Stability Notions for Dicrete Time Systems

Welcome back. In the previous modules, I talked about continuous-time systems and how to design sliding mode control for these systems. So, in this module, I am going to discuss discrete time systems and how one can design discrete sliding mode control. So, obviously, whenever we are going to talk about discrete sliding mode control, one of the crucial things is the stability of the discrete-time system, and once you apply the idea of sliding mode, most of the time our system, even if it is linear, is converted into non-linear, and due to that reason, Some rigorous mathematical background is required to handle the nonlinear discrete-time system. So, in this first module, I am going to discuss the stability concept of discrete-time systems in the first lecture of this module. And then from now on, I am going to deal with the discrete sliding mode control.

So, the purpose is obvious: I am going to deal with the rigorous stability notion of the discrete-time system because the type of system is going to change here, and for that reason, I first have to deal with the stability theory again. And obviously, we are trying to understand the gap between practical implementation as well as the theoretical formulation or foundation. And we are also going to develop the tools for finite-time convergence in the case of the discrete-time system because we are going to deal with sliding mode control. And in sliding mode control, whenever we discuss it in a discrete framework, suppose that the first state is $z_{1,k}$and the second one is $z_{2,k}$.

Again, the idea is exactly the same. We are designing some kind of discrete manifold, and the meaning of discrete is that we have something only defined at discrete points. What is the basic idea? In finite time, I have to converge to this manifold, and after that, in the subsequent instant, we have to maintain along this particular manifold. So, again, in order to converge an infinite number of sequences, I need the concept of finite time stability. So, I am also going to discuss that notion in this particular lecture.

So, what is the key focus? I am going to characterize the stability. So, all notions of stability, like Lyapunov stability, asymptotic stability, and geometrical stability, come into the picture whenever we are dealing with discrete time systems that I am going to discuss, and after that, we are also going to discuss finite time stability. One of the main important issues is that whenever we talk about sliding board control, we are obviously implementing sliding mode control to do more; that is, our plan is to reject meshed uncertainty or minimize unmatched uncertainty, and for that reason, some kind of stability notion in terms of ISS, input to state stability, is also required. So, we are going to extend the notion of stability whenever we have some kind of bounded disturbance, and obviously, the Lyapunov tool is very efficient whether it is the case of linear systems, non-linear systems, discrete systems, hybrid systems, or stochastic systems; everywhere and due to this reason, we are trying to understand the Lyapunov formulation for discrete-time non-linear systems. Whenever we are designing a controller, obviously some kind of criteria that one can verify algebraically is required.

So, using this notion, we are trying to understand. So, I am going to discuss several practical or numerical examples in this particular lecture. So, whenever we are dealing with the non-linear discrete-time system, exactly the same as the continuous non-linear system, some kind of function frequently appears, which is called the comparison function. What is the basic idea, either that of continuous time or discrete time? Whenever you have a higher order system, by using some mechanism, we are somehow converting it into a first order system. But why are we doing that? Suppose that if I have a higher-order system, then I do not know what their convergence behavior is, but suppose that if you have a higher-order system and if you convert it somehow into a one-dimensional system, then things are very easy, because in one dimension, if you are on this real line, one dimension means nothing but some kind of real line, where it might be possible to have continuous points in the case of a continuous-time system; in a discrete-time system, I have discrete points.

So, if you are here, you know that if I have some kind of converging behavior and if this is the equilibrium point, then I will move in this direction. So, you have unique direction from here that only in order to converge here, you can just follow this line. In this way, I can either follow or leave here. So, whenever I have an n-dimensional problem, we are trying to convert it into a one-dimensional problem. And somehow, if you look carefully at the Comparison function or Lyapunov function, their idea is basically based on this particular philosophy.

So, whenever you are going to convert, obviously most of the time you are not getting equality. What does it mean? That whatever equivalent system that comes into picture. That does not come in the form of the differential equation; it will come in the form of a differential inequality. And due to that reason, once the differential inequality comes into the picture, there are several functions, or several kinds of compression functions, that come into the picture such that I can convert the differential inequality into a differential equation. After that, whenever we are formulating some kind of function and suppose that I have to convert to 0, the left-hand side is unnecessary, and for that reason, we have to show that whatever transformation function I am going to use is well defined and bounded.

So, in order to understand these kinds of things, some completion function theory is required. So, another important aspect of compression theory is that whenever you are dealing with some kind of autonomous system and whenever you are moving from an autonomous to a non-autonomous system, it is possible to show that now time explicitly appears. So, now the solution trajectory is actually a function of time. So, somehow uniformity with respect to initial conditions is lost, but still there exist several classes of non-autonomous systems; the meaning of a non-autonomous system is nothing but some kind of system where time is also present and the state is also there. So, whenever you are going to solve this class of systems, it is possible to show that the solution most of the time depends on the initial time.

Now, there exists a class of non-autonomous systems where the solution is uniform with respect to initial time, and at that time it is possible to show that the capturing of that function requires some special treatment, and in that particular treatment, completion functions are very helpful. So, first I am going to introduce the notion of the completion function, and this is applicable for both continuous and discrete cases. So, the first completion function is called the class $\mathcal{K}$function. So, what is the meaning of class $\mathcal{K}$function? We are talking about some kind of function $\alpha(\cdot)$, and whatever argument I am going to give here is again positive. So, in this positive domain of this argument, that domain is 0, and $\alpha(0) = 0$; we know this is 0.

And now, I have a domain from 0 to a, and again that is going to map from some kind of positive side only. So, I have some kind of function $\alpha(\cdot)$, with a domain from 0to a, which is going to map to some kind of positive function. And what is the property of that? That is continuous and strictly increasing. So, if that is the case, we are going to say that kind of function is a class $\mathcal{K}$function. Now, if you increase this, I have taken a single variable.

So, one can assume that $s$is. Now, $s$is defined between 0and ∞. So, now, as $s \to \infty$and if this function is also tending towards ∞, then that is called a $\mathcal{K}_\infty$class function. And if you look carefully, whatever Lyapunov function we are constructing is going to satisfy this particular property. So, for that reason, I have told you that this class of functions is very useful.

It is also possible to show that if you take some kind of positive symmetric definite matrix V, then most of the time that is bounded by two $\mathcal{K}$-class functions. So, $\alpha_1$and $\alpha_2$and x. So, this kind of result you can easily see if you explore the Nonlinear Systems, which is one of the very standard books. So, if you are interested in non-linear systems, even if linear systems, that book is very helpful, and $\mathcal{K}_\infty$is required whenever we are talking about global stability. Time system in order to talk about whether if you start anywhere such that you will converge to the equilibrium point at that time, this $\alpha_1$and $\alpha_2$initially that is class $\mathcal{K}$, now that is converting to class $\mathcal{K}_\infty$.

Similarly, the reverse of class $\mathcal{K}$is another class called class $\mathcal{L}$function. Again, you can see the domain and range, so the same kind of treatment applies here for domain and range as with a class $\mathcal{K}$ function, and what is the property? Now, this is continuous and strictly decreasing, and as $s \to \infty$, this function $\mathcal{L}$ is also equal to 0. Now, actually, this class of functions is very, very interesting; it is the combination of class $\mathcal{K}$functions and class $\mathcal{L}$ functions. So, class $\mathcal{KL}$ is a function that now has two arguments. And with respect to the first argument, this is a class $\mathcal{K}$function, and with respect to the second argument, that is a class $\mathcal{L}$function.

And in one of the examples, you can easily see that as $s$is going to increase, this function is going to increase, provided $t$is constant. Now, you can keep $s$constant and increase t; then what happens? This is decreased to 0, and for that reason, it is called the $\mathcal{KL}$class function. If you have any system like $\dot{x} = Ax(t)$, obviously this is a continuous-time system, and if you solve it, it is not difficult to show that exactly the same class of functions comes into the picture. Here, you can see that if you, this is not a dot because I am solving this. So, basically, if you take the norm here, now automatically x_{t_0}, the norm of this and the exponential, which is always positive, and if you take the norm, then what you can do is take the smallest eigenvalue from here, $\lambda(t - t_0)$, like this.

So, that is always bounded by this. So, now, this is nothing but a kind of class $\mathcal{KL}$function. Here, this is class $\| x(t_0) \|$ that you can just replace with $\| x_{t_0} \|$, and e^t, you can use $e^{\lambda(t-t_0)}$. So, in this way, this class of functions frequently appears in the case of linear systems as well as non-linear systems. This is a very, very important class of functions that is called the generalized $\mathcal{K}$-class function.

So, what we have seen in the case of the class $\mathcal{K}$function is that the class $\mathcal{K}$function is always increasing, but the generalized $\mathcal{K}$-class function does not have such a property. So, basically, for some reason, the strictly increasing property is okay. So, obviously that is somehow a generalization of that, but here you can see that once $\alpha(s_1) > 0$. So, this relation will satisfy, but when $\alpha = 0$in that region, this will satisfy this kind of property. I will, when I take one example, easily help you understand how the basic class $\mathcal{K}$function and the generalized $\mathcal{K}$-class function are different, and these classes of functions are very helpful whenever we deal with finite-time stability.

Whenever a dead zone comes into the picture, it is very important, and this is somehow handled by a function known as strictly Apano. I am going to prove that if a function is defined like this and its domain is from $0$to ∞, then it will represent a generalized $\mathcal{KL}$class function. So, if you look carefully, this is nothing but something like this. So, $\alpha(s)$is continuous everywhere. Now, $\alpha(0)$, if you take it, then I have to first check the mean maxima between $0$and -1.

So, obviously, that is 0. So, that is 0. So, that will follow the properties of the previous definition. This property is also being followed. Now, since $s > 1$, you can see here that $s > 1$; this is actually strictly increasing. Now, $\alpha = 0$; this is 0.

When you put $s$between $0$and 1, then obviously, the maximum is always 0, and there s^2 is 0. Due to that reason, this becomes a constant function, and you can see that these two properties are somehow different from the class. So, what we have seen is that in this particular domain, it is somehow showing non-decreasing behavior. So, basically, whatever I have written is continuous and strictly increasing; that is only true when this condition is not satisfied. So, here if you look carefully, everywhere that is greater than 0means increasing, but here it is non-decreasing.

What is the meaning of “non-decreasing”? It means that it will not decrease, but also not strictly increase; that is the meaning. So, some correction in the definition here: it is strictly increasing only whenever there is some kind of domain. Suppose that I have a domain from

0to ∞that I am talking about. So, some domain that is not strictly increasing, nor strictly decreasing. So, in that particular way, this generalized $\mathcal{KL}$ class function is different from the $\mathcal{KL}$ class function.

An important function that is a generalized $\mathcal{KL}$class function. I have already told you that the $\mathcal{KL}$class function has several advantages because the solution of several linear and non-linear systems is somehow related to the $\mathcal{KL}$class function. So, even if I have a non-autonomous system, one can show that some class of solutions of the non-autonomous system is going to belong to this class of $\mathcal{KL}$functions. Here, $\mathcal{KL}$class functions have important applications whenever we are dealing with finite-time stability. So, what is the main motivation behind finite-time stability? So, we have to maintain a quantity equal to 0.

And after that becomes 0, we have to maintain that throughout the time interval, and for that reason, it is possible to show that the $\mathcal{KL}$class function is not enough. Hong has actually shown that there exists some kind of class function ϕ, called the generalized $\mathcal{KL}$class function, and what the property of this class function is. So, if you fix t, then with respect to x, that is a generalized $\mathcal{K}$class function, and once you fix s, this is nothing but it is possible to show that as $t$tends towards some finite time t, not infinity, then that equals 0, and after that, that is going to maintain, and due to that t-sum, this kind of function is very useful whenever we are characterizing finite-time stability, and obviously, in hybrid systems also and discontinuous-time systems also, this class of function comes into the picture, and these are the properties that are taken from the Khalil book. So, you can see that if you have some kind of class $\mathcal{K}$function like $\alpha_1$and α_2, and $\mathcal{K}_\infty$functions that are $\alpha_3$and α_4, and $\mathcal{KL}$class function β, then they will satisfy several properties, and these kinds of properties, whenever you are going to manipulate the system, are very useful. This $\alpha^{-1}$is defined over $[0, \alpha(a)]$

So, again, that is a class $\mathcal{K}$function, α_3, because α_1, if you see, then that is a class $\mathcal{K}$function. So, inverse also that is going to satisfy the exactly same property. So, similar kind of property. So, this property of the composition also comes into the picture. A picture means that the composition of two class $\mathcal{K}$functions is still a class $\mathcal{K}$function.

The composition of two $\mathcal{K}_\infty$class functions is still a $\mathcal{K}_\infty$class function. And these properties are very useful whenever you talk about Lyapunov stability. So, now let us come to the dynamical system. So, a discrete-time non-linear dynamical system is represented by $z_{k+1} = f(z_k)$, and here now $z$again belongs to subset $\mathcal{D}$, and $\mathcal{D} \subset \mathbb{R}^n$, and now $f: \mathcal{D} \to \mathcal{D}$. Here, how is that basically different from the continuous-time system? If you see a continuous-time system, it is something like

So, you can see here that if $x(t)$ is the position vector, then somehow we are talking about the rate of change of the position vector on the left-hand side of the differential equation. So, physical interpretation is totally different. If $x(t)$ is position, then at that time $\dot{x}(t)$ is velocity. But if you see here for a discrete-time system, then the physical formulation is very much consistent. It means that if z_k is the position vector, then somehow we are assuming that z_{k+1} is the position vector at the next instant I am talking about.

So, that is going to maintain their physical interpretation. So, in that particular way, somehow these two systems are different. Now there are several examples. So, this example is nothing but a linear system. A non-linear system means non-linearity comes into the picture.

Now, similar to continuous-time systems, one can characterize the notion of stability of discrete-time systems by solving this. So, one of the important stabilities that is called Lyapunov stability can also be extended to the discrete-time system. So, the definition is exactly the same as the continuous. So, whenever I have this kind of system, the interpretation is exactly the same as the continuous one that I am going to give the initial condition z_0, and after that, I am going to generate the next instant z_k somehow. So, I have to solve this particular equation, and after the solution, what do I have to do now? I have to select some kind of ball that is related to the output side.

And after that, corresponding to that ball, there exists some kind of ball of initial condition that depends on the ϵ. And due to that reason, for every ϵ, there exists a δsuch that if you start from the δ-ball, you will remain inside the ϵ-ball. So, this is $z_k$and this is the z_0. So, exactly same way we can able to define this is the notion of norm. And obviously, you can see that this is nothing but Lyapunov's stable.

So now, it is your task to prove that this is Lyapunov's table. So, what is the idea? The solution idea is very simple. You can substitute $z_k = 0$.

Then, you will get something like 0.5 z_0. Now, you can substitute the next instance. So, that becomes 0.5, and z_1, you know.

So, basically, this becomes $0.5^2 z_0$. So, in this way, basically, if you come to z_k, then $z_k$is the solution, which is nothing but $(0.5)^k z_0$. Since this lies between 0and 1, you can easily able to show that this is going to converge. Where is that going to converge? So, as $k \rightarrow \infty$, it is going to converge to 0. Similarly, you are able to show here.

Obviously, the coefficient here is 1. So, you are not going to converge to 0, but you are always going to remain bounded, and somehow boundedness is coming from the definition. Now, another stability concept is asymptotic stability, and here, apart from boundedness, I am also going to add convergence. So, similar to the continuous-time system, we are saying that if this $k$is tending towards infinity, then $z_k = 0$. And somehow this is asymptotically stable. Why is it asymptotically stable? We have already seen that as $k \rightarrow \infty$, that is tending towards 0, and it is also possible to show that this is locally asymptotically stable.

So, how do we prove this is locally asymptotically stable? Again, I have to prove that $z \neq 0$; there exists some kind of finite region where I can do the epsilon–delta formulation such that if I start from the ball, I am going to remain in the delta ball, and the proof is very, very easy. Since I am assuming $z_k \in \mathbb{R}$, you can take the absolute value, and it is possible to show that if you bound this side. So, if I maintain $1 - z_k^2 \leq 1$, then obviously this compression system looks like some kind of differential; now we have a difference inequality, and it is possible to show that as $k \rightarrow \infty$this equals 0, and here $\lambda \in (0,1)$, then only this is true, and in this way I can tell that from this I can make some kind of condition on that.

So, this is always $\sqrt{\frac{1}{2}}$. So, if $z$lies in this domain, then only I can be able to talk about this kind of convergence, and for that reason, this particular difference equation is only locally asymptotically stable, and the next is called geometrically stable. So, in the case of a continuous-time system, we are talking about exponential stability; here we are talking about geometric stability, and in this case, there exists a βsuch that it will follow this. So, now it is your turn to check if these two are geometrically stable. Now, there is another very good idea called finite-time stability. So, a solution is called finite-time stable if there exists a neighborhood $\mathcal{N}$and a settling-time function.

So, similar to the case of continuous time, we are talking about the existence of a time that is a function of the initial condition. So, here I am saying that there exists some kind of function called the settling function such that as the next instant comes into the picture. So, at every subsequent instant, I can maintain $z_k = 0$, and again this should be Lyapunov's table. So, it is possible to show that if $z_{k+1} = 0$. So, in all next subsequent steps that is going to maintain that equal to 0.

So, it is very easy to prove. So, you can see that if I put $k = 0$, then $z_1 = 0$. So, in this way, $z_2$is also equal to 0, and $z_k$is also equal to 0. So, in one step that becomes 0, and after that, it is going to be maintained. Similarly, it is possible to show that this is nothing but finite-time stable.

So, you can see how to prove that. The function is defined like this. So, $x > 1$, which is like the signum function, and when that lies between $-1$and 1, then it looks like the linear function. So, as $z_0 > 1$, what happens is that if this is greater than 1, the whole value means that, within one step, I can lie in this particular region. Now, once I am in this region, we know the definition; the definition looks like this. So, basically, our system now looks like this, and it is possible to show that I can converge to 0at most in this number of steps, and here this function is called the ceiling function.

So, the ceiling function means that if you have 0.5, then that is equal to 1. So, obviously a finite number of integers is required to show the time of convergence in the case of the discrete-time system. And this is the Lyapunov theory that supposes there exists a Lyapunov function, and if the Lyapunov function satisfies this kind of condition. And here some kind of operator, the minimum operator, comes into the picture. It is possible to show that if this kind of condition is satisfied, I will discuss this again whenever I come to the third lecture of this particular module.

At that time, I was talking about sliding mode control based on the minima. And this equation will actually play a very important role. And what is the basic idea? You can see here that $L$is some kind of constant. So, $\Delta V_k$is nothing but $V_{k+1} - V_k$; that is, the ΔV_k, and this is the minimum of $V_k$and L.

So, you can start with any initial condition and select the minima. So, initially, if the minimum is actually L, then in L, I will slowly decay, and it might be possible that I am starting with 5.8. The initial condition is that each step $L = 1$.

So, the next step I will come to is 4.7, 3.7, and finally, 0.7comes into the picture, and at that time, $V_k = 0.7$. If you compare the minima, then this is 1, and this is 0.7.

So, that has become $0.7 - 0.7$. In this way, in finite time I can converge to the equilibrium point. And, the time of convergence can also be given again by the ceiling function. So, this is an example. Similarly, it is possible to show that in the case of a continuous-time system, I have two different kinds of algorithms: $\dot{x} = -k$ sign (x). So, this is nothing but their variant, and another algorithm we have seen; then that is nothing but $\dot{x} = -| x |^{\alpha}$ sign (x), where $\alpha = \frac{1}{2}$, or you can also easily take αhere.

So, that variant in the discrete-time system is given by this, and the time of convergence is given like this. And now, this is very very important class, where disturbance also comes into picture. So, here nu is represent the disturbance. And, that disturbance we are assuming that is bounded. So, this external in case of external disturbance and again our equilibrium point is 0 0.

So, at equilibrium, suppose the physical interpretation is that at the equilibrium point we do not have any disturbance; it is possible to show that this class of system is input to a stable state. So, we are going to show some examples and I am also going to provide a definition. So, what is the meaning of finite time input to a state stable? So, we are saying that some system has finite input to a stable state. So, in absence of disturbance, so gamma gamma function whatever we have taken.

So, $v = 0$. So, in the absence of disturbance, this is class $\mathcal{KL}$function. What is the property of the generalized class $\mathcal{KL}$ function? Because the $\mathcal{KL}$class function is not sufficient, I need a generalized $\mathcal{KL}$ class function, because in a finite number of steps I have to converge, and after that, we have to maintain it. And after that, this function is actually a $\mathcal{K}_{\infty}$class function. So, in this way, basically whenever uncertainty comes into the picture, one can show that the finite-time controller is robust in the presence of external uncertainty and disturbance. So, I have taken one example.

So, up to here we have already proved that it is finite-time stable. Now, I have added some kind of bounded noise, and in the presence of bounded noise, it is possible to show that once $v_k = 0$, then obviously there is finite-time convergence; but when $v_k \neq 0$, due to the behavior of the saturation function, it will actually converge to $-v_k + v_k$. If $v_k$is not present, then I will converge from $-1$to 1. Now, using the Lyapunov function, obviously this Lyapunov function is a $\mathcal{K}_\infty$class function, and in worst-case analysis here, due to Taylor's expansion, this term comes into the picture, and it is possible to show this now. By the analysis, this is obviously the ceiling function, and if I have a minimum number of steps that is given by this, obviously, this is a function of the magnitude of the disturbance. We are assuming that the disturbance is bounded by some magnitude that is the infinity norm or some norm you can take.

At that time, it is possible to show how the differential equation is converted to $z_k - z_0$ and $1 - v_k$, due to the consequence of this particular equation. And now, if you further analyze it, then it is possible to show that there exists some kind of time; then I will always remain bounded. In this way, you can show the finite-time input-to-a-stable-state. And again, one can also be able to extend the same kind of concept, even if that satisfies this kind of condition.

I have already discussed here that there are two different kinds of formulation whenever we are talking about finite time stability. So, the first formulation is coming from here. So, that is the minimum operator. Again, the second formulation is coming due to the variant of this. So, with the help of that, we are again able to define the finite time input to state stable.

And in this way, we can establish that a finite time stable system, discrete finite, discrete finite time stable system is robust with respect to a certain class of perturbation. Now, it is time to conclude this lecture. So, I have discussed finite time stability, and after that, we have also seen how one can maintain finite time stability in the case of a disturbance. And obviously, input to the state stability concept comes into the picture in that particular case. And we have established the Lyapunov condition for finite-time stability, and we have taken several non-linear systems to demonstrate the example of the finite-time stable linear or non-linear system.

So, basically, we have unified the approach of the continuous-time system for the discrete time, and we have also seen the ISS whenever the input-to-state stability concept comes into the picture. So, somehow that is equivalent to a linear system that we know, and if the linear system is stable, then a bounded input will give us a bounded output. In the case of the non-linear system, there are only some classes of non-linear systems. We have a very limited class of non-linear systems, continuous or discrete.

If we apply the bounded input, then I will get a bounded output. So, that class of system is called an input-to-state stable system. So, if our system satisfies that kind of property, then we can easily design sliding mode control, and it is possible to show that in finite time I will converge to some manifold, and after that, we can maintain it. And obviously, this will provide the robustness tool and this kind of theory is very useful whenever we are talking about the non smooth class of system. So, with this primary background in the next class, I am going to talk about discrete sliding mode control. Thank you very much.