

Tuning of PID & High Gain Controllers

Welcome back. In the previous classes, I was discussing proportional control, proportional and integral control. And what we have observed is that in the presence of vanishing and non-vanishing disturbances, neither control is efficient in several cases. In today's lecture, I am going to explore the most commonly used control, which is PID control. So, I am going to incorporate information about the future. And after that, we are trying to understand the tuning process and explore whether the PID controller is able to compensate for uncertainty or disturbances.

If not, then what are the future possibilities? So, this is the purpose of the discussion. So, we are trying to understand the tuning process of proportional derivative control and proportional integral derivative controllers, and I am going to use exactly the same framework. I am also going to take the example of a first-order system. So, in the previous class, I took an example of the underwater vehicle.

So, I am going to continue with the same example in this lecture as well. We are also trying to analyze the limitations of the output feedback. So, Suppose that I have a second-order system and I only have information that the position means

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = u + d(t).$$

I am assuming. So, x_1 is being interpreted as a position vector. So, if I have information about the position, is it possible to stabilize the second-order system at the equilibrium point? The answer is negative in this case.

So, we are going to explore the PD controller and it is possible to show that using PD controller one can be able to achieve this objective. The last part of this lecture is based on high gain feedback, and it is possible to show that high gain feedback is closely related to sliding mode control, which is the topic of this particular course. What kind of outcomes are we expecting from this particular lecture? We are trying to understand the tuning process, and we are also able to understand the connection between high-gain feedback and sliding mode control. So let us start today's lecture.

You can see in today's lecture that I am going to start from a very simple example again. So I have taken the mathematical model of the underwater vehicle moving in one direction, and if you recall, we saw the first class at that time when we did mathematical modeling like this. So, this is the thruster force, this is the drag force, and after that, we have the disturbance. And after that, what we have done is design this as a feedforward action. So, this is the drag force.

So, I am going to compensate for the drag force, and after that, we are going to design the feedback. So, here e is nothing but our objective to stop the underwater vehicle. So, $v=0$, and due to that reason, e is exactly equal to v , and now I am going to design this particular

vehicle. So, in today's lecture, I am also going to include information about derivatives. What is the meaning of derivative? We are trying to use futuristic information during the design process.

That is our goal. You can see here that our closed-loop system on the right-hand side of this differential equation is also containing the information of the derivative. So, for that reason, I am going to rearrange this term, and finally, our closed-loop system. You can see that this is exactly like the proportional action, and here we have some kind of disturbance. And this disturbance $d(t)$ may be non-zero, constant, or time-varying.

It is possible to show that $v(t)$ is uniformly bounded again. Or uniformly ultimate boundedness kind of result comes into the picture. So, I suggest that you go to the second lecture where we discussed the proportional control and then try to calculate this bound in the case of the disturbance. You are able to see that the ultimate bound is smaller because this disturbance is scaled out by $1+k_D$, where k_D is positive. So, what are our observations? In the presence of the proportional and derivative controller, we still cannot completely reject the disturbance, but we will achieve some kind of ultimate bound or uniform ultimate bound.

So, that is our observation. It means that the PD controller is also not insensitive with respect to constant disturbances or time-varying disturbances. So, let us try to see the advantages of the PD controller in some other context. What we are basically going to do now is discuss how to control the velocity. Now, suppose in the same mathematical model, the mathematical model is v plus u plus dt ; I am assuming that there is no disturbance for the time being.

So, the mathematical model is just $\dot{v} = u$. Using this mathematical model, I am trying to control the position. So, I have to involve the position. So, I assume that x_1 is the position. So, $\dot{x}_1 = x_2$, x_2 is the velocity, and $\dot{x}_2 = u$.

So, this kind of mathematical model comes into the picture. If I am trying to stabilize the position as well as the velocity, then the first-order mathematical model is converted into a second order. Now, suppose that I only have information about the position. So, x_1 is the position vector. So, after that, I am going to scale it up, and then I am going to utilize this as feedback.

So, you can see the closed-loop system that is a second-order system. So, now whenever I have second order system, so directly I cannot able to apply the philosophy of Newton. What is our philosophy? Philosophy is very simple; if something belongs to the real line, you can just see that quantity and its rate of change; if that is in the opposite direction, then we can always observe the motion, which is the direction of the velocity. So, here v is positive, and \dot{v} is negative. If you are here, then v is negative, but \dot{v} is positive.

So, we will see this kind of motion. But now here we have second order system. So, in order

to apply Newton's philosophy, I have to decouple the second-order system, and it is possible to show that there are several other coordinate frames that exist where these two states x_1 and x_2 are decoupled. So, now, how do we decouple? So, one of the simplest ways you can transform the system into polar coordinates, and due to that reason, I have defined it like this: r is positive, and that is given by $k\sqrt{kx_1}$. So, due to that reason, $\sqrt{(kx_1^2 + x_2^2)}$.

Now, what can you do? Again, you can calculate the rate of change of this r , which is the radius. This means that in a second-order system, the object is represented by both the radius and the angle, θ . So, now I am going to see how the radius is changing and how the angle is changing. So, you can see that after calculating \dot{r} , if I calculate and substitute this closed-loop system, I will get $\dot{r}=0$. What does it mean? r is nothing but a constant quantity.

It means that once x_{10} and x_{20} , that is, the initial condition is fixed, then the value of r is fixed and that is not going to change. Now, let us try to see the behavior of θ . So, again I am going to do exactly the same treatment. So, I have calculated $\dot{\theta}$. I am going to substitute this value from the closed-loop system.

Here I have used one more inequality, that is $\sec^2\theta = 1 + \tan^2\theta$. So, $\sec^2\theta$ is $1 + \tan^2\theta$, and due to that reason, we have x_2^2 by kx_1^2 . And if you rearrange the terms, then finally you will get it like this. It means that now you can see that k is a kind of gain. So, the rate of change of θ is also constant.

What does it mean? θ is actually decreasing with respect to time because I have a negative value. Suppose that the initial condition of θ_0 is 0; then what happens? What kind of observation will we get? r is constant, and I have this kind of motion; due to that reason, I have some kind of harmonic motion, but I cannot move to the equilibrium point. So, what kind of conclusion have we seen from this mathematical treatment? That if I have just information about the position, even if the system is free from disturbance, I cannot stabilize the position and velocity. Let us try another example. Suppose that if I have information about the velocity, what kind of conclusion can I actually draw in this case? Let us try to see that if I substitute $u = -k_1 x_1 - k_2 \dot{x}_2$, then at that time $\dot{x}_1 = x_2$, $\dot{x}_2 = -k_1 x_1 - k_2 x_2$.

You can see that this is already a decoupled equation. So, x_2 is tending towards 0 as t tends towards infinity because k is positive. So, you can just apply Newton's law. And what kind of observation will you get? That x_2 is tending toward 0.

It means that x_1 is constant. It means that using just velocity information, I am also not able to stabilize the system at the equilibrium point. And due to that reason, I need both pieces of information. So how do you get more information? Suppose that if you have only information about the position, then you can design a differentiator. And then you can differentiate this signal and get x_2 information. Another way you can design a kind of observer.

But most of the time, whenever you are observing a signal or measuring some kind of signal with sensors, there is always noise. And if you are going to differentiate that noise, and if the noise is high frequency, what is the meaning of high frequency noise? Suppose noise is represented by some kind of sinusoidal wave, $n(t) = \alpha \sin(\omega t)$, and ω is very high, α is very small, but after differentiation, what happens is that I get $\dot{n}(t) = \alpha \omega \cos(\omega t)$, and this ω , which is very high frequency noise, is going to amplify. During the differentiation process, the implementation of the PD controller still presents some challenges. So, we are going to see how to mitigate this challenge in the subsequent lectures. So, here I am assuming that our differentiator is robust and accurate.

What does it mean? That we can be able to differentiate the signal in the presence of noise as well. Now, if you design control like this, at that time our system basically looks like this: I have \mathbf{k}_1 and $-\mathbf{k}_2$, and here I have \mathbf{x}_1 and \mathbf{x}_2 . So, again, this is a second-order system, so you can apply the transformation the same way as in the previous class, decouple the system, and apply Newton's law. And what kind of result have we actually observed in the previous class, that if the eigenvalues are strictly negative, then both quantities $x_1 \rightarrow 0$ and $x_2 \rightarrow 0$ as $t \rightarrow \infty$. And in this way, I can design the PD controller.

And what are our observations? Both x_1 and x_2 are tending towards 0 as $t \rightarrow \infty$. It means that we can be able to solve a stabilization problem using PD controller. So, every control has its own limitations won't benefit. So, you have to understand each and every controller whenever you have a physical system. So, you have to go through proportional control, proportional integral control, and PID controller and try to see what kind of advantage we will get, and then you can select the final control for your practical implementation.

It is good to simulate this system. And here this exactly looks like the PD controller. But what kind of extra information are we providing here? We have been using the $\text{sign}(x_1)$, and you can see that this is the scaled value of the present information and future information. It is possible to show that both x_1 and x_2 equal 0 an infinite number of times. If there is no disturbance, okay. So, please do this kind of simulation by yourself.

Now, let us try to see the treatment of proportional integral derivative control using the same framework. Here, I am initially assuming that $d \neq 0$ and that it is constant. But you can easily generalize; you can take $d(t)$ as some kind of time-varying signal. Now, here you can see that I have used information from the present, past, and future. And then I am going to apply to the same first-order plant.

And after rearranging, after absorbing this dot term on the left-hand side of the differential equation, I will get this equation. So, you can see that again this looks exactly like a proportional integral controller with a scaled value of the disturbance. So, again, what can you do? You can treat this as an extra state. Take the derivative, apply the transformation, and design the gain. So, if $d(t)$ is constant, you can see that the PID controller is insensitive to disturbances, but if $d(t)$ is time-varying, then you will again get the uniform bound or

ultimate bound similar to that of proportional-integral control.

So, now I hope that you are able to understand the classical proportional control, proportional-integral control, and proportional-integral-derivative control. So, whenever you see the subsequent lecture, we will come up with a new form of the PID controller that is called the non-smooth PID controller. So, before coming there, what I am suggesting is that you can simulate. Simulation will give you a lot of ideas.

And here is how to define this ϕ . So, this is the definition you can apply; you can replace it with this value, and please do the MATLAB simulation. You will be able to observe that again $x_1 = 0$ and $x_2 = 0$. If $d(t)$ is some kind of differentiable uncertainty or noise, then $x_1 \rightarrow 0$ and $x_2 \rightarrow 0$ as $t \rightarrow \infty$. You can also simulate this, and the same kind of observation you can get here. Why have I actually been keeping this kind of slide? Because you are able to understand that sliding mode control is not different from the PID controller, just here we are using the scaled information of the present, past, and future.

So, you can see here that this is the present information, this is the futuristic information, and this is the past information. Now, let us come to some new kind of controller that is called high gain feedback. So, in the starting class, I told you that most of the time in this network era, people are relying on the present information. So, is it possible to mitigate all different kinds of uncertainty if uncertainty is bounded with the help of the present information or not? And somehow this controller, called a high gain feedback controller, is giving a way to show that you are able to nullify any kind of disturbance, whether it is vanishing, non-vanishing, constant, or time varying; any kind of disturbance you are able to reject using high gain. But we have limitations in terms of the actuator because every control we are actually applying through the actuator, and every actuator has some kind of finite bandwidth.

So, particularly nowadays we have a lot of electrical actuators where we are using DC motors. For some kind of motors, you are using. And we already know that they have a finite bandwidth. So, the implementation of this control is challenging in that case. But suppose you have a pneumatic actuator or a hydraulic actuator, you will still be able to apply this kind of control.

So, what is philosophy? You can see here that again I have proportional control and I am scaling it by $1/\tau$, where $\tau \rightarrow 0$. And if you substitute this control, after substitution, what kind of things will I get? So, after substitution, you can see here that I have an equation like this, and since $\tau \rightarrow 0$, I can expect that in the presence of any class of disturbance, you can substitute D here as some kind of constant, time-varying, or any class of disturbance; one can maintain $v = 0$. So, mathematically, it is possible to reject any kind of disturbance with help of the high gain feedback. Why is it termed high gain? Because you can see that as $\tau \rightarrow 0$, the gain $\rightarrow \infty$.

So, I am going to do the MATLAB simulation. So, during the MATLAB simulation, I am slowly decreasing the value of τ . And you can see that in our response, I started from here and they are tending towards 0. If I decrease the τ by a very, very small amount, then exact compensation we can see in the presence of the sinusoidal noise. So, what is the discussion? The discussion point here is that high gain feedback is able to compensate for any kind of disturbance.

You can see that order of the system is reduced by 1. So, we are going to see why I am writing this particular statement. It has become clearer if I take the second order system and control has infinite magnitude. Why? Because $\tau \rightarrow 0$, the value of control is tending towards a very high value. So, let us try to see what kind of observation we will get if I apply high-gain feedback to a second-order system. So, I have taken second order system and I have assumed here that any kind of disturbance either vanishing, non-vanishing or time varying.

So, this dis containing all possibility of the disturbances or uncertainty. Again, I have designed some kind of proportional-derivative control. And then I am scaling it by $1/\tau$. So, why I am telling this as a present information because second order system. So, I have used two different sensors to estimate or measure x_1 and x_2 .

And then I am going to design the control like this. So, again, this is a high-gain feedback controller. Here, I am assuming that τ is very small. So, if you already assume τ is very small, there is no need to write the limit as $\tau \rightarrow 0$. So, a closed loop system is given like this. Again you can multiply both sides by τ and then you will be able to see that we will get this kind of relation.

So, I have defined z in another way; you can define z as a . This term, so this whole term, you can divide as a/z ; you can transform into the z coordinate frame, and then you can maintain τ very, very small. Then you can see that I get this kind of algebraic relation between x_1 and x_2 . Both state x_1 and x_2 , and due to that reason, x_1 can be expressed in terms of x_2 . And after substituting x_2 in place of x_1 , which is equal to x_2 , I will get this kind of differential equation.

And since k_p and k_d are both positive, you can apply Newton's law. Again, if you are here, then x_1 is positive, but \dot{x}_1 is negative, so we will move in this direction. If you solve it, you will get this kind of response. So, this is nothing but exponential convergence. So, as $t \rightarrow \infty$, $x_1 \rightarrow 0$, and x_1 and x_2 are both coupled, and due to that reason, $x_2 \rightarrow 0$. So, what is our interpretation? So, we have extremely fast convergence because τ is very, very small.

So, $1/\tau$ is very, very large, and due to that reason, in just a few moments, you are able to get this kind of algebraic equation. So, the overall system is reduced by $\dot{x}_1 = x_2$ and this equation. So, this is a differential algebraic system, and then I will get the convergence of x_1 and x_2 in the presence of any class of disturbance. And this is exactly the same idea as

sliding mode control. So, in sliding mode control, we are designing as if I have a second-order system.

So, this is the equation of $k_p x_1$ and $k_d x_2$ in the two-dimensional plane. So, by using control, we will converge here, and after that, we will slide here. And due to that reason, there are close connections between high-gain feedback and sliding mode control. After that precision is also high because this is able to mitigate any kind of disturbance.

Only issue is the unbounded control. So, this will give us the next homework: is it possible to design some kind of control that has finite gain and allows us to achieve the same kind of behavior? The answer to this question is yes; using sliding mode control, we are able to do so, which means that the value of the control gain remains finite, and we can compensate for any kind of uncertainty, including any bounded uncertainty, as long as it enters through the control channel, provided our actuator is strong enough. So, what is the conclusion of this lecture that we have explored the proportional derivative and proportional integral derivative control? We have understood the limitations of output feedback. We are able to understand the connection between sliding mode control and high gain feedback, and what our future direction is. Study the impact of actuator limitations on high-gain feedback. Because whenever people are using hydraulic or pneumatic actuators, they are utilizing this kind of concept.

So if you have an actuator like hydraulic or pneumatic, it is good to explore high gain feedback control. So with this remark, I am going to end today's lecture. Thank you very much.