Sliding Mode Control and Applications

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Week-08

Lecture-35

So, welcome back. In the previous class, I talked about homogeneity and its relation to finite time stability. And we have also talked about accuracy. In this lecture, I am going to explain how to design second-order sliding mode control. And one of the very, very popular second-order sliding mode controls is called the twisting algorithm. So, in this lecture, I am going to talk about the twisting algorithm and the application of twisting algorithms to some practical systems.

So, the purpose of the discussion is the twisting algorithm, and this algorithm was basically introduced by Professor Levanth. So, we are going to understand the significance of the name whyy-twisting. We are also going to understand why the twisting algorithm represents second order sliding mode control, not first order sliding mode control. So, what is the limitation of the first-hour sliding mode control, and why are we basically moving to the second-hour sliding mode control that I am going to highlight first? So, we have already seen that one of the very important implementation issues of the first-hour sliding mode control is called the chattering phenomenon.

And why chattering comes into the picture due to discontinuity. So, what is the main philosophy given by Professor Levant regarding whether it is possible to shift discontinuity at the higher derivative or not? If you are able to do then what happens that lower term or lower control become continuous and we can also able to get the insensitivity with respect to some kind of matched perturbation and chattering can be minimized. So, using that state G, he has proposed several second-order algorithms, and after that, they move to the higher-order sliding mode control algorithm. In this class, I am going to limit myself only to second-order algorithms, particularly to the twisting algorithm.

So, this is the limitation of first-order sliding mode control: basically, we are designing the sliding surface, taking the derivative of the sliding surface, and after that, we are utilizing some kind of discontinuous control action.

So, I am assuming that I have a single input single output system. Whenever I have a multi input multi output system, at that time, basically, we are designing some kind of control where something looks like this, and after that disturbance, we have $d(t)$ like this. So, now first-order sliding mode control is only limited to relative degree 1. It means that if you calculate the derivative of the sliding surface, then control will explicitly appear in the first-order derivative. And what is the consequence? If the relative degree is 1, then I can maintain $s = 0$, but most of the case, whatever reduced order dynamics, that is just asymptotically stable.

It is also possible to show that due to unmodeled dynamics, whenever we are going to apply first-order sliding mode control, then that is very sensitive with respect to noise and due to that reason, twisting algorithm comes into picture. Twisting represents the second-order sliding mode control.

Now, once we are talking about the higher sliding mode control, I have to show that if I start anywhere, this is a second-order system. So, suppose that I will take two variables, s and $\dot{s}$. So, I have to show that $s = 0$ and $\dot{s} = 0$, if I start anywhere in this space, in finite time, in any way that will actually converge here. Obviously, trajectories will not cut each other. So, I have to show it like this.

This is going to reduce the chattering through higher-order action, and higher sliding mode control is quite useful for mechanical or electromechanical systems that I am going to highlight. Obviously, whenever we are designing our sliding mode control, I have already discussed that I am selecting the sliding manifold in the higher dimensional space, and after that, I am maintaining the rate of change equal to the rate of change using some kind of discontinuous control.

So, we are designing some kind of discontinuous control here, and we are making sure that in the presence of any kind of uncertainty, uncertainty may depend on the parameter, may depend on the time, as well as the state, such that I can maintain $s = 0$ for an infinite amount of time. And, for that reason, we are stating that sliding mode control, first-order sliding mode control, is also called classical sliding mode control.

Classical means basically this first came into the picture. That is actually maintaining $s = 0$, but $\dot{s} \neq 0$. Why $\dot{s}$ is not equal to 0? Because instantaneous velocity, if you see the signum function at $s = 0$, that is switching with very high frequency and due to that reason, one can also able to interpret that s is position vector, then $\dot{s}$ that is velocity vector.

So, the instantaneous velocity vector is non-zero. Now, during the equivalent control calculation, you please do not confuse, because Utkin is telling that average of this equal to 0, not the instantaneous equal to 0. So, this is the definition of first-order sliding mode control.

Any variable $\dot{s}$ where discontinuity will appear, and what is our main goal? I have to design control, discontinuous control, such that I have to maintain $s = 0$. Now, let us try to understand the motivation behind higher sliding mode control.

For that, I am going to take a single input single output system. So, I have n number of states. I am assuming that $u \in \mathbb{R}$ and s, that is our output, that also belongs to $\mathbb{R}$. And due to that reason, this system is a single input single output system.

I am going to define the relative degree. So, how do you define relative degree? You can take the derivative. So, the derivative of h with respect to x, so $\frac{\partial h}{\partial x}$ multiplied by $\dot{x}$, I am going to substitute. After that, I will expand it. So, this is nothing but the Lie derivative.

Now, suppose that $L_g h(x) \neq 0$. If this is non-zero, then the relative degree of the system is 1. And in that case, you can design first-order sliding mode control such that $s = 0$ and $\dot{s} \neq 0$. And in this way, you can show that $h(x) = 0$ in finite time.

But suppose that this term $L_g h(x) = 0$. So, basically, control is not going to appear in the first derivative. So, for that, I have to take the higher derivative, which means the second derivative. And if you do take the second derivative, then what happens now?

Suppose again this term is zero. So, again you have to move for the third-order derivative. But suppose you are lucky enough that this is non-zero. Then, the relative degree of s with respect to u is 2, and now you can design u to be discontinuous.

Why am I talking about discontinuous? Because whatever system I have considered here, I am assuming that f is not completely known, g is not completely known, and for that reason, I am also assuming that these two terms are not completely known. So, it might be possible that these two terms have some kind of uncertainty.

Practically, whenever we are doing modeling, the model is not perfect. So, I always have uncertainty. So, I have only partial knowledge of these two terms.

Now, I have to design control u. So, obviously, this control is discontinuous control such that I have to maintain $s = 0$ and $\dot{s} = 0$ in finite time.

In first-order sliding mode control, you can see here that $s = 0$, but $\dot{s} \neq 0$. And discontinuous control is going to appear in the first-order derivative, so u is basically discontinuous.

But here what happens is that I have to maintain $s = 0$ and $\dot{s} = 0$ in some finite time $t \geq T$, and after that, I have to maintain $\ddot{s}$ equal to some kind of discontinuous control. And in this course, I am assuming that this discontinuous control is actually designed based on the philosophy of Filippov.

Second-order sliding mode control, for all second-order sliding mode control, we are assuming that whatever this term is, it is uncertain and that is going to bound between

$-\phi$ to ϕ'. And after that control term, I am assuming that that is not going to change the sign.

Several times I am assuming this because if control is going to change its sign, coefficient which is multiplied with control is going to change its sign, then actually the design of this algorithm contains some kind of extra care. And without loss of generality, I am assuming that it is always greater than 0, but obviously that is uncertain.

Now, I have to actually design a discontinuous control u such that I maintain $s = 0$ and $\dot{s} = 0$. So, this is the statement of the second-order sliding mode control.

Suppose that if you have third-order sliding mode control, then $\dddot{s}$ and differential inclusion come into the picture, and the structure is exactly like this. And at that time, you have to design the control, third-order discontinuous control, such that you have to maintain $s = 0$, $\dot{s} = 0$, $\ddot{s} = 0$ in some finite time $t \geq T$, and after that, you have to maintain $\dddot{s}$, which is discontinuous, and you have to design this discontinuous control basically using the Filippov sense.

And in this way, I can able to realize third-order sliding mode control.

Now, the first algorithm of the higher-order sliding mode control, which is also called second-order sliding mode control, is because discontinuity will appear at the second-order derivative. So, this is called the twisting algorithm.

And what is the form of the twisting algorithm? So, you can see that this is just an extension of the first-order algorithm. Now, I am saying that in $\ddot{x}$, obviously, I have some kind of disturbance $d(t)$ here.

And after that, what am I going to do? In place of one gain, now I am going to use two gains. Here, I am representing the gain by b_1 and b_2.

So, $b_1 \operatorname{sign}(s)$ and $-b_2 \operatorname{sign}(\dot{s}) + d(t)$. So, if you look carefully, what is the physical interpretation? I have a second-order uncertain system that is represented in the form of a chain of integrators, a second-order chain of integrators.

Now, I have to design some kind of PD controller. Why I am telling this is just like a PD controller? Because you can see that I am using the information of x_1 and x_2, which is nothing but the derivative of x_1.

So, proportional plus derivative. But how is this basically different from the PD controller? In the PD controller, exact information of x_1 is required, but here only sign information is required. So, even if your information is inexact, but if you are going to somehow take the sign, then you are going to design the control PD controller.

And due to that reason, twisting is nothing but some kind of non-smooth or nonlinear PD controller, you can able to interpret. And this is also one of the motivations because the implementation of the switching gain controller is relatively easy, and due to that, even the twisting controller has become very popular for second-order uncertain systems. Another beauty comes into the picture if you talk about the classical PD controller.

So, X_1 and X_2 are both asymptotically stable if there is no disturbance. What is extra the extra property that comes into picture here, that in presence of disturbance, any bounded disturbance, I can able to ensure x_1 and x_2 equal to 0 infinite time, that is the beauty. But you have to tune the gain. So, gain tuning is like this. Suppose that if you know the upper bound of the disturbance, so for all time if that is bounded by δ, so δ is some kind of upper bound, then it is possible to show that just you have to choose gain simply like this, means

$$b_1 > b_2 + \delta.$$

So, gain tuning is also not very difficult for the twisting algorithm. In this lecture, I am going to show you that if I have this kind of structure, then how to tune the gain of the twisting algorithm. And for that, we are going to do a geometrical analysis that is called the Magellan curve. So, before going to that analysis, I am going to simulate this system, the same system I have taken, and I have simulated it in MATLAB.

Now you can see the structure. How does this structure basically look if you start in the phase plane? Here, this is a phase plane because I am plotting between x_1 and x_2. So, if you start from here, you can see along which axis this trajectory is twisted? This axis, the x_2 axis. And finally, after infinitely many rotations in finite time, that is going to converge to the equilibrium point. So, since this trajectory is going to do infinitely many rotations in finite time, that kind of phenomenon is also called Zeno phenomena, Zeno behavior. And due to that reason, infinitely many rotations in finite time finally make it possible to show that I will converge to the finite time if I start anywhere in the state space.

Still up to 1995, so this problem is nicely solved that if suppose that if I have some kind of uncertain chain of integrator, then how to design control based on switch and gain such that I can able to stabilize X_1 and X_2 in finite time, but if you see carefully the behavior of this control, again this is discontinuous control. And this control is sometimes not suitable for mechanical actuator. So, in 1995, one problem posed in literature that is it possible to design some kind of continuous control such that I can able to maintain x_1 and x_2 equal to 0 and I will not utilize extra information, some kind of differentiator, some kind of observer I am not going to utilize. So, finally, this problem was actually solved during the period from 2010 to 2012, and we are lucky that we were able to solve this problem. Now, based on this particular problem and its generalization, a new era of sliding mode control comes into the picture, called fifth generation sliding mode control.

It means how to realize sliding mode control and higher order sliding mode control using continuous control. So, those kinds of things we are also going to actually discuss. So, actually I was working with one of my friend Ashif Chalanga and we both have actually explore the all kind of algorithm is of fifth generation sliding mode control. Obviously, we are unable to prove it.

So, that is proven by Professor Morino. Now, let us come to the convergence of the twisting algorithm. So, how does one talk about the convergence of the twisting algorithm? So, suppose that if I start with this algorithm, initially I am assuming there is no disturbance. So, if there is no disturbance, then algorithm basically looks like

$$\ddot{x} = b_1 \operatorname{sign}(s) - b_2 \operatorname{sign}(\dot{s}).$$

Now, what I am going to do here, since there is no disturbance, so if you actually consider this gain $b_1 > b_2$, that is enough for the convergence of both s and $\dot{s}$ equal to 0 in finite time. So, here you can see that this is a two-dimensional system.

So, I am able to do phase plane analysis. Suppose that if you start from this point, you can see that I have drawn an arrow in this direction. So, try to understand the direction. Since here I have two components, one is s and $\dot{s}$. So, try to see how basically s is changing. So, whenever we are going to see this system, then $\dot{s}$, s if you consider x_1 and if you consider $\dot{s}$ as a x_2, so basically

$$\dot{x}_1 = x_2.$$

So, basically if I want to move in direction of x_1, so I have to show that that direction of x_1 should be positive because x_2 is positive here because what is x_2? x_2 is $\dot{s}$. So, basically, the arrow is in this direction. Now, if you see the $\dot{s}$ dynamics, so in order to show the direction in which direction basically vector field comes into picture, so for that I have to calculate one higher derivative $\ddot{s}$ and if you see $\ddot{s}$, so basically $\operatorname{sign}(s)$ is positive, $\operatorname{sign}(\dot{s})$ is positive and due to that reason $-b_1 - b_2$ comes into picture and negative due to what is negative direction, this is negative direction and due to that reason overall vector field comes like this. Similarly, here you can able to see that similar way one direction is in this and one is in this because I am maintaining $b_1 > b_2$ and due to that reason this is the requirement of b_1 and b_2 otherwise I cannot able to point arrow like this. Similarly, now this axis and this axis are somehow symmetrical.

So, arrow is just you can put mirror here. So, arrow direction is like that again you can put mirror here then arrow direction looks like this. Okay, so in this way you can see that if you start from here, then you can actually rotate like this. Now, I am going to analyze the actual equation. So, how do we analyze? I am going to take this equation $\frac{d\dot{s}}{ds}$ and here you can see that in first quadrant $s > 0$, $\dot{s} > 0$ and due to that reason $-b_1 - b_2$ because sign is just going to give you the signal is just going to give you the sign.

So, both are positive. So, $-b_1$ and $-b_2$, and after that, it is divided by $\dot{s}$. In this way, I can basically calculate the rate of change of ds with respect to $d\dot{s}$ or $d\dot{s}$ with respect to ds. Similarly, this is the behavior in the fourth quadrant because our trajectory is moving like this from first to fourth, then three and second. So, in a similar way, I have written the differential equation.

Now, this is very, very interesting. What can you do? What we have to show is that we are twisted along this $\dot{s}$ line, and for that reason, you can start from this line. So, what is the coordinate of this line $(0, \dot{s}_0)$, and due to that reason, you can see that what I have done in the first quadrant is that I first multiplied this b_1 by b_2. So, $s\,ds$ I have written, and ds I am going to multiply here, and after that I am going to take the derivative from 0 to $\dot{s}$, because I am assuming that I am going to start from this point p_1. For that reason, I am going to take the integral from 0 to $\dot{s}$. At that time when I was at 0, so somewhere s I am maintaining, and after that at that time s, it means that I have to actually denote this coordinate system s and $\dot{s}$.

So, $\dot{s}$ I have already fixed on the left-hand side, now I am fixing the right-hand side and in this way I will get the expression of s. In exactly the same way, I am going to do it in the fourth quadrant. So, I have fixed some point p_2, and after that, I am going to calculate the equation of this particular equation with the help of the differential equation, and you will easily be able to see it because this is the $\frac{d\dot{s}}{ds}$ in the fourth quadrant. And if you solve it, then you will get this equation. So, once you get these two equations, now try to see the interpretation of this length and this length.

So, basically, here somehow this image is compressed, but if you see carefully, then this point, it is possible to show that length of this point is always higher than this point. Because we have some kind of convergent behavior. So, as time moves, the distance from here to here is always less than 1. So that you can see mathematically, since at this point $s = 0$, the distance is just the distance on the y-axis. So, that distance is $\dot{s}_0$ and here distance is $\dot{s}_1$, and I have from this differential equation, what I have done, I have put this equal to 0, this equal to 0, and in this way I have calculated the $\dot{s}_0$ and $\dot{s}_1$ by doing the square root of that, and after that what I am going to do, I am going to take their ratio, so and ratio is always negative.

What does it is showing, that in each subsequent step, you are going to converge along the y-axis, you are going to twist along the y-axis. And similarly, you can able to extend this reasoning for any step. And in this way, you can able to show that you are going to converge in finite time. So, how do you show that you are going to converge in finite time? So, for that, I have to calculate the time. Again, in order to calculate the time, what am I going to do? I am going to seek the help of the differential equation.

So, since I wrote $\frac{dS}{dt}$ in the first, that is called $\ddot{S}$, which is given by $-b_1 - b_2$, because the $\text{sign}(S_1)$ is positive, and the $\text{sign}(\dot{S})$ is positive. And if you solve this equation, then it is possible to show the term of convergence from here to here that I have represented by T_1 plus. Similarly, you can able to calculate time from S_1 to S_1 dot, that time I am telling that from here to here, because you can see that behavior of first quadrant is exactly like the third quadrant, behavior of second quadrant that is exactly like the fourth quadrant.

So, if you calculate the behavior from here to here, the next behavior is automatically calculated.

And in this way, one can be able to calculate this time. So, please verify these two calculations, and it is possible to show that if you add these two times together, then I have this kind of equation, and if you generalize, then obviously one of the nice geometric series comes into the picture. How many times is it rotated? We do not know. So, I am taking the summation up to infinity. And due to that term, you can see that now infinitely many times I am actually doing summation. It means that our trajectory is going to rotate how many times? Infinitely many times along the y-axis.

And what time is it finite? So, after infinitely many rotations, you can see that time is still finite, and in this way, it is very nice to see that the time of convergence is always finite. So, this behavior is also called Zeno behavior. Now, what do we have to show? this behavior or this proof is only applicable if we are assuming there is no disturbance, because I have started with this system. Now, I have to extend the disturbed differential inclusion, which is our main goal. And now, here I have to select gain in such a way that the exact same kind of property will be preserved.

What basically uncertainty is going to do, they are going to disturb our trajectory, but I want to maintain similar kind of behavior and due to that reason I have selected $K_1 > K_2$, after that based on the upper bound and lower bound we have selected like this is to maintain the trajectory, their direction always normal to the disturbed trajectory. So, basically disturbed differential inclusion that also becomes homogeneous in some sense, and then I can show that I can again maintain a similar kind of behavior. Obviously, in normal cases, I am starting from here; I will start either here or here. So, you have to show both trajectories, both extreme trajectories, and you have to show that the vector field always remains between these two, and due to that reason, finally, I will converge over some larger time.

Obviously, we have uncertainties. So, you will converge in some larger time frame. One more important thing I am going to highlight is here. If you represent the twisting algorithm like this

$$\ddot{x}_2 = b_1 \operatorname{sign}(x_1) - b_2 \operatorname{sign}(x_2),$$

this is the form of the twisting algorithm. It is easy to check that if you scale S_1 by some kind of λ, it becomes 2. x_1 and x_2 if you scale by λx_2. So, this is homogeneous. How is that homogeneous? Here you can see that λ^2 comes into the picture and time is here. So, minus 1. So that is exactly equal to λ^1. Similarly, here is λ^1, and here you do not have any coefficient.

So, $\lambda^1 - 1$, that is 0. So, both the right-hand side and the left-hand side that are going to match here also have $1 - 1$ because ds_2/dt. So, the degree of homogeneity of s_2 is λ^1, and the degree of homogeneity of time is 1. And due to that reason, $1 - 1$ is in the denominator. So, this algorithm is also homogeneous. So, twisting is going to represent some kind of homogeneous differential inclusion.

Now, what is the main idea of the He was suggesting that whenever you are going to use twisting for the first-order system, you can increase the relative degree. How can one increase the relative degree? He was saying what you can do, suppose you have this system. Now, you can assume the right-hand side of this differential equation to be z. So, I have

$$\dot{\sigma} = z.$$

Now,

$$\dot{z} = \dot{u} + \dot{f}.$$

And in this way, now you can design the control such that σ and z both equal 0 for infinite time. So, if σ and z both equal 0 infinitely often, then $\sigma = r$, and you will be able to get it. After that, you can see that your control becomes continuous because you are designing based on the sliding mode philosophy, and obviously, the main theme here is that you have to use the information of $\sigma - \text{sign}(\dot{\sigma})$ with some kind of gain: b_1 and b_2. In this way, control become continuous, but one criticism comes into picture, since I am assuming that $\dot{\sigma}$ is known to us.

So, if $\dot{\sigma}$ is known to us and control is known to us, then somehow uncertainty is known to us. So, if uncertainty is known to us, then any kind of continuous kind of controller can compensate it. In reality, that is not the case because we always have some kind of measurement, disturbed measurement. It is possible to show that in case of disturbed measurement, this algorithm will perform much more better. So, we have actually implemented this on a type of robot, and after that, we have verified it.

So, now what I am going to do is take the example of a wheeled mobile robot. So, that is nothing but some kind of four-wheel differential drive system. This is the mathematical modeling of a differential drive system, where $x_r \in \mathbb{R}$, $y_r \in \mathbb{R}$, the position coordinate with respect to the inertial frame, and θ_r belongs to the orientation of the robot with respect to the inertial frame. Now, what do we have to do? I have two control variables V_R and W_R. So, basically we have to design control variables such that I can able to achieve some kind of desired trajectory that is our aim. So, basically, we have represented it using the second-order dynamics, and after that, u_x and u_y, I am going to design based on the twisting algorithm.

And what is our objective? Our objective is to track

$$x_d(t) = \sin t \quad \text{and} \quad y_d(t) = \cos t$$

so that I can move in any circular path. Now, how do you apply the twisting algorithm? You can take the derivative, and after that, you can actually design $\dot{x}_1$, which is u_x, and you can substitute u_x. Here is one more important point. If you know S_1, it is possible to show in next part of this course, I will apply some kind of differentiator that is designed by the professor Levant such that you can able to calculate $\dot{S}_1$ in presence of some kind of noise also, and this is the simulation.

So, in MATLAB, we have simulated it. So, you can see the trajectory in the x_r and y_r plane like this. And this is the S_1 and $\dot{S}_1$; this is S_2 and $\dot{S}_2$. You are able to see the twisting kind of behavior, and this is nothing but the control input. So, now it is time to conclude this lecture. So, we have actually explored the first second-order sliding mode control that is called twisting, and we have also justified the name; we have seen the Zeno behavior.

After that, in the presence of disturbance, I have to talk about the measurement curve. What is the meaning of the measurement curve? That is nothing but the worst case of phase plane analysis. And after that, we proved the finite time stability based solely on the geometrical analysis. We have not involved the Lyapunov function here. And we have characterized the reaching phase because one more important point here I am going to highlight.

In the case of second-order sliding mode control, what is the sliding phase? That is just equilibrium point. So, the sliding point is just the equilibrium point in the case of all higher-order sliding mode algorithms. And after that, we have already seen that chattering reduction because if you have a first-order system, then control becomes continuous because you are going to apply control on $\dot{u}$. So, u becomes continuous. And obviously, we have also shown you that this is nothing but a homogeneous differential inclusion with a negative degree of homogeneity because, in order to match the left and right-hand sides, whatever time dilation we have done is negative, and for that reason, the degree of homogeneity is negative, and this is very useful for the electromechanical system.

So, now using measurement curve, we have actually developed criteria whenever we have some kind uncertain situation also. So, with this remark, I will end this lecture. Thank you.