

Sliding Mode Control and Applications

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Lecture-52

Welcome back. In the previous class, I talked about the stability theory of discrete non-linear systems. And what was our observation? So, approximately whatever theory is developed for the continuous time nonlinear system is extended for the discrete time systems, and instead of exponential stability, people are just talking about geometrical stability. In this lecture, with the help of whatever previous stability theory we have discussed, we are going to propose a new kind of sliding mode control that is called discrete sliding mode control. So far, we have just discussed the continuous sliding mode control. In this class, we are assuming that our system is inherently discrete and how to develop sliding mode control for that particular class of system.

So, what is the purpose of the discussion? The main purpose of the discussion is why we need discrete sliding mode control. So, if you see the literature, there are several practical systems where control is implemented with the help of a digital computer or macro controller. We know that digital computers and macro controllers are only compatible with discrete time signals, and for that reason, I can justify the requirement of discrete sliding mode control. Most of the time, whenever we are assuming a continuous time system, particularly when talking about sliding mode control.

So, we are assuming that it is due to the discontinuous term. So, control is changing with very, very high switching frequency. But whatever our system, every system has some kind of inertia, some kind of delay, and due to that reason, the performance of the continuous-time system, or whatever theory we are developing for the continuous-time system, is not compatible, particularly during practical implementation. and digital implementation of continuous control. The basic philosophy is that you can develop a theory for continuous time systems, and after that, you can apply the sample and hold; in another way, we first discretize the continuous control, and by using sample and hold, I am going to implement it in the practical system.

But in this particular process, particularly for the sliding mode control, unwanted chattering phenomena come into the picture. And in this way, one of the key features is the robustness of the sliding mode control. So, we are going to compromise with the robustness as well as stability of the overall system. So, what is the practical advantage of discrete sliding mode control over the continuous one? First, before discussing discrete sliding mode control, I am going to tell you how people are basically designing it. First, they are creating

the continuous mathematical model, and after that, they are not discretizing the continuous control, but they discretize the mathematical model.

No mathematical model is exact, and for that reason, we know that after discretization, whatever model we are getting is not perfect either. Now, for that imperfect discrete model, we are going to design the sliding mode control, and for that reason, the name discrete sliding mode control comes into the picture whenever we do that kind of task. So, basically, the lower switching frequency requirement comes into the picture than the continuous one. And these kinds of systems, there are several classes of systems that are inherently discrete in nature. And even if that is not discrete in nature, whenever we are designing control, one of the important factors that we have to measure using sensors.

And most of the time, in order to get the next instant, we have to wait for the next output, and due to that reason, we have control over when we are going to update it since the information has still not arrived, and for that reason, we have to wait. Suppose that I have to control some kind of satellite and we have a ground station. So, whenever we are going to send information to the satellite and it comes back, there is always some kind of time lag in this particular process, even if the satellite system is continuous in nature. And due to that reason, this will justify the practical utility of the discrete sliding mode control. And obviously, one issue is the computer control system because now we are trying to implement whatever things.

So, we are trying to implement using some kind of network where the role of the computer is very important, and for that reason, we need some kind of methodology such that even if I develop a theory for continuous time systems that is implementable in a discrete framework, there are several systems that have sampled measurements, such as biological systems, radar, economics, and obviously, power electronic systems, where we have to control the on-off operation, and at that time, it is again possible to show that discrete sliding mode control is more justified. So, what is the key benefit if we switch from the continuous sliding mode to the discrete sliding mode? Obviously, one of the main prospects is the implementation, because you can develop any kind of theory, but if the implementation is not good, then that theory is not very useful. And due to that reason, the practical applicability of any theory is very, very important. And obviously, we have limitations; theory will develop very quickly, but if you look at the progress in hardware, it is very slow, and due to that reason, there is always some lag between theoretical development and practical development. So, using some new methodology, we somehow have to compensate for that kind of mismatch, and for that reason, we are going to propose the discrete sliding mode control.

Using this particular discrete sliding mode control implementation of digital control in modern systems is basically easier. So, if you see the history of discrete sliding mode control, it is obvious that variable structure control was developed by Professor Emiliano and Professor Utkin. And after that, the variable structure control and discrete variable structure control come into the picture, and their route of development again lies in the same country. The term discrete sliding mode control is introduced by Professor Utkin and Darkunov. So, Professor Utkin is one of the founders of sliding mode control; several times

I have repeated that, and after that, you will be able to see that several people have contributed in this particular direction.

So, I have just given a few names. Obviously, this is not the complete name. So, it might be possible that I missed some names. So, if I miss that name, I have used this kind of word for that. So, the main contribution started from Utkin, Darkunov, and Sarpturk, as well as Spurgeon and Sira-Ramirez.

So, very nice books are written by this particular author, especially for nonlinear systems. And after that, Gao, Bartoszewicz, and Professor Bandopadhyay also contributed a lot towards discrete sliding mode control, and due to their huge contribution in discrete sliding mode control, he became an IEEE fellow in the same domain. So, now the implementation characterization is that the control input is computed only at the sampling instant, and after that, the control remains constant over the entire sampling period. So, we will design the control and sample again; I am going to change the control because everything is discrete in nature. So, if you see the discrete sliding mode control, in discrete sliding mode control, we basically have two different schools of thought.

So, what kind of schools of thought are there in the case of discrete sliding mode control? So, first is switching based. So, what is the meaning of switching based? We are taking the continuous sliding mode control, and after that, we are going to discretize it. So, after discretization, the switching term is again inside that discrete model. And due to that reason, Professor Gao has shown that if you are going to get some kind of reaching law. Actually, the reason I am talking about reaching law is that, if you remember the foundation of continuous sliding mode control, sliding mode control design has two steps.

So, the first step is called the reaching phase, and after that, I have to slide along this particular line. So, here during the reaching phase, I need some kind of discontinuous control, and discontinuous control has two responsibilities. One responsibility is to compensate for the matched uncertainty, and the second responsibility is to take any initial condition; if the trajectory starts from any initial condition throughout this space, then in finite time, I have to move to the sliding surface, which is another responsibility. So, in order to move here in finite time, some high switching frequency is required, and after that, I have to maintain along this line. So, this is also very, very important.

So, in order to maintain this, we have to switch at a very high switching frequency. Now, during the discretization process, since I have to remain in the vicinity of the sliding surface, if you are going to discretize, then a similar kind of requirement comes into the picture. In continuous control, we actually have control at each and every moment, and due to that reason, ideally and theoretically, we can show that the trajectory will converge from any point, and after that, it can be maintained exactly. However, whenever sampling comes into the picture, this kind of behavior emerges, which is called quasi sliding mode behavior. We will characterize that kind of behavior in the subsequent slides.

Another philosophy that is also very beautiful is, what is the suggestion? Since sliding mode control requires some kind of discontinuous control in the case of the continuous, you have already seen that in the case of higher sliding mode control, the discontinuity is

going to shift to the higher derivative. But if you see the discrete-time system, it is inherently discrete or inherently discontinuous. So, if inherently discontinuous, then it is possible that without using the discontinuous control, I can achieve the sliding in the absence of the disturbance. And that kind of approach is basically proposed by Bartoszewicz and Bartolini. So, what is the main philosophy? By using one sample, you can converge here, and after that, you can maintain.

Later, this one-sample theory is actually extended for the multiple samples. So, those are the kinds of things we are going to discuss in this particular lecture and design again. So, the design step is exactly the same as the continuous if we are going to talk about the sliding phase. So, this phase is called sliding, and this phase is called reaching. So, obviously, I have already told you that the reaching phase in the case of discrete-time sliding mode control can be both continuous and discontinuous, and due to that, teaching the reaching phase is particularly different for the discrete perspective.

So, how are we basically designing a stable sliding surface? First, we are assuming that ideal sliding comes into the picture. What is the meaning of ideal sliding? Whatever sliding manifold we are designing is. So, we are going to make sure that the sliding manifold equals 0 in some finite time, and after that, even if it is not exactly 0, we are going to remain in the very close vicinity of the sliding surface after finite time. And obviously, once we are in close proximity to the sliding surface, we have reduced aerodynamics. and that is nothing but some kind of zero dynamics.

So, that dynamics should be stable by the design. So, always we have design parameter. So, we are going to choose the design parameter in such a way that whatever sliding phase dynamics should be stable. I have already discussed that the basic reaching phase is a little bit different because we are able to achieve this with a lower switching frequency, as in the discrete time case we always have the restriction of the sampling time. So, I cannot force sliding with a very high switching frequency, and for that reason, we always have to compromise on the switching frequency.

Obviously, reaching conditions is more complex because we do not have control in each and every place. We have control only at the sampling instant, and for that reason, the theory of discrete-time sliding mode control is different from the continuous one. Now, in the case of continuous sliding mode control, you might have seen that one of the fundamental laws based on discontinuity is expressed like this. So, this is reaching law, and what this reaching law is suggesting is that if you have so many folds and if you start from any initial condition, then in finite time you can move. So, how can you move infinite time or at least how can you attract towards this manifold? So, for that, we are maintaining $\dot{s} \leq 0$, and that is also justified by Newton's law.

Suppose that if s belongs to \mathbb{R} , a one-dimensional manifold, it is represented by the real line, and if you have to move towards $s = 0$, that is equilibrium. So, if $s > 0$, then \dot{s} must be less than 0. Similarly, if you are here and $s < 0$, then \dot{s} should be greater than 0, because the object is finally going to move in the direction of the rate of change; that is the philosophy. And due to that reason, in a continuous time system, since I have control, we

are maintaining this kind of condition for convergence, or this condition is also called the reaching condition, because I have to design the first reaching phase. So, what has this particular Russian scientist done? They have just created the equivalent of a \dot{s} .

So, what is the equivalent of \dot{s} ? That is $s(k + 1) - s(k)$, and sampling is $1, k + 1$ minus k . So, using that, they have actually replaced. So, that is not exactly a replacement. Otherwise, if we want to do exact means replacement, then we have to do $\frac{s(k+h)-s(k)}{h}$ as h tends towards 0. So, those are the kinds of things we have to perform.

But, in the discrete case, I always have a limitation of h ; h is the sampling frequency that is always finite, and for that reason, they have developed a theory for taking the sampling equal to 1. So, in this way, basically, this condition comes into the picture, but it is possible to show that this will not guarantee the convergence, and for that reason, this is only a necessary condition, not a sufficient condition. But I have to actually develop some kind of necessary and sufficient condition such that in the discrete plane, if I start anywhere, after subsequent sampling I can reach the sliding surface. So, I am assuming that control is here, control is here, control is here; in between, I can update the control. So, using this particular necessary condition, we can give the direction of the momentum of the $s(k)$, but that will not guarantee convergence, and after that, Sarpturk's reaching law comes into the picture.

You can see here what Sarpturk's law is suggesting: if you see the $s(k + 1)$ and $s(k)$, then $s(k + 1)$ means the next sampling state, whatever function s is, the sliding variable that is generated. So, that is always less than the previous one, and it is possible to show that this is decomposed into two different conditions. So, this is a necessary condition, and it is the convergence condition. From here, you can easily check that this condition is always satisfied here and here. So, whenever we are designing sliding mode control and if we use this kind of condition, then it is possible to show that if you start anywhere in the discrete plane, after a subsequent time interval, you can move towards the sliding surface.

Milosavljevic's Necessary Condition

$$s(k)(s(k + 1) - s(k)) < 0$$

where $s(k)$ is the sliding surface.

Continuous Counterpart

$$s\dot{s} < 0$$

Sarpturk's Condition

$$|s(k + 1)| < |s(k)|$$

Decomposition

Necessary Condition:

$$(s(k + 1) - s(k))\text{sgn}(s(k)) < 0$$

Convergence Condition:

$$(s(k + 1) + s(k))\text{sgn}(s(k)) > 0$$

Now, another law proposed by Furuta and Furuta is very famous. He is one of the greatest scientists from Japan, and he has also created a type of pendulum called the Furuta pendulum, which is an excellent setup for research purposes. So, now, they have given one kind of Lyapunov-based criteria such that you can be attracted to the sliding manifold. And same alternative form he has actually generated. It is not difficult to show that one can easily get this from here.

Furuta's Lyapunov-type Condition

$$\Delta V = s^2(k + 1) - s^2(k) = 2s(k)\Delta s(k) + \Delta s^2(k) < 0$$

Alternative Form

$$s(k)(s(k + 1) - s(k)) < -\frac{1}{2}(s(k + 1) - s(k))^2$$

Now there is another law called the Gao reaching law. If you see the literature, most of the time people are actually using this kind of reaching law. Why? Because this is very, very simple, and that is also the one that can be obtained by Euler discretization of the continuous reaching law algorithm. Why is this law called the reaching law? You can see that during the previous lectures, I was talking about three different kinds of teaching law. One reaching law that is called the constant rate reaching law.

And second, the reaching law that is constant plus proportional rate reaching law, and the third one is somehow a non-linear reaching law; here, alpha is going to lie between 0 and 1 . So, we have used these three reaching laws, and we are not actually appreciating this kind of reaching law in the case of the continuous time system because, using this kind of reaching law, it is not able to completely reject the mass uncertainty. Why? Because at $s = 0$, the right-hand side of this particular differential equation equals 0 , but we prefer these two. So, by discretizing the second law that is proportional plus a constant reaching law. So, this always gives some kind of constant gain, and for that reason, it is called the proportional plus constant reaching law.

And if you discretize using Euler discretization by taking the sampling equal to 1 , you can automatically see that this expression comes into the picture, where I have $s(k + 1)$, and τ is nothing but the sampling time. So, whatever sampling time you are going to take, in order to maintain discrete time stability, this extra condition comes into the picture. In the case of continuous, you can see that if there is no disturbance, then Q and K , any positive number you take, then obviously this law comes into the picture. And you are able to maintain $s = 0$ in all finite time $t \geq t_s$, but here in the discrete time case you cannot do that because the discrete system has inherent limitations, and for that reason, you have to

maintain this between 0 and 1 . It is not difficult to show why this is the case because suppose that this term will not be present.

So, how do we maintain $s(k + 1)$ in each subsequent step such that $s(k)$ is tending towards 0 ? So, for that whatever value here that is going to lie between 0 to 1 . So, based on that particular theory, this condition comes into the picture. So, what is the attribute of this particular law? Trajectory moves monotonically towards the switching plane; the switching plane is $s(k)$, and after they reach this switching plane, it is possible to show that this kind of action has started. So, basically, I have cross and recross actions started. So, basically, zigzag motion actually started along this particular plane.

Obviously, we have a non-increasing zig-zag amplitude, and the trajectory stays with a specified bend. It is possible to show that as time passes, the initial amplitude of this zig-zag motion is very high. Finally, that is going to become smaller, and it is always going to lie actually in the very close vicinity of this $s(k)$. Obviously, we cannot ensure that $s(k)$ is perfectly 0 , and due to that reason, some special band comes into the picture, and that kind of sliding mode control is called quasi sliding mode control. So, basically, the term "quasi" is used because there is no real sliding, no ideal sliding.

So, in the case of the discrete-time system, if we use the discretization of this law, the proportional and constant rate reaching law, then obviously we will get this law, which has a beautiful property: monotonicity. It means that we are going to monotonically decrease to the sliding surface. And obviously, I will finally converge to the invariant set. So, once I start from that set, I am not going to actually deviate from that set, and that is somehow found to form a bond that is called quasi sliding mode bond. So, I will show how a quasi-sliding mode bond will be obtained after actually implementing this kind of control that is developed by the discretization of the continuous one.

Reaching Law Definition

$$s(k + 1) = (1 - Q\tau)s(k) - K\tau \text{sgn}(s(k))$$

Obtained by Euler discretization of:

$$\dot{s} = -Q\dot{s}(t) - K\text{sgn}(s(t))$$

where τ is sampling interval, $Q, K > 0$, and $0 < (1 - Q\tau) < 1$. So, I am going to start with a discrete time system. So, these kinds of discrete time systems you can obtain in two different ways. So, some systems are inherently discrete in nature; then, obviously, you will get this model. But most of the system is continuous, so you can discretize it, and then you will get this model. And after that, you can design the sliding surface exactly the same as the continuous time system. And how to obtain the control? What can you do if you set $s(k + 1)$ equal to 0 , or $s(k + 1)$ if you calculate not equal to 0 ? So, now I have this kind of system that comes into the picture in the next instant, and I am going to substitute $s(k + 1)$ from the upper equation or from the discrete system equation, and then what is our goal? Our final goal is to match this equation with this equation, and during this matching, whatever extra term comes into the picture, I am going to interpret as a control action.

So, I hope that you can easily design control whenever some sliding surface is given to you. So, this becomes basically $\Phi_\tau x(k)$ and after that $\Gamma_\tau u(k)$, and finally, what do you have to do? Now you have to apply this kind of algorithm. So, $(1 - Q\tau)s(k)$, and after that, the next term is actually $K\tau \text{sgn}(s(k))$. Now, you can just equate, okay, and by equating, you can easily calculate the u , okay. Now, during this process, I am assuming that, by design, this should be invertible.

Discrete SMC Derivation

For system $x(k + 1) = \Phi_\tau x(k) + \Gamma_\tau u(k)$ and surface $s(k) = c^T x(k)$:

$$u(k) = -(c^T \Gamma_\tau)^{-1} [(c^T \Phi_\tau - c^T + Q\tau c^T)x(k) + K\tau \text{sgn}(s(k))]$$

Now, how do you actually calculate the bond? So we have already seen that if you start anywhere monotonically, I am going to decrease, and after that, I am going to cross and recross. So, $s(k + 1)$, and if you see the sign of the next, $s(k + 1)$, and after that I have $s(k)$ and then $s(k + 2)$. So, in order to maintain or to get some kind of band, cross and recross comes into the picture. So, if this side, because whenever we are designing a manifold, by designing a manifold and if $u(k)$ belongs to \mathbb{R} , if I have just one-dimensional control, then this whole plane is separated into two parts. So, one part is $k > 0$, another part is $k < 0$.

So, suppose that $s(k + 1)$ if you are in this part. So, next instant in order to maintain this kind of cross and recross, second part should be positive. So, I am assuming $s(k + 1)$ is positive; then the previous one $s(k)$ should be less than 0 . And obviously, what is our assumption? Our pre-assumption is that all subsequent steps should decrease so that I can maintain my position within the bond, and for that reason, we have used this kind of inequality. And if you rearrange this inequality, then you can approximate the bond like this.

And this bond is called a quasi sliding mode bond. So, whatever law I have discussed is based on the discretization of the continuous. Another one that is just based on the linear reaching law. So, what is the justification for this? Since we know that in the case of a discrete-time system, I never need to and I am never able to achieve the exact sliding; most of the time, I will get quasi-sliding due to that reason. Now, using this linear reaching law, where Θ is nothing but some kind of diagonal entry, all diagonal entries are positive. Then, it is going to lie between 0 and 1 because we are assuming that it is 0 and 1 .

You can easily see from the stability theory that if I have this kind of expression and if λ lies between 0 and 1 , then finally, whatever $s(k)$ comes into the picture is $\lambda^k s(0)$, and due to that reason, we are restricting whatever diagonal element such that this whole reaching law for the nth-order system or mthorder system is actually decoupled into m difference equations. Now, one of the reaching laws that is called Utkin's reaching law is a special case of the above reaching law, and what it suggests is that in one sampling instant you can directly converge here, and after that you can maintain. So, again, if $s(k + 1) = 0$, then obviously, in just one sampling instant, I can converge to the equilibrium point. But in this

particular process, huge control is required because in one step, you are very far from the equilibrium point, and your equilibrium point here is nothing but the whole sliding surface.

So, in one step, you have to actually reach the sliding surface. So, this is not an easy task. A huge amount of control is required in this particular process. And now again, several systems, whenever we are talking about discrete time systems and sliding mode, mean that I am talking about the disturbance. And disturbance we are assuming that is bounded. And again, I am able to design this sliding surface, and I can also do $s(k + 1)$, and then I will design the control.

And how to design control I have already told you; you can actually equate both sides, $s(k + 1)$ can be replaced by the reaching law that is proposed by Gao. And during that process, you can see that one disturbance term comes into the picture if you are going to equate. And, since it contains the disturbance term, we are not able to implement that kind of control. So, if disturbance-free, then you can easily implement the Gao reaching law, but if disturbance comes into the picture, then you cannot implement it due to the reason that $\tilde{d}(k)$ is not known to us.

Linear Time Invariant Model

$$\begin{aligned}x(k + 1) &= \Phi_\tau x(k) + \Gamma_\tau u(k) + \tilde{d}(k) \\y(k) &= Cx(k)\end{aligned}$$

where $\tilde{d}(k)$ is bounded disturbance.
Linear Sliding Surface

$$s(k) = c^T x(k)$$

designed for stable sliding mode dynamics.

Control based on Gao's Reaching Law

Without Uncertainty $\tilde{d}(k)$

$$\begin{aligned}s(k + 1) &= c^T \{ \Phi_\tau x(k) + \Gamma_\tau u(k) \} \\u(k) &= -(c^T \Gamma_\tau)^{-1} [(c^T \Phi_\tau - c^T + Q\tau c^T)x(k) + K\tau \text{sgn}(s(k))] \\(c^T \Gamma_\tau)^{-1} &\neq 0\end{aligned}$$

With Matched Uncertainty ($\tilde{d}(k) \neq 0$)

$$\begin{aligned}s(k + 1) &= c^T \{ \Phi_\tau x(k) + \Gamma_\tau u(k) + \tilde{d}(k) \} \\u(k) &= -(c^T \Gamma_\tau)^{-1} [(c^T \Phi_\tau - c^T + Q\tau c^T)x(k) + K\tau \text{sgn}(s(k))] - (c^T \Gamma_\tau)^{-1} c^T \tilde{d}(k)\end{aligned}$$

Control not computable due to unknown term. For that reason, we have to assume something. So, we are assuming that the disturbance matches, and after that, we are assuming that the disturbance is bounded, with d_0 as the

mean of the disturbance and d_1 as the spread of the disturbance. Then it is possible to show that if you modify the reaching law, you can see here this is the disturbance term, and this is justified because in continuous time systems, whatever discontinuous control we are proposing, we are always proposing it like this, and we are stating that in the presence of this unknown bounded disturbance, $s = 0$ in finite time. Similarly, whatever reaching law I have proposed here is going to contain some kind of unknown disturbance $d(k)$, and what is d_0 ? d_0 is the mean of the maximum and minimum defined exactly here. And in this way, I can ensure the convergence condition. And where $d(k)$ is nothing but $c^T \tilde{d}(k)$, c^T is the design matrix.

So, I have to design c^T in such a way that I can minimize this disturbance. So, based on the disturbance, you have to select some value of c_k . So, that is again a design problem. And now, you can see that inside the control, everything is actually known.

Assumptions

- Matched disturbance satisfies: $d_l \leq c^T \tilde{d}(k) \leq d_u$. - Mean and spread defined as:
 $d_0 = \frac{d_u + d_l}{2}$, $d_1 = \frac{d_u - d_l}{2}$.

Modified Reaching Law

$$s(k+1) = (1 - Q\tau)s(k) - (d_1 + K\tau)\text{sgn}(s(k)) + d(k) - d_0$$

where $d(k) = c^T \tilde{d}(k)$. Ensures:

$$s(k)(s(k+1) - s(k)) < 0$$

Control Input

$$u(k) = -(c^T \Gamma_\tau)^{-1} [(c^T \Phi_\tau - c^T + Q\tau c^T)x(k) + d_0 + (d_1 + K\tau)\text{sgn}(s(k))]$$

where $Q > 0, K > 0$ and $(1 - Q\tau) > 0$. d_0 is known because d_0 is the mean of disturbance. I am assuming that the upper bond and lower bond of the disturbance are known and d_1 is also known. And after that, in order to maintain quasi sliding mode control, I have already told you that we have to cross and recross. So, I have interval $s(k)$; after that, whatever $s(k+1)$. Their sign should be negative, and whatever $s(k+2)$ comes into the picture, that should be positive. So, suppose this is $s(k)$, then the second instant becomes $s(k+1)$, and the third instant becomes $s(k+2)$.

So, if this is positive, then this should be negative, then this should be positive again, and for that reason, this requirement comes into the picture in order to hold the quasi-sliding mode. And based on that, one can easily calculate the bond by proposing this kind of law where disturbance is also included. So, we have calculated $s(k+2)$. So, one can easily calculate $s(k+2)$ and after that substitute whatever term is inside $s(k+1)$, then we can get this kind of expression and we know what $s(k+1)$ is. So, from here, if you substitute

and maintain this $\text{sgn}(s(k+2)) = \text{sgn}(s(k))$, because what I have said is that whatever sign is here, the same sign is going to be maintained.

Quasi Sliding Mode Condition

Requirement:

$$\text{sgn}(s(k+2)) = -\text{sgn}(s(k+1)) = \text{sgn}(s(k))$$

Derivation:

$$\begin{aligned} s(k+2) &= (1-Q\tau)s(k+1) - (d_1 + K\tau)\text{sgn}(s(k+1)) + d(k+1) - d_0 \\ s(k+2) &= \text{sgn}(s(k))((1-Q\tau)^2|s(k)| + Q\tau K\tau + d_1 Q\tau) + (1+Q\tau)(d(k) - d_0) + d(k+1) - d_0 \end{aligned}$$

So, if you are going to substitute that, then I will finally get this kind of bond. So, now, modified quasi sliding mode bond can obtain like this. So please check it. So, whenever we are going to modify the reaching law, it is possible to show that in absence of the disturbance, I have some bond. Now, in the presence of the disturbance, I have this kind of bond that comes into the picture.

Final QSM Band Expression

$$\delta \leq \frac{K\tau + 2d_1}{(1-Q\tau)} = \frac{K\tau + \frac{Q\tau K\tau}{(1-Q\tau)}}{(1-Q\tau)} = \frac{K\tau}{(1-Q\tau)^2}$$

By substituting d_1 , I can finally show that this kind of bond comes into the picture. So, by designing τ very, very small, I can reduce this bond. And it is also not difficult to check that if this is converted into a continuous-time system with $\tau = 0$, then whatever bonds come into the picture are 0. So, this is more practical, maintains robustness properties and obviously, a true sliding mode control comes into the picture and inherent chattering occurs because during the proof, we applied the methodology such that cross and recross occur, and for that reason, I cannot stop the chattering. So, what is the origin of chattering? Because we have just discretized the continuous sliding mode control.

And control cannot change continuously because I always have some kind of sampling restriction, and inherent chattering comes into the picture. And due to that is a next approach that comes into picture that is called the without switching. And what is the main idea? You can keep $s(k+1) = 0$. With $s(k) \neq 0$.

Chattering Origin

$$\begin{aligned} \dot{s} &= -Qs(t) - K\text{sgn}(s(t)) \\ &\Downarrow \\ s(k+1) &= (1-Q\tau)s(k) - K\tau\text{sgn}(s(k)) \end{aligned}$$

- $\text{sgn}(s(k))$ changes abruptly near $s(k) = 0$ - Control cannot change continuously - Results in inherent chattering

So, that kind of concept is actually driven by deadbeat control. So, what is the basic idea of deadbeat control? So, you can directly converge to the origin in one step. So, similarly now we are not going to converge to the equilibrium point. What are we basically going to do? We are going to converge to a manifold, and after that, we will subsequently move to the equilibrium point. So, in that particular way, the deadbeat controller is different from the sliding mode controller.

Now, what have I done again? I have taken the LTI system. So, I have taken a continuous LTI system, and after that, we performed the discretization. So, this is the equivalence between continuous and discrete time systems, and after that, you can see that we have designed the control law, which is very easy. What do you have to do? You just have to keep $s(k+1) = 0$, and then whatever control comes into the picture, you have to calculate. It is possible to show that once you apply this kind of control. So, this particular control, since we have to reach in one step, and due to that reason, most of time, actuator saturation comes into picture.

System Representation

Discrete-time LTI system:

$$x(k+1) = A^*x(k) + B^*u(k) + D^*r(k)$$

where:

$$A^* = e^{A\Delta t}, B^* = \int_0^{\Delta t} e^{A(\Delta t-t)} B d\tau, D^* = \int_0^{\Delta t} e^{A(\Delta t-t)} D d\tau$$

Control Law

$$\begin{aligned} s(k+1) &= CA^*x(k) + CB^*u(k) + CD^*r(k) = 0 \\ u(k) &= -(CB^*)^{-1}(CA^*x(k) + CD^*r(k)) \end{aligned}$$

Control Decomposition

$$\begin{aligned} u_{eq}(k) &= -(CB^*)^{-1}s(k) - (CB^*)^{-1}((CA^* - C)x(k) + CD^*r(k)) \\ s(k+1) &= s(k) + (CA^* - C)x(k) + CB^*u(k) + CD^*r(k) \end{aligned}$$

And due to that reason, now it is possible to show that one can able to modify the control. So, that modification is suggested by Professor Utkin, that if your, this is the saturation limit of your actuator. So, if the equivalent control is less than your actuator saturation limit, then you can apply this. Otherwise, you can just give the direction.

So, this is nothing but unit-vector-based control. So, this is just going to give the direction, and every time I am going to utilize the maximum saturation limit of the actuator. And obviously, we have this; then I have to apply this control, and I have to show that the norm of $s(k+1)$ is less than $s(k)$. So, those are the kinds of things we have to show. So, this is the convergence condition. And if we are able to maintain this kind of convergence condition, then it is possible to show that $\|s(k+1)\| < \|s(k)\|$.

Implementation Challenge Potential for excessive control in one step Solution

$$u(k) = \begin{cases} u_{eq}(k) & \text{if } \|u_{eq}(k)\| \leq u_0 \\ u_0 \frac{u_{eq}(k)}{\|u_{eq}(k)\|} & \text{if } \|u_{eq}(k)\| > u_0 \end{cases}$$

Ensures $|s(k+1)| < |s(k)|$.

Control Analysis

For $\|u_{eq}(k)\| \geq u_0$:

$$s(k+1) = s(k) + (CA^* - C)x(k) + CD^*r(k) - [s(k) + (CA^* - C)x(k) + CD^*r(k)] \frac{u_0(k)}{\|u_{eq}(k)\|}$$

Norm Analysis

$$\|s(k+1)\| = \left\| s(k) + (CA^* - C)x(k) + CD^*r(k) - [s(k) + (CA^* - C)x(k) + CD^*r(k)] \frac{u_0(k)}{\|u_{eq}(k)\|} \right\|$$

where:

$$\|u_{eq}(k)\| = \left\| -(CB^*)^{-1}s(k) - (CB^*)^{-1}((CA^* - C)x(k) + CD^*r(k)) \right\|$$

Norm Inequality Analysis

$$\|s(k+1)\| \leq \|s(k)\| + \|(CA^* - C)x(k) + CD^*r(k)\| - \frac{u_0(k)}{\|(CB^*)^{-1}\|}$$

Convergence Condition

$$\begin{aligned} \|(CB^*)^{-1}\| \|(CA^* - C)x(k) + CD^*r(k)\| &< u_0(k) \\ \Rightarrow \|s(k+1)\| &< \|s(k)\| \end{aligned}$$

So, this is sarpturk reaching law. If I have a disturbance, then it is possible to show that I am assuming that the disturbance is a mesh disturbance, and in the framework of the mesh disturbance, the system is written like this: this is the sliding variable, and now whenever you are going to maintain $s(k+1) = 0$, then obviously again the control is going to contain some kind of uncertainty, and due to that reason, again this control is not implementable; and due to that reason, we have to modify it. And after the modification of the reaching law, it is possible to show that I can again maintain the modified reaching law where d_0 and d_1 are the spread and mean. And then finally, one can able to propose this kind of law based on the disturbance. So, here $s(k+1)$ is not exactly 0, but we are in the

vicinity of the quasi-sliding; again, things are exactly like the quasi-sliding mode, and we are going to convert to the vicinity of the sliding surface, and control can be given like this.

System Representation

$$x(k + 1) = \Phi_\tau x(k) + \Gamma_\tau \omega(k) + D_\tau f(k)$$

Matching condition:

$$D_\tau = \Gamma_\tau \tilde{D}_\tau$$

State equation becomes:

$$x(k + 1) = \Phi_\tau x(k) + \Gamma_\tau (\omega(k) + \tilde{D}_\tau f(k))$$

Sliding function:

$$s(k) = c^\top x(k)$$

Control Law Development

Using reaching law $s(k + 1) = 0$:

$$u(k) = -(C^\top \Gamma_\tau)^{-1} [C^\top \Phi_\tau x(k)] - \tilde{D}_\tau f(k)$$

Not feasible due to uncertain term
Modified Control Law

$$u(k) = -(C^\top \Gamma_\tau)^{-1} [C^\top \Phi_\tau x(k)] - \tilde{D}_c(k)$$

The last reaching law proposed by Professor Bartoszewicz is somehow adaptive in nature. So, they have actually added some extra term $s_d(k)$, and in order to design $s_d(k)$, you have to satisfy this kind of condition. So, in this particular slide, I am not going to give the proof, but you will be able to check the paper that is retained by Bartoszewicz. And after that, it is possible to show that s should satisfy this kind of condition. And we have to maintain $s_d(k)$ condition such that $s_d(k) - s_d(k - 1)$ and the next sampling instant k is always greater than 0 .

So, it is possible to show that $s_d(k)$ actually converges monotonically. And each step that is going to decay by this, and if this is less than to decay, then it is exactly equivalent to Utkin's law. So, again, quasi sliding mode control comes into the picture. By using this approach, you can obtain the quasi sliding mode band, and after that, it is possible to show that control can be achieved in this manner. Additionally, $s_d(k)$ is given in this way, and $s(k)$ is always less than d_1 ; this is the plot of that.

So, now it is time to conclude this lecture. So, what have we seen? We have seen a comprehensive analysis of discrete sliding mode control. We have actually developed several modified reaching laws for robustness purposes because our main goal is to compensate for uncertainty. So, exact compensation for uncertainty cannot be achieved in

the discrete case. Why? Because control is only available at a discrete instant. We have also demonstrated the effect of chattering reduction by adjusting the value of the gain as well as τ .

And how can one basically reduce the chattering? By replacing the discretized control with Utkin's control in a continuous way or without any switching function, I can reach the sliding manifold. So, these kinds of things are very suitable for industrial applications, and obviously, by using this approach, one can also minimize the control effort, which will provide some flexibility in the design process. So, with this remark, I am going to end this lecture. Thank you very much.