Sliding Mode Control and Applications

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Week-08

Lecture-36

Welcome back. In the previous class, I was talking about the second-order algorithm, and one of the second-order algorithms is the twisting algorithm. What we have seen is that in the presence of disturbance as well as uncertainty, the twisting algorithm gives us finite-time convergence for any second-order system. We have also taken the example, and in order to prove the convergence in the previous class, I used the geometrical method. Another way of proving the convergence of any non-linear algorithm or non-linear system is based on Lyapunov analysis. So, in this class, I am going to extend the Lyapunov analysis for the same twisting algorithm.

So, the purpose of the discussion is to analyze finite time stability because we have already seen that this is a second-order higher sliding mode control algorithm that is called twisting, and now in this class, I am going to analyze the stability based on Lyapunov theory. So, basically, this lecture is based on the weak Lyapunov function. What is the meaning of a weak Lyapunov function? So, suppose that if we construct some kind of Lyapunov function for some algorithm without uncertainty, and once uncertainty comes into the picture, if that Lyapunov function is not valid, then that class of Lyapunov function is called a weak Lyapunov function. What is the meaning of a strong Lyapunov function? So, if system do not have any uncertainty or disturbance and same Lyapunov function will work for both case, under uncertainty or without uncertainty.

So, that kind of Lyapunov function is called a strict Lyapunov function. In this, I am going to limit myself only to the weak Lyapunov function. Those who are interested in the strong Lyapunov function or a strict Lyapunov function for the twisting algorithm, please visit the work done by Professor Morino. So, he has developed several different algorithms for the twisting algorithm, the super twisting algorithm, and all Lyapunov functions. After that, we are going to apply the lessons in the variance principle.

So, basically, whenever we are talking about sliding mode control, we have observed that the system dynamics are discontinuous. So, first, I have to extend the concept of Lasell's invariance to the discontinuous system. It is possible to show that for a continuous system, the same Lasell's invariance principle which is applicable for continuous systems cannot be directly applied to the discontinuous system. Now, after that, we are going to establish the stability condition under which the twisting algorithm is actually finite-time stable. So, some key focus areas again.

In this particular lecture, I am going to take advantage of two concepts: asymptotic stability and homogeneity. So, first I am going to review the homogeneity again. Several times I have reviewed these concepts because they are very useful whenever you are dealing with sliding mode control and higher order sliding mode control based on the homogeneity principle. And obviously, we are trying to understand the construction of the Lyapunov function for non-smooth systems. And obviously, our main intention here is to prove finite time convergence.

We have already defined what the meaning of the homogeneous function is. So, suppose I have some function map from $\mathbb{R}^n$ to $\mathbb{R}$ and n here, I have n number of the vector. So, basically, if I am going to scale each element by some kind of weight, the same weight, then that is called a homogeneous function if this relation comes into the picture for all $x \in \mathbb{R}^n$. And what is the meaning of weighted homogeneity? We have seen that in weighted homogeneity, I am going to assign the weight again, but for each coordinate system, I will assign a different weight. That is the difference between homogeneity and weighted homogeneity.

And this particular weight is also called dilation. After that, you can see that I have constructed several examples here so that you are easily able to understand the difference between a homogeneous function and a weighted homogeneous function. So, this function is homogeneous of degree 1, and here the weight of homogeneity, actually whatever the scaling factor is, a scaling factor is exactly the same. So, each coordinate system is scaled by λ in this particular example. Kind of things I have done for the second example.

In third example, you can see that if you scaled each vector x_1 by some λx_1 and x_2 by λx_2, then you can see that this function is no longer homogeneous function. So, one more important point I am going to highlight is here. In the previous class, I used κ in place of λ_1. So, commonly people are utilizing κ or sometime λ and due to that reason, means in order to make whole theory unified, I am going to use both kind of notion that λ as well as κ.

So, here you can see that this is not actually homogeneous if you are going to scale by the same amount, but if you scale by λ^2 and λ^1, then you can see that this function becomes weighted homogeneous, weighted because we have given different weights here; this weight is 1, and the degree of homogeneity is 3.

Similarly, you can see that the degree of homogeneity here is 2. Now, once we define the homogeneous function and after that we are defining the homogeneous vector field. What is the meaning of a homogeneous vector field? So, basically, whenever we are talking about a vector field, that is a mapping from the same space $\mathbb{R}^n$ to $\mathbb{R}^n$. So, I have here f, f is going to map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Now, we are telling that this vector field is called homogeneous provided they will satisfy some kind of property with respect to the weighted dilation and that is $m + r_i$. In the previous lecture, what we have observed that r_i is coming from the dilation of the time.

Suppose that I have a system. Our goal is to select a_1, a_2, and β in such a way that this whole second-order system becomes homogeneous. So, in order to do that, I am going to scale out x_1 by $\lambda^{r_1} x_1$ and x_2 by $\lambda^{r_2} x_2$, and after that I am going to substitute inside this particular homogeneous function.

So, basically this is a vector field, but if you are going to see the individual components, then they are nothing but some kind of homogeneous functions. Now, since I have talked about the overall homogeneity, so I have to actually talk in terms of $m + r_i$. So, if you do that, it is possible to show that

$$r_2 = m + r_1, \quad m + r_2.$$

If you solve these two equality, then you will get the condition like this and where r_2 you can free to select.

Now, whenever we are dealing with the higher error sliding mode control, obviously differential inclusion comes into the picture. So, for that, Filippov methods come into the picture, and whenever you are going to construct the Filippov differential inclusion, the Filippov differential inclusion is actually now defined as a homogeneous differential inclusion.

So, the right-hand side of the differential equation, I am talking about

$$\dot{x} \in f(x).$$

If you scaled x by different dilations

$$x_1 \mapsto \lambda^{r_1} x_1, \quad x_2 \mapsto \lambda^{r_2} x_2, \quad \ldots, \quad x_n \mapsto \lambda^{r_n} x_n,$$

then this particular differential inclusion is homogeneous.

If $x = 0$ is the equilibrium point of this differential inclusion, and if $x = 0$ is globally asymptotically stable, then since the system is homogeneous, $x = 0$ is globally finite-time stable.

Now, let us come to the twisting algorithm. Suppose there is no disturbance. In that case, this is nothing but a PD control. We use $\operatorname{sign}(\sigma)$ in the proportional part and $\operatorname{sign}(\dot{\sigma})$ in the derivative part.

Now, assume

$$\sigma = x_1, \quad \dot{\sigma} = x_2.$$

Then the system can be written as

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -b_1 \operatorname{sign}(x_1) - b_2 \operatorname{sign}(x_2).$$

Now, scale

$$x_1 \mapsto \lambda^2 x_1, \quad x_2 \mapsto \lambda x_2.$$

Then you can see that the degree of homogeneity is -1. Hence, twisting algorithm has negative degree of homogeneity.

Now, consider the Lyapunov function

$$V(x) = |x_1| + \frac{1}{2}x_2^2.$$

This function is positive definite but not continuously differentiable everywhere. Taking derivative for $x_1 \neq 0$,

$$\dot{V} = \operatorname{sign}(x_1)x_2 - b_2 x_2 \operatorname{sign}(x_2).$$

This is negative semi-definite.

Now, we use LaSalle's invariance principle. Suppose we have a compact, positively invariant set $\Omega \subset \mathbb{R}^n$ and

$$\dot{V}(x) \leq 0.$$

Let

$$E = \{x \in \Omega : \dot{V}(x) = 0\}.$$

If the largest invariant set $M \subset E$ is the origin, then the equilibrium point is asymptotically stable.

Now, consider the pendulum system. Let θ be the angle, m the mass, and l the length. The equation is

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}.$$

Dividing,

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{k}{m}\dot{\theta}.$$

Define

$$x_1 = \theta, \quad x_2 = \dot{\theta}.$$

Then

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -a \sin x_1 - bx_2.$$

Consider the energy-like function

$$V(x) = \frac{1}{2}x_2^2 + (1 - \cos x_1).$$

Its derivative is negative semi-definite. Hence, by LaSalle's invariance principle, the equilibrium point $x = 0$ is asymptotically stable.

Now, I have to extend LaSalle's invariance principle for the discontinuous system.

We have already defined what the meaning of the homogeneous function is. So, suppose I have some function map from $\mathbb{R}^n$ to $\mathbb{R}$ and n here, I have n number of the vector. So, basically, if I am going to scale each element by some kind of weight, the same weight, then that is called a homogeneous function if this relation comes into the picture for all $x \in \mathbb{R}^n$:

$$f(\lambda x) = \lambda^m f(x).$$

And what is the meaning of weighted homogeneity? We have seen that in weighted homogeneity, I am going to assign the weight again, but for each coordinate system, I will assign a different weight. That is the difference between homogeneity and weighted homogeneity.

And this particular weight is also called dilation. After that, you can see that I have constructed several examples here so that you are easily able to understand the difference between a homogeneous function and a weighted homogeneous function. So, this function is homogeneous of degree 1, and here the weight of homogeneity, actually whatever the scaling factor is, a scaling factor is exactly the same. So, each coordinate system is scaled by λ in this particular example.

In the third example, you can see that if you scaled each vector x_1 by some λx_1 and x_2 by λx_2, then you can see that this function is no longer a homogeneous function. In the previous class, I used κ in place of λ_1. So, commonly people are utilizing κ or sometimes λ. So, here you can see that this is not actually homogeneous if you are going to scale by the same amount, but if you scale by λ^2 and λ^1, then you can see that this function becomes weighted homogeneous, weighted because we have given different weights here; this weight is 1, and the degree of homogeneity is 3.

Similarly, you can see that the degree of homogeneity here is 2.

Now, once we define the homogeneous function and after that we are defining the homogeneous vector field. What is the meaning of a homogeneous vector field? So, basically, whenever we are talking about a vector field, that is a mapping from the same space $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$f : \mathbb{R}^n \to \mathbb{R}^n.$$

Now, we are telling that this vector field is called homogeneous provided it satisfies a certain property with respect to the weighted dilation:

$$f_i(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^{m+r_i} f_i(x).$$

In the previous lecture, what we have observed is that r_i is coming from the dilation of time.

Suppose that I have a system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -a_1 x_1 - a_2 |x_1|^{\beta} \operatorname{sgn}(x_1).$$

Now, our goal is to select a_1, a_2, and β in such a way that this whole second-order system becomes homogeneous.

So, in order to do that, what am I going to do? I am going to scale x_1 by $\lambda^{r_1} x_1$ and x_2 by $\lambda^{r_2} x_2$ and substitute inside the system. Since I have talked about the overall homogeneity, I have to talk in terms of $m + r_i$. If you do that, it is possible to show that

$$r_2 = m + r_1, \qquad \beta r_1 = m + r_2.$$

Solving these two equalities gives the homogeneity condition, where r_2 is free to select.

Now, whenever we are dealing with higher-order sliding mode control, obviously differential inclusion comes into the picture. So, we write

$$\dot{x} \in F(x),$$

where $F(x)$ is a Filippov set-valued map.

If this differential inclusion is homogeneous and if $x = 0$ is the equilibrium point, then it is possible to show that if $x = 0$ is globally asymptotically stable, then it is globally finite-time stable. This concept comes from dilation retractability plus contractivity. Now, let us come to the twisting algorithm. In the disturbance-free case, this control law is given by

$$u = -b_1 \operatorname{sgn}(\sigma) - b_2 \operatorname{sgn}(\dot{\sigma}).$$

Assume

$$x_1 = \sigma, \qquad x_2 = \dot{\sigma}.$$

Then the system can be written as

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -b_1 \operatorname{sgn}(x_1) - b_2 \operatorname{sgn}(x_2).$$

Now, scale x_1 by $\lambda^2 x_1$ and x_2 by λx_2. You can see that the degree of homogeneity is -1. Hence, the twisting algorithm has negative degree of homogeneity.

Consider the Lyapunov function

$$V(x) = |x_1| + \frac{1}{2}x_2^2.$$

This function is positive definite but not continuously differentiable. Taking the derivative for $x_1 \neq 0$, we get

$$\dot{V} = \text{sgn}(x_1)x_2 + x_2\dot{x}_2.$$

Substituting the dynamics yields

$$\dot{V} = x_2 \left(\text{sgn}(x_1) - b_1 \,\text{sgn}(x_1) - b_2 \,\text{sgn}(x_2)\right).$$

This is negative semi-definite. Therefore, we use LaSalle's invariance principle. If $\dot{V}(x) \leq 0$ and the largest invariant set contained in

$$E = \{x : \dot{V}(x) = 0\}$$

is the origin, then the equilibrium point is asymptotically stable.

For the twisting algorithm, analysis of the Filippov dynamics shows that if

$$b_1 \geq b_2,$$

then the only positively invariant set is $\{(0,0)\}$. Hence, asymptotic stability is guaranteed.

Since the system is homogeneous with negative degree, asymptotic stability implies finite-time stability. So, what have we seen? We have analyzed the twisting algorithm and its finite-time stability using Lyapunov analysis. We have shown that the degree of homogeneity is -1, extended LaSalle's invariance principle to non-smooth systems, derived the gain conditions, and finally applied the theory to a practical system.

With this remark, I am going to end this class. Thank you very much.