Sliding Mode Control and Applications

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Lecture-34

So, welcome back. In the previous class, I was talking about homogeneity, weighted homogeneity, as well as differential inclusion. So, basically, we are defining the stability of differential inclusion, and now in this lecture, I am going to talk about how finite time stability and homogeneity are related. And this will give us the room to prove finite-time stability just by knowing the asymptotic stability of the differential inclusion. So, for the purpose of discussion, again our lecture is mainly focused on differential stability inclusion, and here I am going to talk about homogeneous differential inclusion. In this lecture, we are going to understand why I am talking about homogeneous differential inclusion, because this class of differential inclusion has several beautiful properties.

in the in case of the practical situation because we already we are well aware that whenever we have system then we are keeping sensor to measure the output and based on that output we are basically designing the control system and most of time sensor is going to introduce some kind of noises. So, it is possible to show that if our controller that is based on homogeneous differential inclusion theory, then always I can able to get higher accuracy. So, what are the key properties of this particular lecture? So, obviously, again I am going to talk about dilation transformation, and it is possible to show that due to dilation transformation, in the case of noise also, I am able to preserve the symmetry or preserve the scaling. After that, we are going to talk about the connection between asymptotic stability, finite time stability, and the negative degree of homogeneity, and it is possible to show that if the origin is asymptotically stable and the system has a negative degree of homogeneity, then it leads to finite time stability, and this is a very strong result, because just by proving asymptotic stability, I am able to approach the finite time stability.

Obviously, during the analysis, we are going to utilize the Lyapunov framework. It means that the algebraic framework, as well as the geometrical framework, is called a contractive set. So, by merging these two concepts, I am going to prove the stability of the homogeneous differential inclusion. In the previous class, I was talking about how basically this particular differential equation is homogeneous. And after that, what we have done, similar concept we have extended for the differential inclusion.

So, this is called a differential inclusion. Again, in low main language, we are saying that at some point, which is at a distance of 0 measures, there are infinitely many differential equations. So, this particular differential equation is homogeneous of degree q and dilation d_κ, and how we are defining this is basically κ.

So, one thing I am going to clarify here is that, if you see the literature, people are either using d_κ or d_λ. So, sometimes interchangeably, I am going to use κ and λ for the scaling purpose.

And what is the meaning of this? It means that each coordinate system is scaled by ξ_i; ξ_i is the coordinate. So, I am going to keep scaling like κ, and m_i is the degree of homogeneity of each coordinate system ξ_i. Or in the same language, I can also utilize like this.

So, if the right-hand side of this differential inclusion will satisfy this property, then we can tell that this differential inclusion falls in this very, very special class of differential inclusion that is called homogeneous differential inclusion. And what is the property of that kind of differential inclusion? That is invariant under this kind of transformation.

So, here, since the left-hand side of the differential equation contains derivative with respect to time, so if you scale the time with κ^{-q} and x by d_κ, at the same time, even if you know the solution and if you scale the solution by this particular transformation, it is possible to show that I can move anywhere inside the space.

It means that somehow locally you can analyze the stability, and after that, you can do this kind of transformation. So, the same kind of stability and the same kind of property you can ensure throughout the state space.

So, what is the key point of this particular slide? Homogeneous differential inclusions are invariant under the scaling transformation. So, even if some noise enters inside this class of system, I am going to show you that what class of noise again preserves this kind of structure.

In this class, I am mainly going to focus on finite-time stability, because you are already well aware that even in first-order sliding mode control, when we are just talking about one dimension, what our main goal is.

If I have a first-order differential inclusion, some kind of differential inclusion is written like this. So, I have to prove that in finite time, I can move to the equilibrium point.

Similarly, in higher-dimensional space, in the case of the first-order sliding mode control, you can see that I have to steer the trajectory in finite time, and after that, I have to maintain it.

So, in order to steer the trajectory from here to here, I need first-order sliding mode control if we are going to design some kind of higher-dimensional manifold using classical sliding mode control. So, everywhere I need some kind of finite-time stability.

And, what is the idea of higher-order sliding mode control? In higher-order sliding mode control, suppose I have a two-dimensional space. So, basically, we are trying to move directly to the equilibrium point. So, we are not going to converge to any manifold; directly, we are going to converge to the equilibrium point in finite time.

And due to that reason, the concept of finite-time stability is very essential whenever we are talking about higher-order sliding mode control.

And due to that reason, what we are going to do, whenever I have some system, suppose that system looks like $\dot{x} = u$, is that I am going to design some kind of control, discontinuous control, such that this equation will convert into a differential inclusion.

So, due to that reason, I am going to select that function of x, and after that, I am going to ensure these two conditions. So, the first condition is Lyapunov stability, and in the previous class, I already talked about Lyapunov stability.

What is the meaning of Lyapunov stability? Whenever we have this kind of differential inclusion, the physical interpretation of this is, I am giving the initial condition x_0 and I am generating $x(t)$. And after that, what I am going to do, I am going to create some kind of ball with respect to x and the center of the ball at the origin, such that if $x(t)$ remains in some ball whose radius is ϵ, then we are writing for every ϵ, there exists a δ that is a function of ϵ, greater than 0, such that this condition is satisfied.

So, $x(t_0)$ is going to start from this particular ball, the δ ball. So, this is the kind of relation we are showing. So, this is the meaning of Lyapunov stability in the case of differential inclusion.

Now, apart from that, we are also talking about convergence and finite-time convergence. So, you can select any radius, a proper radius, and you can initialize your differential equation from that radius, and we are going to guarantee that in finite time t, we are going to converge to the equilibrium point.

This is a very strong result that a homogeneous system with a negative degree of homogeneity asymptotically implies finite-time stability. So, I am going to prove this, but you can easily apply this kind of result in several places whenever you have a homogeneous differential inclusion.

And obviously, what is key in this lecture? I am going to talk about convergence to the equilibrium point in the case of differential inclusion in finite time.

So, I have actually created a second-order differential inclusion. Why is this a differential inclusion? Because I am assuming that $\text{sign}(x_2)$ at $x_2 = 0$ lies between $-|x_2|^{1/2}$ and $|x_2|^{1/2}$. Now we have constructed one kind of Lyapunov function. If you look carefully at this differential inclusion and if you scale x_1 by some kind of κ, here I am taking $\kappa^3 x_1$, and after that x_2 by κ, where $\kappa > 0$. You can easily see that in the left-hand side of the differential equation, I have $\kappa^3 \dot{x}_1$, and on the right-hand side, I have $\kappa^2 x_1 x_2$. And here time is also there, and due to that reason, 3 can be written as $2 + 1$, or equivalently, $2 = 3 - 1$. So, -1 is nothing but the degree of homogeneity.

Similar kind of things here you can able to see that k power square. So, $\dot{x}_2$ and after that here in this side, since x_1 is κ^3, so 1/3 and 1/3 is going to cancel out. So, I will come here. So, on this side, I have κ. So, I will write $2 - 1$. So, the same degree of homogeneity is going to appear here: 3 and 2, and an extra -1 comes into the picture, and due to that reason, this whole differential equation has homogeneity equal to -1.

If you check the Lyapunov function, the Lyapunov function is also homogeneous. How am I to tell that it is homogeneous? You can apply the same kind of scaling, and you can easily see that basically $\kappa^4 V$ comes into the picture by scaling x_1 and x_2, and for that reason, the Lyapunov function is also homogeneous. If you check, then the derivative of the Lyapunov function is also homogeneous. Now, here one more important point you have to you have to become very careful, since $\dot{V}$ does not contain any information of x_1. So, what happens is that $x^2 = 0$ or for all $x \neq 0$; in this case, I cannot comment on anything.

I cannot able to comment about asymptotic stability, but if you put this kind of point inside this differential equation, you can see here since $x_2 = 0$, so $\dot{x}_2 = 0$, and due to that reason there is no other choice $x_1 = 0$ also. So, this principle is called LaSalle's invariance principle. How are people basically applying LaSalle's invariance principle? First, you can derive the weak Lyapunov function; after that, you can come up with this kind of invariant set, which means those can offset if the trajectory starts. They are not going to leave, and after that, you can put that point inside the main differential equation, and then you can conclude the asymptotic stability of the whole system. Now, this is also finite-time stable because the degree of homogeneity, as we have just seen, is negative.

Now, whenever we are proving asymptotic stability and homogeneity, negative homogeneity equal to finite-time stability, at that time, I need some kind of geometrical property. So, I have to show that whatever set I am going to consider, that set is dilation detectable. So, what is the meaning of dilation detectable? What can you do? You can scale the set. Obviously, the set is formed due to the state of the system. So, suppose if you have n number of state inside the system, so you can scale each state and after that, it is possible to show that if κ lie between 0 to 1 and if set is dilation detectable, then that will satisfy this kind of condition.

So, using one example, I will show it to you in the next slide, and after that one another important aspect that is contractive. So, what is the definition of contractive? That if there exist a compact set D_1 and D_2, which D_2 is the interior of D_1, and if D_1 is dilation detectable, so it is possible to show that all trajectory starting from D_1 that is going to remains inside D_2 in some kind of finite time. So, here basically if you see this slide, this slide contains two important points. The first point is direct dilation detectability.

The second one is contractivity. So, with the help of these two, we can show that our set is going to actually become smaller and smaller and finally, we can converge to the equilibrium point in finite time. So, that kind of key insight comes from this particular slide. So, I am going to take this example, exactly the same example, and after that, I am going to verify all three properties. We are going to talk about the compact set, how to construct a compact set, how to discuss direct dilation detectability, and after that, how to calculate the time of convergence.

If you see carefully, then this differential equation is actually weighted homogeneous differential equation, because if you scale x_1 by some kind of $\kappa^3 x_1$ and after that x_2 by some kind of $\kappa^2 x_2$, then you can able to see here that degree of homogeneity of this equation is nothing but -1. Now, I have applied this kind of transformation; I have a scale degree of x_1. I have already explained this, and this is the degree of homogeneity. Now, if you see the previous differential equation, basically this is differential inclusion, because I have told you that $x_2 = 0$, I have something like $x^{-1/2}$ to $x_1 x_2^{1/2}$. This is a plus sign, and so here at this particular point, this particular term is going to lie anywhere between these two terms.

I am going to do a uniform dilation. So, if you do dilation like this, time if you scale by -1, this is actually the same as the degree of homogeneity of the system, and x_1 if you scale by κ^3 and x_2 by κ^2. So, similarly, if you see the literature, people are using λ. λ is more convenient because several places are using it, but Professor Levant has used κ; due to that reason, I will preserve this.

Now, after that, what am I going to do? I am going to make a set. And what is our basic idea? I am trying to construct a set that is also homogeneous. Because if we are going to construct the set in that particular way, then I can easily be able to scale that set up to any large value. That is our main goal. So, how do you construct this? So, in order to construct that set, you can see that I know the degree of homogeneity of x_1 and x_2, and for that reason, I have adjusted the power so that if you set $x_1 = \kappa^3 x_1$. So, actually, cubes cancel out and κ^2 comes into the picture.

From here also, x_2 is mapped by $\kappa^2 x_2$. So, κ^2 will come into the picture. In this way, whatever set I have constructed, whatever ball I have constructed, that ball is somehow homogeneous and its radius is r_1. Now, I can easily able to construct a smaller ball if I just consider $r_2 < r_1$. Now, it is possible to show that D_2 lies in the interior of D_1

because both balls are exactly the same; the radius is different, and due to that reason, both are going to lie.

It is also possible to show that this ball, being compact, means closed and bounded. So, bounded is guaranteed by $\leq r_1$ and how to talk about the closeness. So, it is possible to show that this system is asymptotically stable. So, if you start anywhere, you will converge. Now, I am going to show you the direct dilation detectability of the sets.

I am going to prove this. So, obviously, whatever set I have constructed, it is a homogeneous set or homogeneous ball I have constructed. So, if you scale it, you can see that κ is going to lie between 0 to 1. And due to that reason, it is possible to show that if you perform a dilation of each coordinate of set D_1, then that is going to be actually less than or equal to r_1.

Now, next I have to show the finite-time convergence. Suppose I have constructed some kind of set in two-dimensional space, and I have another set. The first set has a radius r_1 and the second one is r_2. Now, I have to show that if I start from here, as time progresses, I converge to this set. Again, I have created a homogeneous Lyapunov function. After taking the derivative, by LaSalle's invariance principle, it is possible to show that $x_1 = 0$ and $x_2 = 0$.

Time of convergence. I am going to do worst-case substitution, and in place of $x^{3/2}$, I am going to substitute this radius. Since this radius is constant, that term will come out, and then V_0 will appear. In this way, you can see that the time of convergence is also finite.

It means that what happens is that if the trajectory starts anywhere from the ball, it is going to come to the smaller ball in finite time. And, this kind of construction, if you create another ball of r_3 radius that is inside the second ball, again that will converge in finite time to the second ball. And, in this way, finally, we are going to converge to the equilibrium point in infinite time. So, this is the summary. We have already started with this system, homogeneous system.

We have constructed some kind of dilation detectable set, and after that, we have actually calculated the finite-time convergence. So, in this way, you can prove the finite-time stability. Now, based on the previous observation, I am going to talk about the properties of homogeneous Filippov inclusions. What is the meaning of Filippov inclusion? We have already seen that whenever discontinuity comes into the picture, using the convexification of the vector field, we are going to define some kind of unique vector field that is tangential to some kind of sliding manifold. So, basically, the key concept is homogeneity with a negative degree.

Finite-time stability and robustness with respect to perturbations. So, these three things I am going to actually explore in case of the differential inclusion and we are trying to understand the equivalence of stability. So, we actually started with asymptotic stability, uniform asymptotic stability, and finite-time stability. So, how is that basically related to what I am going to discuss? Obviously, the effect of perturbation is very important because we are going to use several physical circuits whenever we are designing the closed-loop system, and uncertainty obviously comes into the picture; due to that reason, we have to show the accuracy. So, it is very easy to show accuracy if I have some kind of homogeneous differential inclusion.

So, a statement of a theorem is like this. If $\dot{x} = f(x)$, that is some kind of differential inclusion and if that is homogeneous Fillippov differential inclusion with negative degree of homogeneity $-p$, then it is possible to show that definition of global uniform asymptotic stability at 0, global uniform finite-time stability at 0 and contractive property all are equivalent. So, either you can prove one or this or this, all are exactly equivalent. Another important aspect you can see here is that the maximal settling time is locally bounded by a homogeneous function of the initial condition of degree p. Also, suppose that degree p is negative. So, it is possible to show that somehow the time of convergence is inversely proportional to some power, and that power is nothing but p, such that that much time you have to wait.

That is the physical interpretation of the time of convergence in the case of a homogeneous differential equation. And the proof is very, very simple because the definition of globally uniform asymptotic stability at 0 and globally uniform finite-time stability at 0 implies the definition of contractivity. How are we basically proving contractivity? We are making some kind of compact set; inside the compact set, we are making another compact set, and after that, we are trying to show that if a trajectory starts from the first compact set, it will actually lead to the second compact set in finite time. In this way, if you cascade the sets, we will finally converge to the equilibrium point, and for that reason, these two properties imply contractivity. The definition of globally uniform finite-time stability obviously implies asymptotic stability.

What is the meaning of asymptotic stability? As $t \to \infty$, I will converge to the equilibrium point. So, even if I will converge at t equal to some finite time t, and after that I will maintain the equilibrium point, so that is somehow one of the cases of uniform asymptotic stability. And homogeneity transformation shows that the trajectory concentrates on the smaller ball, because I have already told you what the beauty of homogeneity or a homogeneous system is: whatever property they contain locally, the same kind of property can be extended globally.

Now, robustness with respect to perturbation is very, very important. So, it is possible to show that if the homogeneous differential inclusion is in the Fillippov sense, and if the degree of homogeneity is negative, then it is robust with respect to some kind of small perturbation. Mathematically, I am going to prove this. So, before that you can see that I have taken this system, and after that I have applied some small perturbation here.

You can add this small perturbation here, and you can see that if ε is very, very small, then basically the property of this differential inclusion is not going to change.

So, obviously, it is possible to show that since we have a homogeneous differential inclusion, it is actually insensitive with respect to homogeneous perturbations only. This slide is very very important because that also tells us that during the measurement, if I have some kind of noise or delay, then how basically homogeneous Fillippov differential inclusion is going to take care about that.

So, I have started with the differential inclusion in the Filippov sense and after that degree of x_i, each coordinate system I have scaled by m_i and degree of time is p. And after that, what I am assuming is that the noisy measurement of x_i is obviously coming from the sensor. So, they always have some kind of measurement noise. So, I have represented measurement noise like this.

Now, I have to show that our system in case of noisy measurement is not going to actually lose the finite-time stability and that is going to settle in the nearby of the equilibrium point in finite time. So, those are the kinds of things I am going to show.

So, I have represented the delayed equation by $f(\Delta x)$. If $\delta = 1$, then it is possible to show that the system reduces to the nominal case. And since the system is homogeneous, whatever property one contains, the same kind of property δ is going to contain.

Now, this is the very, very important theorem. The theorem suggests that if you have a differential inclusion, and after that, if the degree is m, the degree of t is $-p$, and if $t \geq \delta^{-p}$, then it is possible to show that x_i will remain inside a ball of the form $\gamma\delta^{m_i}$. This ball is independent of the initial condition.

So, this theorem suggests that finite-time stability under noisy measurement still remains finite time, but we are going to converge to some kind of ball, and that ball is independent of the initial condition. Using this, we can show that higher sliding mode control is robust with respect to noise or delayed measurement and hence has greater accuracy.

So, with this remark, I am going to end this lecture. Thank you very much.