Newton's Inspired Tuning of ON OFF Controller - 1

So, welcome back. So far, we have understood the limitations of proportional control, proportional-integral control, and proportional-integral-derivative control. I have also discussed the high gain feedback and what we have observed: that high gain feedback is able to compensate for any kind of uncertainties and disturbances. In today's lecture, I am going to discuss on-off control, and it is possible to show that on-off control is nothing but a class of sliding mode control provided you switch with extremely high speed. So, we are trying to understand the sliding mode control in the context of the on-off control where the switching frequency is extremely high. So, we are first trying to understand the tuning process of on-off control; after that, we will observe the disturbance.

It is possible to show that if you have a first-order system and if you apply on-off control based solely on the information of the present, then you will achieve finite time stability. What does it mean by finite-time stability? You can start anywhere in the one-dimensional line; you can be able to converge to the equilibrium point in finite time. In spite of any kind of uncertainty as well as the disturbances. And you will be able to maintain the equilibrium point once you achieve it.

What does it mean? That switching term, the equivalent value of the switching term, is exactly going to compensate for the disturbance; then only am I able to say that $v = 0$ and the average rate of change of $\dot{v}$ is also equal to 0. Why I am talking in terms of v is because in the previous class we used a very simple system. So, we have an underwater vehicle, and we are moving the underwater vehicle in a straight line, and we have to control the velocity. I am going to take the same system again in this class because our main intention is to understand the tuning process as well as the behavior of on-off control in the presence of uncertainty and disturbances, one of the very fundamental concepts known as the Comparison Lemma. And we are trying to understand how to understand the Lyapunov stability with the help of the Comparison Lemma as well as the Newtonian framework.

So, if you have already studied non-linear control, you might have seen that the proof of Lyapunov stability involves a lot of things, but here, from the intuition point of view, you can easily understand the framework in terms of Newton. So, I am going to explain the Lyapunov stability framework in terms of Newton and what our outcome is. So, you are able to understand the insight into the disturbance observation property. Now, this observation property is very, very crucial because you can not only compensate for the disturbance with the help of on-off control, but you can also understand what class of disturbance it is. What exact form of disturbance is hitting your system, provided the disturbance will enter through the control channel? So, that behavior is also modeled as a fault.

So, if you have some system running in a well-behaved fashion and suppose some attack, I am assuming that your system is actually on the network. So, using this philosophy, you are able to understand what kind of fault is affecting your system. So, in finite time, you can be

able to estimate that fault. And finally, you can clearly understand the connection between Lyapunov stability and Newton's framework. the today's discussion.

So, what is the main intention of today's discussion? I have exactly the same system again. So, I have a first-order system $-v \mid v \mid$. So, this is a drag term. We have thruster forces and disturbances. And we have designed u again as some kind of feedforward control as well as feedback.

It means that $\bar{u} = v \mid v \mid + u(v)$, exactly the same as in the previous lecture, and $u(v)$, which I am interpreting as feedback, and in this first-order system, I am assuming that I have a sensor that is able to estimate this velocity, and I am going to utilize this information. But here you can see that I have defined the on-off control not exactly in the classical way. So, $v > 0$, which is $+1$, is okay; $v < 0$, which is -1, is also okay. But you can see that $v = 0$; I am assuming that it can take any value between -1 and 1. It means that at $v = 0$, I have infinitely many differential equations, and this is called the differential inclusion.

And we are going to understand the property of the physical interpretation of $v = 0$, where the sign(v) lies between -1 and 1. So, again, I am going to keep the framework exactly the same as in the previous classes. Since I have a first-order system, I have actually drawn a real line, and you can see here that v is positive. The $\dot{v}$ calculation I have done with the help of this particular equation. Initially, you can assume there is no disturbance..

So, $d(t) = 0$. So, the sign of v is positive. So, $\dot{v} = -k$. Even if you have a disturbance, what you can do is keep the value of your gain greater than the value of the disturbance. Again, you will be able to maintain $\dot{v}$ negative.

It means that our trajectory will move in this particular way. A similar kind of conclusion: here, $v < 0$; therefore, the sign(v) is negative, so a negative times a negative becomes positive. So, if this is positive, either $d(t)$ is negative or positive, $\dot{v}$ remains positive and we can move in this direction. Here, we have one more extra advantage comes into picture. What kind of extra advantages? When we explored the proportional control at that time, we were basically using this action, where control is the scaled value of the present information.

But as we move towards the equilibrium point, what happens to the rate of change of the velocity? So, here velocity is what I am assuming as some kind of quantity. Like quantity. Suppose that you have to control the temperature control system and $\dot{t} = kt$. So, again you can assume this temperature control system t as a position-like vector and $\dot{t}$ as a velocity-like vector. So, everywhere whenever I have some kind of mathematical equation and if that is expressed in terms of a state space.

So, without any loss of generality, I am assuming x as a position vector and $\dot{x}$ as a velocity vector. So, I am using the same kind of terminology everywhere in order to apply Newton's

framework. So, you can see here that in this case, what happens? That as we are moving closer, and suppose for the time being, we don't have any disturbance. So, the rate of change is constant. What does it mean? Suppose you want to move from town A to town B.

This is town A and town B. If your velocity, or rate of change, is constant, you can expect that in finite time, even if the rate of change takes any value, you will be able to reach town B. The same kind of observation applies here. So, what kind of extra things come into the picture here besides proportional control? In finite time, I can steer the trajectory from anywhere in this one-dimensional space to the equilibrium point. Now, not only am I able to steer, I am also able to maintain it.

So, let us see the simulations. Simulation will give you the correct ideas. So, I have taken $d(t) = \sin t$. Thus, I am applying a sinusoidal disturbance and assuming that the value of gain is exactly equal to 1 because k should be greater than or equal to the disturbance. So, I have taken $d = \max(\sin t) = 1$; for that reason, I have selected $k = 1$.

You can see that the initial condition I have taken is 1. So, in finite time, I can move here, and once $v = 0$, you can see that our controller starts switching with a very, very high frequency, and due to that reason, the plot looks like this. Now, if you pass this control through some kind of low pass filter, you can see that you can estimate the disturbance exactly. And this is called the disturbance observation property, and for that reason, because of the kind of claim I am making, if you are here, then you can maintain your position here in spite of the disturbance. What does it mean? It means that rate of change.

somehow equal to 0 here. But whatever the construction of this function, you can see here that $v = 0$; this is not equal to 0; it can contain any value between -1 and 1. If you define $v = 0$, it means that at the origin v does not equal 0 due to the disturbance. For that reason, I have to define it like this. What is the physical interpretation of this? So, if you have learned or explored physics, you have already seen that we have basically two different classes of velocity. So, this velocity is called either average velocity or instantaneous velocity.

So, in this case, instantaneous velocity is known to be zero. But we are not moving, so the average velocity is equal to 0. So, since the average velocity is equal to 0, for that reason we cannot move, and if I take the average of this component, then I will get the exact value of the disturbance. So, how do you get that? So, the same kind of things I have written here. We have two different kinds of velocity, and I have to maintain $v = 0$ in spite of the disturbance.

That is our aim. But in reality, what happens is that the switching function has some imperfections. So, exactly I cannot able to maintain here and due to that reason some bandwidth come into picture, suppose that is δ. Now, every plant behave like a low pass filter. One can actually assume that whenever we are at $v = 0$, then the control component has two different kinds of components. One is a high-frequency component, and one is a

low-frequency component.

And since every plant behaves like a low pass filter, the average value is exactly like the low pass filter characteristic. So, mathematically, you can design a low-pass filter. So, this is the equation of a low-pass filter. Here, tau is the time constant of the low-pass filter, and z is the output. So, you can design a low-pass filter, a low-pass filter here.

You can apply the same input that you have applied to the first-order plant, and you can measure it here. You are able to show that the value of z is exactly equal to the distance. So, this is the filtering characteristic, and here whatever time constant of the filter that you have to maintain should be as small as possible. If that is small, then you can exactly estimate the disturbance.

another characteristic. So, one characteristic of the on-off control to estimate the uncertainty is that we are also able to understand that in the presence of any class of bounded disturbance, just what I have to maintain? I have to maintain the gain of the controller so that it is greater than the value of the disturbance. Mathematically, we are writing the supremum of $\dot{d}(t)$. It means that you can, if you have information on the largest value of the disturbance, even if you do not know $\dot{d}(t)$, you can be able to exactly compensate with the help of the on-off control, provided we have a first-order system. For second order and third order, we are going to see in the next lecture what kind of conclusion one can draw. Now, another characteristic we will get is finite time stability, and this is a very, very crucial characteristic of the sliding mode controller.

So, we are trying to understand how to calculate the time of convergence. We have already understood in a geometrical way that since $v > 0$ and $\dot{v} < 0$, and the rate of change is somehow some kind of gain k, even if I have a disturbance, then $k - d(t)$ kind of things come into the picture. So, some value of finite gain causes me to converge here in finite time. Now, I am trying to calculate the time. So, in order to calculate the time of convergence, I am again going to use Newton's framework.

I have told you that in this module, using Newton's law, you can understand everything, okay. So, how do you calculate the time of convergence? Suppose that I start from here. So you can just calculate the distance. So, distance: how do you calculate the distance in one dimension? You can apply the idea of absolute value, that is | x |. And again, you can apply Newton's philosophy: $d > 0$ and the rate of change of distance along the system; here, I am assuming initially that there is no disturbance, which should be less than 0.

What does it mean? Since d is positive, the lower limit is 0, and I have to maintain $\dot{d} < 0$, so I can move in this direction. And if this motion happens in finite time, then I am saying that $\dot{v} = u$ is finite-time stable. u is nothing but an on-off controller. So, you can see that this kind of distance has some limitations because it is not differentiable at $v = 0$. And due to that

reason, I am going to maintain the same property because at $v = 0$, distance is equal to 0.

So, the same property is positive. And now, this particular way of defining distance is differentiable almost everywhere, or it is everywhere. This is almost everywhere because it is not differentiable at $v = 0$, but it is differentiable at each and every point. $\dot{d}$ you can calculate. Now, you can see that $\dot{d} < 0$. After that, what I am going to do, in place of $\dot{v}$, is substitute terms in terms of $\dot{d}$, and you can see here that $\dot{v}$ is nothing but $2d$ times the square root of v, equal to $\sqrt{2d}$.

So, the same kind of substitution I have done here, and after that, now you can apply the variable separation method you have already learned during your 10 plus 2 preparation. And after that, you can see here that the time of convergence is finite. Why is the time of convergence finite? k is finite, d is the distance, and the initial condition is supposed here; then you can calculate this distance, so that is also finite. One more important thing you have to understand here is that at $t = 0$ this is d_0, and due to that reason, I have done the integral from 0, and I am assuming that in infinite time I will reach here due to the geometrical interpretation. Due to that reason, at time t I am at 0, and after that, I am going to calculate the time of convergence.

So, in this way, without knowing the Lyapunov analysis or anything, just with the help of the distance and applying Newton's law, you can calculate the time of convergence. But here, once disturbance comes into the picture, you can see that in the presence of disturbance, I have an extra term that is $d(t)$. So, $d(t)$ $\dot{v}$ comes into the picture. Now, if you apply the norm inequality or absolute value inequality, you will get this kind of term.

Now, this is no longer a differential equation. This is a differential inequality. So, we do not know how to solve differential inequalities. And due to that reason, we take the help of this lemma called the Comparison Lemma. So, what is the basic theme of the Comparison Lemma? It is possible to show that if we have to solve some kind of differential inequality, I can equivalently solve the differential equation and be able to comment on the differential inequality. So, those are the kinds of things we are going to see in subsequent slides.

And in this way, in the presence of disturbance, you can also see that the time of convergence remains finite. Now, this is the Comparison Lemma. And almost everywhere, whenever you are going to talk about stability, you have to apply the Comparison Lemma. And due to that reason, I am commenting here; the Comparison Lemma is a pathway to Lyapunov stability because, in Lyapunov stability, you might have seen that I am selecting some kind of energy-like function that is always positive. This energy-like function should be 0 at 0, and after that, $\dot{V} < 0$.

So, again, differential inequality comes into the picture. Sometimes we are expecting $\dot{v} \leq 0$, again differential inequality. Sometimes we expect $\dot{v} \leq \alpha v$. Sometimes you can see that I am talking about $\dot{v} \leq \alpha v^{\beta}$, where β lies between 0 and 1.

So, we have differential inequality everywhere. So, whenever we have a differential inequality, we need to know how to solve it, and this lemma is actually providing the way. You can see that during Lyapunov analysis, what we are basically doing is converting an n-dimensional system into one dimension with the help of some energy-like function. So, due to that reason, the comparison lemma results. If we are just developing for a scalar system, that is enough, and due to that reason, in this slide, I have taken the scalar system like this, and this is the initial condition. You can see here that I am talking about these two guys that are continuous in t and locally Lipschitz in u.

Why these two guys? Because not every mathematical differential equation represents some kind of reality, as several differential equations do not have a unique solution. It might be possible that they have multiple solutions and they have no solution. But if you are going to restrict these two assumptions, then it is possible to show that this equation has a unique solution. What is the meaning of the maximal interval of the existence of a solution? We are assuming the solution of this particular differential equation since I have the local Lipschitz condition.

So, locally I can assume I have a solution. So, I am assuming that I have a solution up to this particular time interval t, and you can also extend it to infinity. So, for that, you have to do extra analysis. Now, suppose I have this kind of differential equation and differential inequality; you can see that this differential equation and this differential inequality are exactly the same. The only difference is that here I have $u(t)$, here I have $v(t)$, and this kind of inequality I have. So, it is possible to show that I have always had this kind of relationship.

And using this relationship, you can comment on the differential inequality. It means that even if I have a differential inequality, I can solve it and comment on the solution. So, you can see here that this is actually a non-linear system. Now, the solution of a non-linear system is very challenging.

So, you can try to convert this into a linear-like system. So, what have I done? I have defined one $v(t) = x^2(t)$, and again $\dot{v}(t)$ I am going to define, and you can see that this is nothing but if you remove this inequality. So, I have some kind of linear equation, and I know how to solve linear equations. So, you can apply the Comparison Lemma, and in this way, without solving the non-linear equation, just by solving this linear equation, you can understand the bounds of the solution of the non-linear equation. Most of the time, an exact solution is not required for control engineering or analysis purposes. Due to that reason, it is better to avoid the exact solution because I do not know the exact model of any system.

For that reason, it is better to avoid the exact solution. So, if I know how to do an approximate solution, that is enough. So, in order to apply the completion lemma, I need two conditions: the local lift-cheatness condition and the requirement that a solution exists in some time interval. So, how do we do that? So, this is the definition of local Lipschitzness.

So, if you want to learn more about this, it is good to explore the book on non-linear control systems written by Khalil. So, effects is locally Lipschitz provided they will satisfy this kind of condition.

Okay. And in the previous slide, I have taken this system. So before applying the Comparison Lemma, I have to show that this function is locally Lipschitz. So, how can we prove that? You can see that if this function is differentiable, then it is very easy to calculate this inequality. What will I do? First, I will differentiate, and then I will integrate. And in this way, I can show that L is given like this, provided the interval lies between $-m$ and m.

And you can also show the maximum interval of the existence of the solution. Since this is a first-order equation, you can draw a straight line and apply Newton's philosophy. Here I have $x > 0$, and $\dot{x} < 0$. So, all trajectories will converge towards the equilibrium point.

And similar kinds of things here, $x < 0$, but $\dot{x} > 0$. So I can move toward the equilibrium point. In order to apply the Comparison Lemma, we need two different kinds of conditions. So, the first condition is the existence of the solution. So, we have already seen in the previous class and previous slide that the local Lipschitz condition is satisfied, provided we will see the solution in this particular compact interval.

One more important thing one can establish is the asymptotic behavior. You can see that this is a first-order system, and one trajectory of a one-dimensional system can be expressed geometrically using this straight line because I want to apply the idea of Newton to show that solution exist for any initial condition. So, how do we show that? You can take $x > 0$, and $\dot{x}$ you can see here as $-(1 + x^2)x$. So, $\dot{x} < 0$ and $x > 0$; due to that reason, I am able to show that all trajectories will converge towards the equilibrium point. Similarly, if you start on the left-hand side of this real line, then $x < 0$; what about $\dot{x}$? That is positive, so we can move in this direction.

So you can start in any compact interval. What is the meaning of interval? The subset of the real line shows exactly similar behavior. So, both conditions are satisfied, and for that reason, I am able to apply the compression lemma. So, before applying the compression lemma, you have to be very, very careful. Now, I have already told you that you can apply the compression lemma not only for the first-order system. Any order system you are able to apply, and you can comment on the stability; why one is able to comment on the stability by seeing the solution.

So, I have actually defined energy as a function. You can see here that $v > 0$ for all values of x_1 and x_2, and $v = 0$ only when $x_1 = 0$ and $x_2 = 0$. Now, in order to apply Newton's philosophy, what do I have to do? I have to calculate the rate of change of $\dot{v}$, and after that, I am going to substitute from this differential equation, and then I will get this. Now, I have to manipulate this. So, you can see here that I have one term that is

$$\frac{x^2}{(1+x^2)^2}.$$

It is possible to show that that is upper bounded by 1/2. So, you can just calculate the maxima of this and you will be able to confirm this inequality. So, after substituting this, I will get it. Since I have a second-order system, I can now calculate the norm in various different ways. So, one of the ways to calculate a norm is called the one norm.

So, one norm is defined like this. How do you calculate the 2-norm? So, the 2-norm is defined as $\sqrt{x_1^2 + x_2^2}$. It is possible to show that in the finite-dimensional case, the 1-norm and the 2-norm are always related by this. So, these kinds of things you can check from any standard mathematics book. And due to that reason, I have substituted the x^2. Why? Because you can see here that V is given by the 2-norm, and for that reason, whenever the 1-norm appears, I am going to convert it into the 2-norm, and then I will get this kind of inequality.

Now, I have to solve this equation. So, this equation is no longer a linear equation. Why? Because I have this kind of term. For that reason, I am going to do another transformation. So, I have defined $w = \sqrt{v}$. After that, I calculated $\dot{w}$, and you can see that I am perfectly landing with some kind of linear differential inequality.

But what kind of difficulty is it? Since during this transformation I have used a square root of v that is not differentiable at $v = 0$, I also cannot leave this point. $v = 0$. So, for that, we are applying the idea of the Dini derivative. This is called the Dini derivative. So, you can see that the definition of the Dini derivative is exactly like the derivative, but here I am just talking about 0^+.

And after that, since I know that $t + h$, this is $w(t)$. So, when $v = 0$, I have to substitute. So, basically this value is equal to 0, and due to that reason I have this kind of relationship. And I know the differential equation, and for that reason, you can substitute the right-hand side of the differential equation and apply $x = 0$ because $v = x_1^2 + x_2^2$. So, $v = 0$ means $x_1 = 0$ and $x_2 = 0$, both equal to 0.

And for that reason, this term is equivalent to $\dot{v}(t_0)$. And due to that reason, you can see that I have this expression; I have added it, and I have subtracted it. So, $t + h - t$, this term is almost constant. So, $t + h - t = h$, and h is going to cancel out, so this term cancels out. Now, you can see that I have this kind of term. So, you can apply the definition of continuity, and then it is possible to show that at $v = 0$, this derivative is also equal to 0.

So, what is the meaning of continuous? The continuous function I have already talked about states that if some function f is continuous, then you can do the zooming. So, $t - \tau$ is the

dependent variable and this is the independent variable. So, if a function is continuous, then for every ε there exists a δ, and you can apply this limit as $h \to 0$, and in this way, you can establish that the Dini derivative, the upper Dini derivative, is equal to 0. So, everywhere now this particular inequality will be satisfied, and due to that reason, I now have an equivalent differential equation like this.

So, I can apply the completion lemma and I will get the solution. So, in this way, even if you have second order, third order, or fourth order, you will be able to solve it, but you have to manipulate a lot of inequalities. So, now it is time to understand the connection between the Newtonian framework, the completion lemma, and Lyapunov stability. So, what is the meaning of Lyapunov stability? So, one way to establish the stability, asymptotic stability, and exponential stability is to solve the differential equation. And using solution, you can able to comment about the behavior of the system.

Another way is to map everything into one dimension. So, how do you map? You can apply some kind of idea of energy; energy is a scalar quantity. So, I am assuming that I have this system, and most of the time, the non-linear model is not valid everywhere. So, we are assuming that the non-linear model is valid here, and after that, I have applied this kind of transformation. And now you can see here that I have designed the transformation in such a way that $v = 0$. It means that the equilibrium point of $\dot{x} = vf(x)$ and v; these two are going to coincide.

And after that, what am I going to do? I am going to maintain this. And since $v = 0$, I already know this is equal to 0. So, due to that reason, I have to maintain $v(x) > 0$ for all known values of $x \neq 0$. And then I will apply $\dot{v} \leq 0$. What is the physical interpretation? In terms of Newton, I have $v(x)$; this is positive.

So, now I just have to maintain $\dot{v} < 0$. And then I will apply the inequality framework whenever it comes into the picture; I will apply the Comparison Lemma, and then I will be able to show that either system is stable, asymptotically stable, or exponentially stable. What is the meaning of asymptotic stability? You can see that $\dot{v}(x)$ should be strictly less than 0. It means that everywhere our velocity and direction of velocity should be in this way. If $\dot{v} \leq 0$, it means that it might be possible that I stop converging, and due to that, there is instability, but still, I am very close to the equilibrium point. What is the meaning of exponential stability? You can see that I have a linear-like differential inequality.

So, this is equivalent to some kind of differential equation like $\dot{v} \leq -\alpha v$. And if you solve it, you can get exponential convergence. And due to that reason, the name exponential stability comes into the picture. So, actually, Lyapunov stability can also be understood in the framework of Newtonian. What do you have to do? You have to apply the transformation.

You have to convert your system to one dimension if a differential inequality comes into the picture. Then you have to apply the comparison lemma to solve that differential inequality.

If a differential equation comes, then that is not a problem. You can solve it directly and conclude. What is the conclusion of this particular lecture? So, we have understood the framework of on-off control, particularly for first order system.

We have understood why on-off control is able to compensate for any class of disturbance. We have also been able to understand the completion lemma and what the relation is between the completion lemma and Lyapunov stability. Stability is one of the key factors. So, using the completion lemma, you can do it. You are able to explore the disturbance observation property because fault-tolerant control or fault detection is one of the very exciting areas of research.

You are able to apply this kind of idea to that particular class of the system. So, with this remark, I'm going to end this part of the lecture. Thank you very much.