How to Implement Super Twisting Controller Based on Sliding Mode Observer

Welcome back. In the previous class, I talked about the differentiator design. We have discussed how to design differentiators based on the extension of the super-twisting algorithm. And it is possible to show that I am able to calculate rth order differentiation with the help of the algorithm proposed by Professor Levant. In this lecture, I am particularly going to look into a very important problem, and it is possible to show that several papers which appeared before this paper somehow give us the wrong result. So, basically, we are going to look into some kind of problem where, suppose that if I have just a second-order system, and if I create some kind of super-twisting-based observer.

I want to implement some kind of controller that is, again, a super twisting-based controller. So, suppose that I apply continuous control to design a sliding mode where I only have information about the output, and from that information, I can obtain the information about another state, and then I will design sliding mode control. Then what kind of extra care do we have to take in advance? Otherwise, we will get some kind of mathematically wrong result. Even if the mathematically wrong result is not visible, obviously, I am not able to prove the stability of the overall closed-loop system.

So, those are the kinds of things I am going to actually highlight in this lecture. So, for the purpose of discussion, obviously, I am talking about output feedback stabilization because placing sensors is one of the very, very difficult tasks, and due to that, whenever we are designing control, we are trying to minimize the number of sensors. So, for that, we have two solutions. One solution is based on the differentiator design; another solution is based on the observer design. So, suppose that I have a perturbed double integrator system; it means that I have an uncertain double integrator system, and I want to design some kind of controller based on the super twisting algorithm.

Meaning is like this: suppose I have a second-order system. So, you can just assume that I have this kind of structure $f(x) + g(x)\,u$, and here I am assuming this is also uncertain, and this is also uncertain. Now, I want to stabilize $\sigma$and $\dot{\sigma}$asymptotically. So, how do we solve this problem? You can design a sliding manifold that will contain the information of $\sigma$and $\dot{\sigma}$. So, I can design a manifold like $c_1\,\dot{\sigma}$.

And after that, what will I do? I will take the first derivative. So, if I will take the first derivative, then obviously, control will explicitly appear. Now, you can design some kind of control that is based on the super-twisting algorithm $\sigma^{1/2}$. So, here basically I have to stabilize $s$because once $s = 0$, then $\sigma = 0$ and $\dot{\sigma} = 0$ both equal to 0 asymptotically, and this equals sign (s), and after that I have another term that looks like the integral that is k_2 sign (s), and here I have again $\int s\,ds$.
So, in this way, if I have some kind of double integrator system, then I can implement a controller called a super-twisting controller.

And what is the beauty of the super twisting controller? Over the first-order sliding mode controller, I am able to generate continuous control. Now, in this lecture, what kind of challenge are we going to see? We are trying to prove that we are designing a super twisting controller based on the super twisting observer. So, we have already discussed what the meaning of super twisting observer is. Then it is possible to show that I am not able to achieve second order sliding mode control using continuous control. So, this is the claim, and we have to prove it.

So, obviously, several papers before 2014 actually made this kind of mistake. So, we are going to highlight that mistake, and after that, we are going to give some kind of solution. So, this work is basically done under the guidance of Professor Bijnan Bandopadhyay, and Dr. Ashif Chalanga has actually contributed to this work. And if you see Google Scholar, this is one of the very highly cited works.

Try to understand the problem. Again, the problem statement is very, very simple. You can take any second-order perturbation system that is $\dot{x}_1 = x_2$ and $\dot{x}_2 = u + \rho_1$.
So, for simplicity, I am assuming there is no disturbance associated with the u, but you can again develop the same kind of theory, exactly the same kind of theory, even if you consider some kind of disturbance that is actually multiplied with this control, and obviously, that disturbance will always remain positive such that the sign of the gain of the controller will not change.

Here, I am assuming that $\rho_1$is a non-vanishing uncertainty. What is the meaning of non-vanishing uncertainty? Those classes of uncertainty which will not become 0at the equilibrium point. What is the equilibrium point in this system? I assume that $x_1 = 0$and $x_2 = 0$ is the equilibrium point of the system. Another important assumption I am going to make here, since I am going to design some kind of controller or observer that is based on the super-twisting, is that whatever disturbance exists, its derivative is actually bounded.

So, it might be possible that the disturbance is unbounded, $\rho_1$is unbounded. If it is continuous and the derivative is bounded, then I can still apply this kind of concept. So, what is our control objective? The control objective is very simple. I am assuming that I have information about the output. It means that I have information about x_1.

For the first task, I have to reconstruct the remaining information. So, I have a second-order plant. So, how much information is required to talk about the stability of the whole plant? So, two pieces of information are required. So, now by using the information of y, the first step is to design the information of x_2. And once I get that estimate in finite time, then I can easily design a super-twisting controller.

And using this kind of method, it is possible to show that I am able to maintain $x_1 \to 0$and $x_2 \to 0$ as $t \to \infty$. So, I am just talking about asymptotic stability. So, the first step, since I have to recover the information, is that I am going to design the observer. So, I have a structure like $\dot{x}_1 = x_2$ and $\dot{x}_2 = u + \rho_1$. So, how do you design an observer? So, what can you do, just like Luenberger? You can just copy this dynamic.

So, I have copied one $\hat{x}_1 = \hat{x}_2$, and this is nothing but a correction term that is designed based on the super twisting algorithm. exactly the same way, since ρ_1 is not known to us, and for that reason, I am going to copy this dynamics. And again, this extra term is nothing but a kind of correction term. And how do we design that correction term? Since I want to convert the whole structure like super twisting in the error coordinate frame, and for that reason, in terms of the error, I am going to add these two correction terms. So, you can see here that $e_1 = y - \hat{y}$ is nothing but the information of the output minus the estimate of the output.

So, I am going to substitute here, and after that, I am going to define the error dynamics. How do you define the error dynamics? You can take the derivative of e. So, $\dot{x_1} - \dot{\hat{x}_1}$, and if you substitute $e_2 = x_2 - \hat{x}_2$, then you can easily see that the error dynamics come like this, where ρ_1 is nothing but the disturbance. Now, if I design these two gains k_1 and k_2, you have several different ways you can select the gain; one of the ways you can apply is Professor Levant's theory. And optimally, you can design k_1 and k_2; then it is possible to show that both $e_1 = 0$ and $e_2 = 0$ in some finite time $t \geq T$.

So, what is the physical interpretation whenever $e_1 = 0$ and $e_2 = 0$? It means that x_1 is going to converge to its estimate and x_2 is also going to converge to its estimate exactly. Now, since I know the exact information of $\hat{x}_2$ and $\hat{x}_1$, I am going to design the sliding manifold. So, since x_1 information and output information are already coming from the sensor, and x_2 is coming from the computer, I am going to design the sliding surface using their linear combination because I want to apply the super twisting controller. And as you know, that super twisting controller is directly applicable for first-order plants or relative degree 1 plants. But what kind of difficulty do I have here? If you consider $y = x_1$ as an output.

So, $\dot{y} = \dot{x}_1$and $\ddot{y} = \ddot{x}_1$. So, control explicitly appears. Relative degree 2, and for that reason, super twisting is not directly applicable. So, how does one apply super twisting? Again, similar to classical sliding mode control, you can design the sliding surface; that is, I have taken here a very simple sliding surface, which is a linear combination of $x_1$and $\hat{x}_2$, and after that I will calculate the time derivative, and once the time derivative is calculated, I am going to substitute the control. Now, here is one important thing you can see: that $\dot{x}_1$and $\dot{\hat{x}}_2$come into the picture.

What is $\dot{x}_1$? If you look carefully, since the observer and controller are running together, once the observer and controller are running together, $\dot{\hat{x}}_2$, because I am assuming that the information of $x_1$is exactly available. So, once the algorithm converges, the error is exactly equal to 0. So, an error equal to 0means that somehow this first correction term is equal to 0, but I cannot guarantee that the second correction term is also equal to 0,
because that is a discontinuous term. So, the average value of the discontinuous term is

equal to 0, but the exact value is not 0, because this term is also going to compensate for the disturbance. So, the value of this term is exactly equal to this some kind of disturbance.

So, this term is always finite, and for that reason, I cannot remove this term. I hope that you are able to understand the interpretation, because here
$e_1 = 0$. So, obviously, this term will not actually come into the picture, but k_2 sign (e_1) always comes into the picture if I take $\dot{\hat{x}}_2$.
And now here, if you just design the super-twisting controller, how do you design the super-twisting controller? You can take the information; I want to maintain $s = 0$ and $\dot{s} = 0$, both equal to 0, after $t \geq T$, after a finite time. So, I have to design the controller which contains a proportional structure as well as some kind of discontinuous integral structure, and I can put a $c_1\, \bar{x}_2$ super-twisting kind of controller because I have to compensate for this term, and if I substitute this, you can see that automatically this term will remain.

So, closed-loop system, now I cannot able to give guarantee that $s = 0$ and $\dot{s} = 0$.
Why? Due to this term. Due to that reason, the whole closed-loop system is no longer stable. What are people doing? It is possible to show that the average value of this e_1 is somehow equal to ρ. So, one can substitute ρ and then proceed. But most of the time in the literature, people do not care about this term; they assume that once this observer converges, I will get exact information.

So, I have information $x_1$and x_2, which is exact information. And once I have exact information, then $\dot{x}_2$they are going to substitute $u + \rho_1$. What is a dangerous point? Somehow this guy, because of being a differentiator and observer that are running together, cannot prove the convergence now if this is not equal to 0.
Now, it means that the overall structure looks like this, where L is a kind of fictitious term.

Due to this particular term, I cannot realize the second-order sliding mode about this $s = 0$ and $\dot{s} \neq 0$ for an infinite time $t \geq T$. For that reason, the second-order sliding mode will never start. And due to that reason, what we have to do is redesign the control; whenever we are designing the observer-based control, I have to redesign the control so that these kinds of terms are compensated. Otherwise, the closed-loop structure looks like this, and I cannot give a guarantee, and if I am not able to give a guarantee, it is because $\dot{x}_2$ now is nothing but what you can see again; $\dot{x}_2$ is coming from here and will not contain any uncertainty, and for that reason, I am talking about that. So, basically due to this particular term, I am not able to realize second-order sliding mode control, and somehow, basically mathematically, I am not able to prove the convergence of this algorithm.

So, this kind of control people have applied and practically that also works. Why? You can easily see practically why this is working, because what happens is that once this observer converges, this observer at that time k_2 sign (ρ_1) is actually equivalent to ρ_1, which is not exactly equal to ρ_1. And at that time, if ρ_1 is differentiable, then I can proceed further, but mathematically, it is not correct to substitute this with some kind of continuous control, because that is just an approximation. So, mathematically, the implementation of this strategy is not sound because some kind of non-differentiable term comes into the picture whenever you are designing some kind of observer-based control. So, what is the beauty of observer-based control? You can see that the closed-loop system is somehow free from the disturbance.

That is another beauty. So, now how do we design some kind of control such that mathematically you are able to guarantee that second order sliding mode control comes into the picture? So, for that, you have to add some extra terms. But whenever you are going to add this extra term, another difficulty comes into the picture. Now, your controller, super twisting controller is no longer continuous controller. Why? Because this is continuous, because $s^{1/2}$, so $s = 0$, this is 0, and this is actually the integral of some kind of term which is only discontinuous at $s = 0$.

So, this is also continuous. So, this term is continuous. And obviously, this is also continuous, but this guy k_2 sgn(e_1), as $e_1 = 0$, is going to oscillate with a very high switching frequency. So, this is discontinuous. So, overall control is discontinuous. So, even if you apply a super twisting controller, you are somehow applying discontinuous control.

So, chattering is not going to be minimized. What is the main strategy to implement super twisting control? Somehow, using continuous sliding mode control, I want to achieve sliding and classical sliding. But that kind of framework is not possible due to this term. If we remove this term, then obviously, mathematically, that is not the correct one. Mathematically, this term is non-differentiable. So, I cannot be able to do this term here, and after that, I can be able to design λ_2 here.

So, now how do I design some kind of continuous control such that in a classical way I can design the sliding surface and achieve higher sliding mode control? So, for that, we have to actually look into this problem again. So, first I will give the proof of this particular proposition. If control is discontinuous, then you can see that exactly I have closed loop system which is free from the disturbance. And in the language of state s, I somehow have a super-twisting-like algorithm. So, here $s = 0$ and $\dot{s} = 0$ both in finite time $t \geq T_1$, and here I have reduced our dynamics similarly to the first sliding mode like this.

So, if I design $c_1 > 0$, it is possible to show that obviously $s = 0$ and $\dot{s} = 0$ and c_1 tends towards 0. So, there is no other choice tending towards 0. So, in this way, if you incorporate this term, then obviously, mathematically, you can prove that second-order sliding mode control comes into the picture. But somehow this control during implementation due to this particular term create chattering. So, somehow our objective is not fulfilled and how people are basically implementing it in literature.

So, I have already told you that people are assuming that they are designing an observer, and after a finite time, they are assuming that x_1 is exactly equivalent to $\hat{x}_1$ and x_2 is exactly equivalent to $\hat{x}_2$ because I have finite-time convergence $t \geq T$, and they are not considering the dynamics of the observer, because this is the dynamics of the observer. So, during the differential process, this k_2 sgn(e_1) also comes into the picture. So, they are neglecting, and after that, they are basically designing the sliding mode control based on the super twisting algorithm, and they are also providing the proof. So, mathematically it looks fine. But actually, what kind of negation comes into the picture, because we are getting information about x_2 every time from the observer? So, $\hat{x}_2$ always has some kind of dynamics running inside the computer because I do not know the uncertainty; ρ, I do not know.

And due to that reason, I always have to rely on the information from the computer that is coming. And due to that reason this term, I was never able to ignore, k_2 sgn(e), I was never able to ignore. So, in this way, people are talking about super twisting output feedback control, and somehow, mathematically, this is not consistent. So, obviously, the implementation challenge comes into the picture, and because if you are exactly assuming $x_2 = \hat{x}_2$, then finally, things are okay. When $\hat{x}$ comes into the picture, people are substituting $\hat{x}$, and after that, they are implementing it.

So, x is not explicitly considered because we are neglecting this discontinuous term. So, whatever control I am assuming, applying that is approximate control. So, we are never able to get exact sliding, and due to inexact sliding, obviously our accuracy is going to degrade. Now what I am going to do is give you the way such that, again for a second-order plant, it is possible to show that if you have just the information of output $y = x_1$, then you can design an observer and you are able to apply a super twisting controller. But here again you have to become a little more careful because whatever higher sliding mode observer I am going to implement, okay.

So, the order of the observer is more than what we require. So, what I am basically trying to say is that I just need information on x_2. So, second order is enough, but what is another requirement? I also need continuous control. So, for control purpose our control should be continuous and due to that reason you have to increase the order of the sliding mode observer. So, this is the structure of the sliding mode observer, exactly like the differentiator.

So, now $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, and you can actually now give one fictitious state and you can define error like $e_1 = x_1 - \hat{x}_1$, $e_2 = x_2 - \hat{x}_2$, same like the previous one. Now, here correction term is everywhere. That is in terms of e_1 only, okay. And now it is possible to show this with the help of homogeneity as well as the contractivity property. So, basically, whatever term I am talking about, $\hat{x}_3$, here I have k_3, and if you are going to copy this, basically, I have to show that e_1, $k_1 e$, to the power of e^2, k_2.

So, this should somehow be in the error coordinate frame; this should be negative. So, $k_3 < 0$, you have to keep it negative. So, either you can put it negative or you can make it negative. So, both are equal. So, actually what happens is that we are assuming here $-x_3 + \rho_1$

= z_3 = e_3, and due to that reason, I am going to be consistent with the positive gain.

So, gain adjustment can be done based on the definition of error. So, here this construction gain should be positive; otherwise, I cannot be able to get some kind of structure where this correction term, you can see, is based on the degree of homogeneity. And using the same approach as the differentiator design, it is possible to show that e_1 = 0, e_2 = 0, and e_3 = 0 in finite time if there is no measurement noise. So, by designing gain k_1, k_2, and k_3 exactly like the robust exact differentiator, the structure is exactly like the differentiator in the error coordinate frame. And in this way, you can be able to get the convergence; you are going to ensure the convergence of e_1, e_2, and e_3.

Now, once e_1 = 0, e_2 = 0, and e_3 = 0, you can see that x_3 is exactly converged to the value of the disturbance. It means that you can also be able to estimate the disturbance based on this particular process exactly. No low pass filter is actually required in this particular process. Now, I will get x_1 and x_2. So, again, what I am going to do is propose that the sliding surface x_1 information is here.

So, I basically have to design the $\hat{x}_2$ information. So, after that, I will take the derivative, and I have to do it for infinite time $t \geq T_2$. When the observer actually converges, then only can I get the exact information of $\bar{x}_2$. And due to that reason, generally it is guidelines like this that whatever observer loop should be very, very faster than the controller loop. And since the observer is finite-time stable, I can make their loop very, very fast. And after a finite time, it is possible to show that the gain design for the observer and controller is separated.

So, in finite time, the observer separation principle also comes into the picture. So, basically, what I am going to do now is? I am going to take the derivative. $c_1\hat{x}_1$, so $\hat{x}_2$. I am going to substitute all other terms here if you see the observer. So, $e_1 = 0$, so this term will disappear, and for that reason, $\hat{x}_1 = \hat{x}_2$.

After that, $e_1$also disappears because $e_1 = 0$. So, in this particular problem, I can keep here. That is nothing but like this and that. So, now you can either put this term or not put this term; our analysis is not going to create any kind of difficulty. Now, what happens is that we have already written that this term is exactly equal to 0once the observer converges.

So, finally, control can be given in this way. And if you see this control, then this is a continuous term. This is also continuous, and this is also continuous because the integral of a discontinuous term is continuous, and for that reason, $e_1 = 0$ in finite time. In this way, if you implement the higher-order sliding mode observer, or higher-order super twisting mode observer, because this looks like super twisting only and super twisting-based observer, or higher sliding mode observer, then you can ensure that your control looks continuous, and I can achieve it.

That $s = 0$and $\dot{s} = 0$in finite time. And once $s = 0$and $\dot{s} = 0$, then what happens? $s$is nothing but $cx_1 + \hat{x}_2$, and now $s = 0$. So, basically, $\dot{x}_1 = cx_1$, c_1x_1. Now, this is asymptotically stable. So, $x_2$is also asymptotically stable. So, in Simulink, easily you can be able to implement this particular way; you can design the Simulink block.

And after that, you can implement whatever proposal I have given, which means sliding mode super twisting-based control for a second-order system based on the higher-order observer or differentiator. A similar kind of methodology you can apply, suppose that if you want to implement some kind of second-order super twisting-based controller, then again you have to increase the relative degree of the observer. So, what is the key observation? The sliding super twisting control based on a higher sliding mode observer is continuous; the design of the higher sliding mode-based super twisting controller requires tuning only the observer gain according to the first derivative of the disturbance, with no explicit gain of λ_2 with respect to the tuning, which means that you can observe the convergence process. So, this is the convergence process, and we are saying that for convergence now, we are just looking for the error dynamics.

So, the error dynamics of the observer. So, you just have to tune the gain of k_3. So, if you tune the gain of k_3, then every structure is obviously stable. So, the same kind of things we have written here, and completion with the static output feedback. So, basically, a static output feedback is somehow propagated wrongly in the literature because it assumes that the exact information of x_2 is available, which is not the case, because the observer and controller are always running together, along with another key observation.

Is whatever proposal we have given that is mathematically sound, and after that, obviously, one can be able to tune the observer-based controller, and then the precision of the sliding manifold can also be improved by creating a higher-order sliding mode observer-based STC compared. Compared to the static output feedback-based STC. So, precision will also increase. And after that, obviously, the stabilization problem states that the system remains much closer to the origin in the presence of the disturbance.

And obviously, we only discussed the closeness of the equilibrium point. Because most of the time what happens is we have measurement noise, and due to that reason, we are discussing in terms of the closeness. Now, we have implemented this kind of algorithm on one of the setups that are available in the lab of IIT Bombay, which we have already discussed regarding the control of the industrial emulator, and we somehow have two disks. So, one is a drive disk and another is a load disk that is coupled through some kind of timing belt. And suppose I have to control the angle of the load; then basically our system looks like this: if the belt is rigid, what is our goal? I have to actually control the position of $y = x_1$, and precision is very, very important because most of the time, with very, very high precision, we have to do this job. So, one can design the third-order super-twisting observer-based sliding mode control, and it is possible to show that you can achieve very high accuracy.

So, we have also implemented this kind of algorithm in practical setup and practical result is actually given in the paper. So, if you have interest, then please go through the paper. That is actually accepted in IEEE Transactions on Industrial Electronics and is a very highly cited paper. So now it is time to conclude this lecture. So, this lecture somehow tells us that if the order of the sliding mode controller and the order of the sliding mode observer are both the same, then some difficulties come into the picture.

So, whenever you are designing an observer waste controller, or particularly if you want to design some kind of continuous control, then you have to increase the order of the observer. And in this way, you can be able to extend this kind of result for any order system. And what is the future work? Now you can take any higher order system and experimentally you can able to validate this particular process. So, with this remark, I am going to end this class. Thank you very much.