to talk about the minimum function-based reaching law and what our main goal is: to eliminate chattering, prevent excessive control, and handle both perturbed and unperturbed systems. And obviously, I have to maintain theoretical properties like stability, convergence, and settling time. And we are also going to validate through the pendulum system's example.

And what is the key advantage of the discrete sliding mode control? We already discussed in the previous class that this is easier to implement. And obviously, there are several benefits that come into the picture because whenever we discretize the continuous time theory and apply it to discrete time systems, we always have suboptimal behavior. So, how to prevent that behavior? It is possible that by using discrete sliding mode control we will be able to do it. And obviously, we can also reduce the chatter and maintain stability while lowering the switching frequency compared to continuous operation, which is essential for several sample data systems. So, in discrete sliding mode control, I have already told you that we have two schools of thought.

One is a switching-based approach and another is a non-switching-based approach. In the previous class, I discussed this in very, very detail. These two are the representative papers you can explore. And obviously, a non-switching based approach comes into the picture because discontinuous behavior is inherent to any discrete time system. Now, whenever we talk about Gao reaching law, in the previous class we saw that we applied some kind of  $\tau$ .

So, at that time,  $(1 - q)\tau$  comes into the picture, but suppose that if the sampling instant is 1, if I take it, then obviously that term will not appear, and in this way, this law comes into the picture by discretization of the constant rate reaching law. I am going to maintain between 0 and 1, and you can see here nicely that if you start anywhere monotonically, I am going to decrease, and after that, I am going to cross and recross. So, that is the behavior during the simulation. And this is based on the second school of thought that if you start from here, then in one step I will reach the sliding surface, and after that, we can subsequently move to the equilibrium point. Now, this is a very, very interesting question about what kind of modification I will make in the first approach.

Because if you see that I have the signum function, then using the signum function, I can somehow reject the uncertainty that is the key factor. So, how can I modify this approach such that not only one sampling period, but also subsequent sampling periods will attract towards the sliding surface, and after that, we can maintain it? And, during this particular process, it is possible to show that one kind of concept that is inspired by the optimization technique, as well as the work of Professor Haddad, is that the difference equation with minima comes into the picture. And I have already discussed that whenever we are talking about discrete time sliding mode control, it is exactly similar to continuous time sliding mode control; at that time, I have a difference equation with minima that comes into the picture like a differential equation with a discontinuous right-hand side. And in this way, theoretically, you can justify both approaches, and based on that, we have recently had this book accepted. I hope that by June this week, this book will come online and be published with my student Parijat.

And on a Springer. So, you can easily explore this particular book. So, the whole presentation is based on this particular book only. So, again, I am going to talk about the discrete-time system.

$$z(k + 1) = Az(k) + bu(k)$$

$$\text{where } z \in D \subseteq \mathbb{R}^n, k \in \mathbb{Z}_{\geq 0}, u \in \mathbb{R}, A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^{n \times 1}.$$

We have designed this sliding manifold. Either you can design linear or non-linear, but you have to maintain this. So, you have to maintain the sliding surface equal to 0. So, that is our core objective, and it is possible to show now that this law is exactly the same as the continuous law. Here I just have the  $s(k + 1)$  term, and you can see that I am going to multiply the minimum function here. So, what is beauty that I have to see the minimum between  $s(k)$ , and  $\gamma$ ? So, suppose that  $s(k)$  initially is 7.5 and I have to reduce  $s(k)$  to 0, and suppose the sampling instant is 1, and at each sampling instant, I am going to reduce it by 1. So, at point  $k = 7$ , I am going to substitute here, and you can see that every time

the minimum is 1, and due to that design, I am going to decrease by 1. So, next sample 6.5, then next sample 5.5, 4.5, 3.5, and finally, you can see that I will land at 0.5. Now, from 0.5, I have to decrease to 0. So, how do you do that? So, you can see that 0.5; now the minima is going to change. So, initially, the minimum is decided by this  $\gamma$ ;  $\gamma$  I have selected is 1. Now, minima is basically selected by the minimum between these two values, 0.5 and exactly using the signum function, I can converge to this manifold. So, this is the basic core idea, and after that, one can be able to design the sliding surface, and everything is exactly like the classical presentation of discrete-time sliding mode control.

Switching function:  $s(k) = \phi(z(k))$

Sliding hyperplane:  $\phi^{-1}(0) = \{z \in D : \phi(z) = 0\}$

Reaching law:

$$s(k+1) = s(k) - \text{sign}[s(k)] \min(|s(k)|, \gamma)$$

where  $\gamma \in \mathbb{R}_+$ , which is chosen by the designer.

Linear switching function:  $s(k) = c^T z(k)$  where  $c \in \mathbb{R}^{n \times 1}$  with its elements chosen such that  $c^T b \neq 0$ .

So, now suppose that if you have some system like  $z_1(k+1) = z_2(k)$  and  $z_2(k+1) = u$ , then how do you proceed? You can design  $c_1 z_1(k) + z_2(k)$ . And after that, you can design an algorithm like this in this way. So, if you design, then exactly  $x$  is equal to 0. So, if  $s = 0$ , then I have  $z_2(k+1)$ ; this should be  $k+1$  I have to write here.

So,  $k+1$ . So, finally, what happens to  $z_2(k+1)$  is nothing but I have  $-c_1 z_1(k)$ , and if  $c_1$  lies between 0 and minus 1 to 1. So, if that lies between -1 and 1, it is possible to show that  $z_1$  equals 0. So,  $z_2$  is also equal to 0. So, in this way, I can be able to design the whole sliding mode control for the discrete time system. So, I have taken a very, very simple example.

Again, correction is  $z_1(k+1) = z_2(k)$ ,  $z_2(k+1) = u$ . So, this basically resembles the ideal continuous sliding mode control without uncertainty, with no need for cross and recross, and in this way, I can exactly reject the chattering and unnecessary control effort because in the subsequent time interval I am going to converge, but in Utkin's approach, I have to converge within one approach here. So, a huge control effort is required. and this is somehow the definition that why this condition ensure the convergence and we have already seen that if  $|s(k)|$  is greater than gain. So, you can select based on the actuator limit of your system; after that, I will land up to this, and when  $|s(k)|$  is less than or equal to  $\gamma$ , then obviously,  $s(k+1)$  is equal to 0.

So, in this way I can actually be exactly able to converge to the equilibrium point. An equilibrium point is nothing but a sliding surface in this particular case because one important thing I am going to highlight is that in the case of sliding mode control, the whole sliding surface somehow created becomes the equilibrium point, but the largest invariant set is the only equilibrium point. So, all trajectories finally converge to the equilibrium point, but ultimately they will end up at the largest invariant set, which is the equilibrium point. So, now again  $s(k+1)$  you can calculate and then equate  $s(k+1)$  with this

particular reaching law, and then in this way you can easily design the control action. And this is the reaching law we have already mentioned, and it is possible to show that by using this reaching law, you can achieve finite time convergence and a finite time converging function.

So, a finite number of steps is given by what is called the ceiling function. So, by using the ceiling function, you can give a finite time step, and the proof is very easy: if  $|s(0)|$  is less than the gain. So, within one step, I can converge to the equilibrium point. But if  $|s(0)|$  is greater than  $\gamma$ , at that time, I need the subsequent step, and you can see that in the subsequent step, finally, I will reach this condition, case 1 condition. So, in this way, it is possible to show that  $|s(0)|$  divided by  $\gamma$ , and since the  $|s(0)|$  function may not be an integer, every time I am going to decrease by some kind of integer amount, and for that reason, a ceiling function is required in order to show the convergence time.

Definition DTS satisfies the reaching condition of the sliding mode iff for some  $k \geq 0$ , the following holds:

$$\begin{aligned} |s(k)| > \gamma &\Rightarrow |s(k+1)| < |s(k)| - \epsilon \\ |s(k)| \leq \gamma &\Rightarrow s(k+1) = 0 \end{aligned}$$

where  $\epsilon \in \mathbb{R}_+$ , is arbitrarily small.

Control Law Derivation

Calculation of  $u(k)$  : Taking increment of  $s(k)$

$$s(k+1) = c^T A z(k) + c^T b u(k)$$

Comparing this with right-hand side of the reaching law

$$u(k) = (c^T b)^{-1} \{-c^T A z(k) + s(k) - \text{sign}[s(k)] \min(|s(k)|, \gamma)\}$$

Rewriting the reaching law

$$s(k+1) = s(k) - \text{sign}[s(k)] \min(|s(k)|, \gamma)$$

### Convergence Theorem

If in the reaching law,  $|s(0)| > \gamma, \gamma > 0$ , then for some  $k \geq K(s_0)$ , the absolute value of the sliding function of the system ultimately becomes zero, where  $K(s_0) = \left\lceil \frac{|s_0|}{\gamma} \right\rceil$ .

Case 1 ( $|s(0)| \leq \gamma$ ) :

$$s(k+1) = s(k) - \text{sign}[s(k)] |s(k)| \Rightarrow s(k+1) = 0$$

in one time step ( $K(s_0) = 1$ ). Case 2 ( $|s(0)| > \gamma$ ):  $s(k+1) = s(k) - \text{sign}[s(k)] \gamma, |s(k+1)| = |s(k) - \text{sign}[s(k)] \gamma|$ . Since  $|s(k)| > \gamma: |s(k+1)| \leq |s(k)| - \gamma$ . Recursively:

$$|s(k)| \leq |s(k-1)| - \gamma \leq |s(k-2)| - 2\gamma$$

$$\leq |s(0)| - k\gamma$$

When  $(|s(0)| - k\gamma) \leq \gamma$ , we get  $s(k+1) = 0$  for all  $k \geq \left\lceil \frac{s_0}{\gamma} \right\rceil$ .

Now, this is a mathematical model of the pendulum, and it is possible to show that I have taken some kind of bounded uncertainty here. Suppose for the first simulation that there is no uncertainty and that you select these kinds of parameters. So, finally, you can see the behavior. So, this behavior is called unperturbed behavior. What is the meaning of RL1? So, we have just proposed some kind of reaching law here.

So, this is the reaching law. So, we have given some kind of name RL1, because the two reaching laws we have proposed are inspired by the difference equation with the minima. Now, this is basically the unperturbed case again; you are able to see the UK and sliding surface. So, we can exactly converge to the equilibrium point, and this many samples are required. So, the first sample, the second sample, and so on. Now, it is possible to show that you are able to come up with another algorithm.

So, in sliding mode control, basically, we have two different algorithms. Several times I have discussed that  $\dot{s} = -k\text{sign}(s)$ , and another beautiful algorithm, obviously, that is not robust with respect to the uncertainty, is  $\dot{s} = -\alpha s$ . Now, if I do not have any uncertainty, using continuous control, I can easily move to the equilibrium point exactly. So, this basically reaching law is actually inspired by here. Again, I have converted the discontinuous part into a difference equation with minima.

And again, observation is exactly like this because suppose that our initial condition may be some kind of integer multiple of the step or a non-integer multiple of the step. So, an integer multiple of a step, I will reach here, and after that, I am going to maintain it, and it is possible to show that tuning this parameter is necessary because the number of parameters is higher here. So, you can tune more parameters, and after that, you can accelerate your convergence speed. And obviously, this is also very convenient with respect to the actuator limit. And again, the approach is exactly the same as the previous one: here we are going to calculate the minima between this and this, and once the minima come into the picture, then the signum  $s(k)$  and the magnitude of that will be multiplied.

So, suppose that if this is a minimum, then exactly  $s(k) - s(k)$  comes into the picture. This is minima, then we are going to decrease. So, exactly same way I can able to establish the convergence condition. And how do we obtain the law? Again, you can define the sliding surface as  $s(k) = c^T z(k)$ . You can take the increment, and after taking the increment, you can design the control law.

And it is possible to show that this much time step is required to exactly convert to the sliding surface. And here again, this is the ceiling function. So, how do you drive this ceiling function? So, to obtain this you can start with the reaching law. If this condition comes into the picture, then within one step you will be able to convert, because at that time the minima become  $s(k)$  and the signum of  $s(k)$  and  $\gamma$  will cancel out.

So, basically, this is  $s(k+1)$ . So, exactly like Utkin's law. Now, if this is the case, then in subsequent steps I am going to decrease. So, one can easily apply the theory of subsequent

steps, and finally, we are going to show that this much of convergence time comes into the picture. And this is the simulation result again for the same system. So, I have taken the initial condition, and I have applied it to the pendulum system.

And this is again the sliding surface behavior and control behavior. Now, obviously, whatever things we propose should be equally implementable in case of uncertainty. So, I am assuming that the uncertainty bound is given like this, where this is the upper and lower bound. Again, I have proposed the switching surface, which is basically also called the sliding surface, and after that, we are going to implement the first algorithm that is called RL1, reaching law 1, and at this time, I am going to add some kind of perturbation inside this law. So, what is our main goal? I have to now show that obviously  $s(k)$  is not equal to 0, but  $s(k)$  is going to remain in some vicinity.

So, that vicinity is given like this, and this kind of sliding mode we have already seen is called the quasi sliding mode control. So, basically, in the absence of disturbance, I will get exact sliding, but in the presence of disturbance, I will get the quasi-sliding mode. And this is the again condition. So, you can easily understand that since the minimum operator is here, if  $|s(k)|$  is greater than the minimum of gain,  $\gamma$  is gain, and gain of  $\gamma$ , then I am obviously going to reduce  $\gamma$  step. But once, due to this uncertainty, I have to stop somewhere.

So, this is the stopping condition. Beyond that, I cannot reduce further. Again, I can easily control the drive. What can you do? You can take this reaching and then you can just equate it with whatever law I have proposed. So, this newly formulated law allows you to approach it, and in this way, it is possible to show that I am able to reach this step, where  $\gamma$  is nothing but the bound of the uncertainty we have, assuming that this is the upper and lower bound. So, again, proof is exactly the same, but basically we are going to do if  $|s(0)|$  is less than  $\gamma$ .

So, finally,  $s(k + 1)$ , this term becomes 0, and finally, our upper bound is given like this. And in just one step, but if  $k$  is greater than  $K(s_0)$ , then at that time I will get this kind of bound; however, if  $|s(0)|$  is greater than gain, then several subsequent steps are actually required every time. This particular guy is trying to adjust this perturbation such that we can move in a favorable direction. And finally, what happens? Using the subsequent step, I can reach the sliding surface. And if you do this kind of manipulation, it is not difficult to show that the time of convergence is given like this.

So, please just do the calculation with pen and paper; you can easily get it. And, this kind of thing will be achieved in finite time because in a finite number of steps I am going to converge to the quasi-sliding mode band. And here you can easily see how the quasi sliding mode band comes into the picture, because I have a sinusoidal disturbance and the effect of the disturbance also appears on  $z$ . So, in the phase plane, I am not exactly going to converge to 0; I am going to converge to some kind of set that is represented like this, and this is nothing but the control input, which is behaving just like the perturbation. So, now this is actually the last part of this particular lecture. Now, what I am going to do is, again, apply the second RL2, which means reaching law 2, and I am going to make a similar kind of modification here. This law has advantages over

the previous one because I have two tuning parameters, and it is possible to show the same: if there is no uncertainty, then you will converge exactly, but when some uncertainty comes into the picture, you will always converge to the quasi sliding mode band. And how do you calculate the quasi sliding mode band? You can calculate the controls. How do you calculate the control? You can take this system, design a sliding surface like  $c^T z(k)$ , and after that, you can take the increment; finally, it is possible to show that by equating with this reaching law, you can get this kind of control. Now, in presence of disturbance, obviously number of step you have to wait little bit more.

So, it is possible to show that after this step,  $s(k)$  becomes ultimately bounded, meaning all trajectories remain bounded, and after that, it is not going to leave the set. So, again you can do the analysis; again you can apply the minimum operator theory, and it is possible to show that finally, one step is taken if this condition is satisfied. So, within one step, you can converge to this point, but if this condition is satisfied, then you have to actually wait for the subsequent step so that you can move to case 1 again. And, for that reason, in every step, in order to maintain monotonicity, what we have to do is ensure that every gain compensates for the uncertainty. And, in this way, if you manipulate it by yourself, it is possible to show that you will get this finite time.

And this is the expression of finite time. So, again we have done the simulation, and you can see that again in the simulation. So, this is a graph between  $k$  and  $z_1$ . So, the effect of sinusoidal things comes into the picture, and this is the phase plane analysis of  $z_1$  and  $z_2$ . So, we are always our means final condition is going to lie inside this invariant set and this is the control effort.

Now, it is time to conclude this lecture. So, we have developed some kind of novel reaching laws. And after that, what is one of the main conclusions of this? Exactly the same as a differential equation with a discontinuous right-hand side, it is possible to show that in a discrete-time difference equation, a minimum comes into the picture. And using that, I am also going to show you how one can design twisting and super twisting in the next lecture. And obviously, a chattering one can compensate, and because this is inherently discrete in nature, we have implemented that on a pendulum system verified in both perturbed and unperturbed cases, and somehow the performance of this controller is extremely good either in the presence of disturbance or in the absence of disturbance.

So, with this remark, I am going to end this lecture. Thank you very much.