Adaptive Sliding Mode Control

Welcome back. In the previous class, I was talking about higher sliding mode control and what our motivation was for higher sliding mode over classical sliding mode because we have observed that due to the discontinuous control during physical implementation, a chattering kind of phenomenon comes into the picture and what our observation was that. There are two reasons for the chattering. One is discontinuous control, and another is the magnitude of the discontinuous control. So, in previous lectures, we have seen how to make control continuous based on higher-order sliding mode control. In this lecture, I am going to tell you how to minimize the gains of classical sliding mode control or how to optimize the gains of classical sliding mode control.

That will give you the new idea that is somehow the combination of two existing ideas called adaptive control and sliding mode control, and for that reason, I am going to deal with adaptive sliding mode control. So, for the purpose of discussion, we have already seen that conventional sliding mode control, and obviously, during the conventional sliding mode control, we have talked about several limitations of the conventional sliding mode control. And after that, obviously, one of the main restrictions is that in order to mitigate the uncertainty, it means that you can suppose a first-order system

$$\dot{s}(t) = -k\,\mathrm{sgn}\,(s) + d(t).$$

So, now in order to maintain $s(t) = 0$in some finite time $t \geq T$, I have to make sure that

$$k > \sup_{t}\ |\, d(t)\,|.$$

For all t, we have to maintain this, and for that reason, in order to implement this kind of robust control, we are assuming that prior knowledge of the uncertainty bound is required. Now, suppose that if uncertainty fluctuates initially, whenever we start a system, then uncertainty is very, very high. After some time, uncertainty is going to become minimal or less. So, if I somehow adapt this gain $k(t)$, then it is possible to show that initially this gain is high if uncertainty is high, but once the uncertainty bound is lower, then this gain will change to a lower value.

And obviously, during chattering analysis, we have found that the value of the chattering magnitude is directly proportional to this gain. So, in this way, we can basically minimize the chattering as well as the control effort, and we can also minimize it because k is directly proportional to the control effort. So, how are we basically able to solve about two problems? So, for that, we are going to talk about the fusion between two different non-linear control techniques. One is adaptive control, and the adaptive control literature is already well developed; after that, we are going to add the idea of sliding mode control. So, in order to address the problem of classical sliding mode control, we are going to discuss mainly two approaches.

So, one approach is to gain increasing adaptation. So, for this particular approach, there is no need for the bound of the knowledge. Another one is the gain optimization, and basically, here we are going to deploy some kind of equivalent control. It is possible to show that in this particular approach, an upper bound of gain is again required. After that, what are we going to do? We are going to fuse these two methods, and after that, we are going to propose a new method.

Apart from that, we are also going to talk about the real sliding so that I can avoid the above three methods. So, let us start with the non-linear uncertain system, and basically, this is a non-linear uncertain system because f is also uncertain and g is also uncertain. And what is our assumption? We are assuming that we have just a single-input single-output system. It means that $y = h(x)$; I have, or you can also assume that y is nothing but some kind of sliding variable $s(t, x)$, and x may also be time dependent; that does not matter. After that, what is our aim now? Suppose that if the relative degree of this system is 1, what is the meaning of relative degree? If I differentiate y, then control will explicitly appear, just like this.

And since $\gamma(t, x)$ or $\gamma(x, t)$ is nothing but these two terms, and obviously, here $f(x)$ is uncertain, due to that reason this whole term is uncertain, and a similar kind of thing happens with η because here I am assuming that g is also uncertain. Now, these are the assumptions. We are assuming that whatever this term is, it is bounded, and after that, this term is also bounded, and I have already given you the physical interpretation of this bound. Here, you can see that $\gamma(x, t)$ contains any value between $-\gamma_m$ and $+\gamma_m$, but here I do not want our control to change the sign, and for that reason, we are assuming that the minimum bound of the matrix through which control is going to enter is also bounded. It is better to keep this sign here, then that is physically more relevant; or if this is a vector, then the absolute value can be put here, but since we are assuming that the relative degree is 1, due to that reason, it simply means the norm is replaced by the absolute value.

And we are assuming that this unknown will exist even if it is not known, but some unknown bond does exist. Why is this kind of restriction required? Because, finally, I want to design the discontinuous control in the Filippov sense. And whenever we are going to talk about the Filippov solution, the right-hand side of the differential inclusion should also be bounded. So, in order to maintain existence and uniqueness, these kind of this kind of restriction is required. And obviously, our control objective is to design sliding mode control based on global stability without the knowledge of the uncertainty.

So, here in this process, we are going to utilize the concept of adaptive sliding mode control. So, let us first see some definitions. So, suppose I have a nonlinear uncertainty system and I have to maintain some sliding variable whose relative degree equal to 1 equal to 0 in finite time. So, at that time, we are assuming that whatever set comes into the picture is a locally integrable set in the Filippov sense. What is the physical interpretation of this? Because you can see that whenever we are calculating the derivative and the right-hand side, I somehow have a differential inclusion.

So, how do you track σ? Because somehow by controlling $\dot{\sigma}$, you can control σ. So, a solution should exist, and for that, this condition is enough to guarantee that we have a unique solution, even if we do the Filippov regularization in case of the discontinuity. And obviously, ideal sliding mode cannot be achieved in real applications because we have several imperfections during the modeling as well as the implementation. And due to that reason, most of the time we are happy with the real sliding mode control. So, how is the real sliding mode different from the ideal sliding mode control? So, by designing discontinuous control for ideal sliding mode control, we want to maintain $\sigma = 0$.

This is not feasible, and for that reason, we will move to the real sliding, which basically allows some kind of tolerance bound. So, basically, the behavior of the system on this particular set, which is defined as $x \in \mathcal{X}$, and here, basically, in this particular system, I am assuming that since this is a nonlinear system, the state space is limited. And due to that reason, we have this kind of assumption that whatever state is going to belong to some set $\mathcal{X}$. So, based on that, we have constructed this

$$\mathcal{S}_\delta = \{x \in \mathcal{X} : |\ \sigma(x)\ | \leq \delta\}.$$

And now, we are saying that here the real sliding mode comes into the picture.

It means that in this particular set, we are going to define integrability in the sense of Filippov. And obviously, real sliding mode accounts for the practical imperfection and ensures that the system remains close to the sliding surface. And most of the practical application is enough. So, this definition is basically given by or separated from ideal sliding mode by the professor Levant. Now, in this lecture, I am going to talk about the dynamic gain adaptation.

So, for that, we are going to talk about two approaches. So, one approach is given by Huang, and what is the beauty of this approach? This does not require knowledge of the uncertainty bound. And obviously, we are slowly increasing the control until the sliding mode is established. And the second approach is basically based on the knowledge of the uncertainty bound, and here we are going to utilize the concept of equivalent control to minimize the control gain. So, what is the meaning of equivalent control? We have already seen that whenever we have first-order sliding mode control, $\sigma = 0$, but $\dot{\sigma} \neq 0$, and the average value of $\sigma$is equal to 0.

So, by taking the average of this, it is possible to show that the discontinuous term, whatever comes into the picture, is exactly equal to the value of the disturbance. So, in this way, basically, we are designing the equivalent control, or in another way, $\dot{\sigma} = 0$, and whatever control that comes into the picture, we are interpreting that that much control needs to actually establish the sliding mode. So, both approaches are basically responsible for adapting the gain dynamically, and after that we can be able to see that the closed-loop system is also globally stable. So, what is the first approach I am going to take? I am going to start with exactly the same system. So, let us come back to the system.

Here, I now have a first-order system because we are talking about first-order sliding mode control. So, now I am going to design this gain. So, you can see here that this gain is designed as $K(t)$, and $K(t)$can be treated here as an extra state. Why? Because K is now time-dependent and is also varying with respect to some differential equation. So, either you can put $t$here or not; that does not matter.

And here, α_K is some kind of constant, and the initial condition $K(0) > 0$ is assumed. So, by solving this equation, you can see that the rate of change of Kdepends on the absolute value of σ. So, now, if $\sigma \neq 0$, $\alpha_K$is always positive. So, $K(t)$ is going to increase. And what is the result? If you apply this kind of sliding mode control, then I have to prove that there exists a finite time $t_F$such that $\sigma = 0$, and after that we can maintain $\sigma = 0$throughout the next time interval, beyond t_F.

So, this approach does not require any prior knowledge of the uncertainty. So, you can see here I am not assuming that $K(t)$and the uncertainty are somehow related. And after that, one of the difficulties here is that sometimes overestimation of control comes into the picture, and this leads to an increase in chattering. So, what is one of our main motivations to decrease the control gain? Obviously, the second motivation: design some kind of sliding mode control where uncertainty-bound knowledge is not required.

Uncertainty is not known whenever we are considering this sliding mode control, but we are assuming its bond is known. So, at least we will get one advantage whenever we apply this approach: the first advantage is that an uncertainty bond is not required, but again sometimes overestimation of gain comes into the picture, and this will increase the chattering because chattering is directly proportional to the Gain of the sliding mode controller. And obviously, this is applicable only to the ideal sliding mode control. And in several imperfections, we have real sliding mode control. So, we are going to see that how to modify this such that this is also applicable for the real sliding mode control in next subsequent slides.

So, now, what is the key point? The approach dynamically increases the control gain and increases the control gain, but it may overestimate and cause chattering; first, I have to prove this. So, this is nothing but a theorem that if I substitute this kind of control into this system where $\gamma$is, and here I have—let us come back to see that term—here I have $\eta$and u. So, basically, time and the state I am going to drop here.

So, $\eta$and u. So, now, I have to design u, and I am going to design it as

$$u = -K(t)\,\mathrm{sgn}\,(\sigma).$$

And I am not assuming any bounds on $\gamma$and ηin this particular process. So, I have to prove it. So, what have we seen? I have two states here: one state is σ, and $\dot{K}$is also involved, and for that reason, one additional state comes into the picture. And for that reason, the

Lyapunov function now contains two pieces of information. One important thing here is that the sliding mode comes into the picture provided this gain $K(t)$becomes sufficiently large.

to compensate the disturbance. And due to that reason, we are assuming that at the equilibrium state or a steady state, this $K(t)$should converge to some kind of $K^\star$. And this $K^\star$is ideal for starting the sliding; that is our main intention. And due to that reason, we have shifted $K(t)$by $-K^\star$. So, now, if you see the equilibrium point, then that is

$$\sigma = 0 \text{ and } K(t) = K^\star.$$

What am I going to do? I am going to take the rate of change, and after that, I am going to substitute $\dot{K}$. So, once you substitute $\dot{K}$and the value of $\dot{K}$, we know how to basically adopt. So, I am going to apply the adaptation law, then I will get this kind of expression. And after that, I am going to manipulate this expression. So, once you manipulate this expression, it is possible to show that since the assumption is that $\gamma(x, t)$is not known, we are assuming that some bound exists here.

Similarly, $\eta$is not known, but some bound exists here. So, during the proof, we have to show that we have to utilize this kind of concept to show that there exists at least some $K^\star$. So, this is just for the existence of $K^\star$. So, please do not be confused, because we are not utilizing this concept directly; we are just involving it in improving the process.

We are somehow guaranteeing that if you are going to adopt a gain like this and the control is this, then this Lyapunov function is always decreasing. It is possible to show here that if you maintain $K^\star$, it means that $K(t)$is going to increase, and once it crosses some threshold that satisfies this equation, then obviously, $\dot{V} \leq 0$, and Lyapunov stability comes into the picture. So, if $\dot{V} < 0$, then $\sigma \to 0$as $t \to \infty$, and after that, the adaptation process will also occur because $K(t)$is going to increase up to $K^\star$. So, using the Lyapunov process, I can be able to give a guarantee of the gain adaptation process. Now, this is the second approach, and this second approach comes from the solution that is defined in terms of Utkin's regularization or Utkin's equivalent control.

So, what is the philosophy of Utkin's equivalent control? That once sliding is established, it is possible to show that the average value of $\dot{\sigma}$is equal to 0. And how do you calculate the average? You can pass that signal term through a low-pass filter. So, what I am going to do here is define $K(t)$, and obviously, $K(t)$is again defined based on a differential equation. You can see here that $\rho$and $\dot{\rho}$are coming from the low-pass filter. So, somehow dynamics is again involved here, but this gain $\beta > 0$.

So, we have to select $\beta > 0$such that I can somehow establish the sliding. And due to that reason, once this condition is satisfied, automatically the sliding mode comes into the picture. Otherwise, I cannot be able to pass this through the low-pass filter, and at that time, I will not be able to get the value of ρ. And what is the result? Again, I am going to show that there exists a finite time such that the sliding mode can be established. So, this requires knowledge of the uncertainty bound.

Obviously, this approach reduces the chattering by minimizing the control effort, but one extra difficulty that comes into the picture is that transient phenomena arise, because the tuning of the low-pass filter is challenging, as the value of τshould be very, very small. So, due to the dynamics of the filter, some transient response comes into the picture when sliding is about to start. So, basically, why this control, with this kind of equivalent control, is responsible for the decrement of the Lyapunov function that I am going to prove. It means that the overall system is stable. So, again I am going to start with exactly the same Lyapunov function.

I am going to take the derivative of the Lyapunov function. The dynamics of the sliding mode, and after that $K(t)$ sgn (σ), I am going to substitute. In the adaptation process, you can see here that what I am going to do, since $\dot{K}$comes in and here $\dot{\rho}$is in the filter equation, and due to that reason I am able to take the derivative. And β, which is constant; due to that reason, the derivative of βequals 0, and due to that reason, this term comes into the picture. But in a steady state, if you see carefully, then what happens is that the value of $\rho$is exactly equal to sgn (σ). And due to that reason, we are assuming that during the sliding mode, when it comes into the picture, $\rho =$ sgn (σ), and at that time, the derivative of this is equal to $\dot{\rho} = 0$.

So, once sliding is established, then I have to prove that the stability of the whole system comes into the picture. So, it is not difficult to prove again that if I select

$$K^{\star} > \frac{\gamma(x,t)}{\eta_m},$$

then one can show here that $\dot{V} < 0$,

which will ensure stability once the sliding process starts. So, $V < 0$, $\sigma \to 0$, and the adaptation law ensures that $K(t)$ adjusts and contracts the uncertainty. So, basically, using the Lyapunov function, I am going to ensure two things: finite-time convergence and the adaptation law, which we have proposed to contract the uncertainty and also maintain overall stability. So now, during this particular process, you can see that transient phenomena occur due to the interaction between the control law and the low-pass filter, because the control law is designed based on the low-pass filter.

So, we have to consider filter dynamics, control gain adaptation, and system response, and due to that, somehow the transient is affected, and now we have to tune the filter dynamics in a proper way. Another difficulty with filter dynamics is that delay is also always associated. So, delay always causes the output to lag behind the actual switching signal, and the transient may not be immediately reflected in the system, which will cause either an increase in $K(t)$ or some kind of small temporary loss of the sliding. So, $K(t)$ is basically depending on ρ, and $\rho$is not in a steady state. So, $K(t)$ may be tuned properly; otherwise, some overestimation or underestimation of the gain basically comes into the picture, and somehow the oscillation phenomenon comes into the picture again.

So, chattering is somehow not reduced. So, the transient response of the system is influenced by the filter time constant. And if you are going to take τvery, very small, then this will reduce the time delay but may cause high-frequency noise to pass through. Similarly, if $\tau$is slower, then what happens at that time is that the signal is smoothed, but the transient duration increases. So, in this way, we have to make some kind of trade-off between τ; the selection of $\tau$is very, very important whenever we are trying to implement this kind of adaptive sliding mode to a practical system, and for that reason, some trade-off is required. So, what are the challenges of these two methods that we have discussed so far? So, in the first method, the overestimation of the gain comes into the picture.

In the second method, the uncertainty bound is required. And obviously, transient phenomena come into the picture due to the deployment of a low pass filter. So, what is our next objective? We are trying to solve these two problems. So, the first problem, obviously, is that we do not want the uncertainty bound, and I also want to avoid the overestimation of the control gain. We want some kind of reasonable magnitude of control so that I can also maintain the chattering level as low as possible. So, what are we basically going to do? We are going to combine the two methods mentioned above, and one approach comes into the picture until the sliding mode is established; once the sliding mode is established, it is then shifted to Utkin's approach, an equivalent control-based approach.

So, the next solution I am going to aim for is to avoid the overestimation of the gain, and obviously, I am going to eliminate the need for the uncertainty bound. So, the need for an uncertainty bound is a requirement of approach 2, and the first approach is actually dangerous due to overestimation of the control. For that reason, you can see that once $\sigma \neq 0$, we are going to utilize the first approach, and once $\sigma = 0$, then we shift to the low-pass-filter-based approach proposed by Professor Utkin. And we are trying to show that if you design this kind of control, then I can achieve stability in finite time, and after that, I will continue it. So, what is the discussion? The discussion is that $K(t)$ is responsible for the sliding.

So, $\sigma \neq 0$. So, now $K(t)$ will increase, and obviously, $\sigma = 0$comes into the picture. And once $\sigma = 0$comes into the picture, then at that time this gain becomes active. So, a similar kind of thing is written here. And after that, suppose in between, if sliding is destroyed, then again this condition comes into the picture, and again the gain is going to increase. So, in this way, I can optimize the gain or adapt the gain such that I can maintain the sliding. So, the first approach uses the concept of equivalent control, and equivalent control basically introduces low-pass filter dynamics.

So, here they are also going to introduce some kind of low-pass filter dynamics. So, now, what are we going to do? We are going to propose a different approach called the second approach that does not estimate the boundary of perturbation and uncertainty but guarantees real sliding mode control. So, several practical system just we want real sliding. It means that our sliding variable should remain in some kind of neighborhood.

So, suppose that this is a sliding variable. So, I just want that $\sigma$is maintained near 0. So, for that, now you can see that here a different control is proposed. So, as $k > \mu$, that is, the gain is greater than μ. So, at that time, here a small parameter ε comes into the picture. So, at that time, the gain can be given like this. And once $K < \mu$, at that time, $\mu$is shifted to some kind of constant gain.

So, you can see that if $K = \mu$, at that time I have some, and $\mu$is nothing but some kind of constant. So, as $K < \mu$, at that time I have some kind of constant differential equation that comes into the picture, and here I have a state-dependent differential equation. So, we are going to show now that the actual real sliding mode comes into the picture in this particular process. Obviously, we do not have any ideal sliding mode control in this particular process. So, once sliding is established, we have this condition where the gain has actually declined and depends on the smallest level to ensure the accuracy of the σ stabilization.

What is the advantage? It is possible to show that in this process, we can adjust the gain and actually have no need for the estimation of the bond of the uncertainty. But what is a trade-off? Here we are just going to establish the real sliding. So, we have to prove that real sliding mode comes into the picture and that the epsilon parameter here is responsible for the real sliding, which I am going to show you mathematically. So, what is completion? In the first adaptive law, the usage of a low pass filter requires the tuning of tau and guarantees ideal sliding. What is the second adaptive law? No low-pass filter dynamics, easier to implement and guaranteed real sliding.

So, whenever we adopt this approach, we show that this kind of thing basically comes into the picture. So, now again I am going to start with the same system; sliding mode is exactly the same. Now, we are going to design the control that we have defined by the second method. So, we have to prove now that the sliding variable satisfies $| \sigma(t) | \leq \delta$ for all $t \geq t_F$, and here δ depends on ε, and this is the ε.

And obviously, these two are the parameters. So, ε and α_K are nothing but the gain of the adaptation. So, again I am going to start with exactly the same Lyapunov function, but here you can see that initially I am going to use α_K, and here I am going to use some kind of θ; $\theta > 0$, I am assuming. I have calculated $\dot{V}_1$, and after the calculation of $\dot{V}_1$, I am going to keep this dynamics. And, once I substitute that dynamics and apply the inequality, it is possible to show that if I define this term ζ, then from here to here, I will get this kind of uncertainty or inequality.

So, please check this with pen and paper. You can easily see this, but what is the difficulty here? ζmay be positive or negative. So, if $\zeta < 0$ here, then this term becomes positive, and at that time it is possible to show that $V \neq 0$ exactly. And obviously, at that time we are going to convert to $|\sigma| \leq \varepsilon$, and that time is given by this. So, in this finite time, we are able to maintain this. But once $|\sigma| \geq \varepsilon$, at that time this kind of Lyapunov function comes into the picture, and due to that, in infinite time we are able to achieve or converge to this particular bound. So, if $|\sigma| \leq \varepsilon$, ζcan be negative, making Vsign-definite, and at that time it is possible to show that if $|\sigma|$increases beyond ε. So, at that time, this kind of differential equation comes into the picture, and obviously, this tells us that $V \to 0$ as $t \to \infty$ whenever this particular condition is satisfied.

So, finally, we will converge on this particular band. It is possible to show that you can calculate the size of the band; what you can do is keep the worst-case analysis. I have here a differential equation, so you can take the worst case of this, and after that, you can see that I have just a linear equation. Two linear equations I have, and if you solve these two linear equations, then it is possible to show that σbecomes some kind of addition of cosine and sine terms with some amplitude, which is δ. That is actually the magnitude of that, which is given like this. So, in this way, I can prove that $|\sigma(t)| \leq \delta$.

So, please check this by yourself; the solution is not so difficult. So, in this way, what happens is that if I adopt the gain like this, then basically real sliding mode control comes into the picture, and $|\sigma| \leq \delta$, and that δ is a function of the εthat comes into the picture. So, the conclusion is that a real sliding mode is established in finite time within the domain given by this K, which adapts to the uncertainty without prior knowledge of the bound, and the parameter εcontrols the accuracy of the stabilization. It means that you can tune this parameter to control the accuracy. It means that you can go to the very fine vicinity of the equilibrium point.

Here, the equilibrium point is the sliding surface $\sigma = 0$. So, now it is time to conclude this lecture. So, what have we seen? So, we have seen how to fuse the adaptive control technique with sliding mode control, which will eliminate the need for a priori uncertainty bounds and also provide us with a significant reduction in control effort. Obviously, after that, we have combined these two methods, and we are actually achieving both kinds of sliding: real sliding as well as ideal sliding. So, with this remark, I am going to end this lecture. Thank you very much.