Sliding Mode Control and Applications

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Lecture-33

So, welcome back. In the previous class, I talked about the higher-order sliding mode control. Particularly, I am talking about the definition of higher-order sliding mode control. Now one of the job that is remaining that how to design higher order sliding mode control and for that I need sufficient background and one of the philosophy on which it is possible to show that you can able to construct any order algorithm for higher order sliding mode control that is based on the weighted homogeneity. So, in this class, first I am going to start with homogeneity, and after that, I am going to introduce the notion of weighted homogeneity. Then I will also talk about differential inclusion and how to extend the notion of stability for differential inclusion.

So, for the purpose of the discussion, I am going to talk about a very specific kind of symmetry that is called dilation symmetry. It is possible to show that symmetry leads to some kind of invariance under the scaling transformation. And this will provide us with a way such that the local analysis can be easily extended to the global analysis. This is the actual idea of the dilation symmetry.

Now, I am going to talk about, next I am going to talk about the homogeneous differential equation. So, what is the meaning of a homogeneous differential equation? So, here I am assuming that $x \in \mathbb{R}^n$ and that this particular right-hand side of the function is going to satisfy this kind of property, provided all states here are going to be scaled by the same amount. Now, we are going to introduce the next concept that is called weighted homogeneity in this particular lecture, where we are trying to understand that if we will not do the homogeneous scaling of the all state, then what new property that comes into picture that is called non-uniform scaling of the state variable. And obviously, our ultimate aim is to understand or develop a theory for differential inclusion because higher order sliding mode control is governed by some kind of differential inclusion, and then I have to construct the control. So, during the construction of the control, one of the prime objectives is stability.

So, all control systems, whenever we are designing, and after the substitution of control, should satisfy this property that is somehow called the extension of the continuity property, which is called stability. So, that I am going to introduce in this lecture and next lecture basically I am going to talk the relation between stability and homogeneity. So, in this lecture, I am just going to introduce the concept of stability, but I am not going to establish the notion of the relation between weighted homogeneity and stability. So, let us first talk about symmetry. So, obviously, the meaning of symmetry is nothing but some kind of invariance under the specific transformation.

And if you talk about the mathematics or somehow the physical object, it is possible to show that commonly we have four different kinds of symmetry or transformation. So first are rotation, translation, reflection, and dilation. So using this figure, you will be able to understand everything. This is translation, this is rotation, this is reflection, and this is nothing but dilation.

So, if you see the property, these three are very, very special. What is the property of this? They will preserve their size as well as their shape. So, you can apply any geometrical transformation; they are going to preserve size as well as shape, and that is clearly visible from this particular figure. Now another kind of symmetry that is called dilation symmetry and that is just going to maintain the shape, that is not going to maintain the size. You can see here that the shape is exactly this shape and that this shape is equal, but the size is not equal.

And this idea somehow gives us a way to talk about homogeneity. So, basically dilation symmetry is somehow directly related to the homogeneity and what homogeneity is telling that we can able to get invariance under the uniform scaling. So, suppose that you have some kind of function f , and that function is called homogeneous if and only if it satisfies this kind of condition

$f(\lambda x) = \lambda^{\nu} f(x)$,

and it is possible to show that all linear functions can satisfy this kind of property. Similarly, you can extend this kind of concept for homogeneous mapping. So, suppose that the app contains n number of variables that will map to m-dimensional space.

So, now it is called homogeneous if you scale each vector. So, suppose that you have m number of vectors up to x_n; now I am going to scale each vector by some kind of λ, and if λ and ν come into the picture. So, ν is nothing but the degree of homogeneity. And if $m = n$, then I am going to define this as a vector field. So, particularly whenever we are going to define the differential equation, you can see here in the differential equation, this particular function is going to map from n dimensions to n dimensions.

So, at that time we were describing this as a vector field. So, several times I have actually used this terminology without defining it. So, now I hope that you are able to understand what is meaning of the vector field, means some kind of mapping from same space to same space, then we are using this kind of language. Now, I am going to give an example. So, you can see that homogeneity, although whenever we are defining the property of linearity, at that time we are talking about additivity, we are talking about homogeneity, and that is only valid for linear systems or linear functions.

But here, it is possible to show that if you relax the additivity property. So, homogeneity is equally applicable to nonlinear mapping as well. So, I have two state x_1 and x_2 that will maps like this and you can see that here that is homogeneous with degree 1. So, here mapping from $\mathbb{R}^2$ to $\mathbb{R}^1$ happens and this function is non-linear. So, now this particular dilation is invariance under this particular transformation.

Similarly, even if now we have discontinuous mapping, this is discontinuous mapping, because $x_1 + x_2 = 0$, then I have this kind of definition, otherwise I have 0. So, it is possible to show that the homogeneity degree is $-\frac{1}{2}$; you can easily calculate it. How do you calculate? You can scale x_1 by λx_1 and x_2 by λx_2. With the same amount, it is possible to show that $\lambda^{-\frac{1}{2}}$ comes into the picture, and this is discontinuous at the origin. So, what is the meaning of these two examples? The concept of homogeneity can be extended to non-linear systems as well as discontinuous systems.

So, this is a very, very well-established theorem. So, this is also called the Euler theorem. So, it gives us the necessary and sufficient conditions for some function to be homogeneous. So, if you calculate the partial derivative, take the summation, and if that satisfies this kind of relation, then the function f that maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ is homogeneous. In this particular class, only those kinds of functions that are differentiable fall into consideration, because in order to check the homogeneity, I have to take the differentiation, and some kind of regularity property comes into the picture.

It is possible to show that if the homogeneity degree λ is less than 0, then this function is discontinuous at the origin; a similar kind of thing you can locate here. Non-Lipschitz at the origin at that time, this ν comes into play between 0 and 1, and if that is greater than 1, then the function is continuous and differentiable. So, you can be able to see from this particular example. Now, I am going to take another example, and this example is that of some kind of differential equation, and here you can see that right hand side of the differential equation, I have some kind of function that is x^ν and ν is nothing but some kind of quantity that can be expressed in $\frac{p}{q}$ form, where p is nothing but some kind of odd integer, q is the even integer and if you solve it.

Then you can be able to get this kind of solution. So, here since x is homogeneous, it is possible to show that the solution also satisfies some kind of symmetry, and this kind of symmetry is called dilation symmetry. It means that if you know solution locally, so you can able to extend this solution throughout the state space, that is the meaning of the dilation symmetry, and due to that reason homogeneity is very very important property. So, now here they will satisfy dilation property with respect to some special kind of scaling. So, here you can see that time is scaled by $\lambda^{1-\nu}$, and ν is nothing but the power of this, and all initial conditions should be scaled by the λ.

So, scaling basically depends on the ν. Now I am going to generalize the theory based on the previous example, which is the general form of a homogeneous differential equation, given like this. Because most of the time, whenever we are dealing with control systems, we start with some kind of system that is called a force system, $\dot{x}$ into $u(t)$. And after that, we are designing a control that is a function of the state. And after that, once I substitute here, it is possible to show that the system finally looks like some kind of function, like $g(x)$.

So now, if this closed-loop system satisfies some kind of property like homogeneity, then it is possible to show that if I analyze the behavior nearby the equilibrium point, the same kind of behavior can be observed. To guarantee that the same kind of behavior will repeat if I extend the solution in some special way, such as scaling or applying some kind of dilation symmetry, I will be able to extend the solution throughout the state space. So, definition is suppose that some kind of function that map from $\mathbb{R}^n$ to $\mathbb{R}^n$. So, this is vector field. So, in place of f, you can also be able to replace g with a continuous, homogeneous function of new degree.

So, at that time it is possible to show that solution will satisfy this kind of property. Now, you can easily see that if I suppose I have some kind of autonomous system. So, this is a linear time-invariant system. And if you solve it, then the solution looks like this. And if you apply scaling, then it is possible to show that it will satisfy the scaling dilation property or dilation symmetry again.

Here I have a homogeneous degree equal to 1. You can see a similar kind of thing with the algorithm we have used several times; here, I am assuming $x \in \mathbb{R}$ and there is no disturbance. So, using this, I can give a guarantee that if you start in one-dimensional space anywhere, in finite time, you are going to converge. So, that is also confirmed from the solution, and it is possible to show that this particular right-hand side of the differential equation is homogeneous with degree 0 if you scale λx by λ into x. So, λ is positive, so that is not going to change the sign, and due to that reason, I am able to write $\sin x$ by $\sin(\lambda x)$.

So, that kind of things I am able to write and due to that reason that will satisfy the homogeneity with respect to 0 because nothing will come out and it is possible to show that solution is also satisfy the homogeneity or a scaling property. Now you can, if you take some kind of fractional power of $\frac{1}{3}$, and at that time, a new kind of differential equation comes into the picture, and it is possible to show that the solution of this differential equation has some specific property. It means that all solutions are going to converge after this finite time, and the finite time actually depends on the initial condition.

So, in this way, homogeneity is somehow related to stability and finite time stability. Similarly, if $\nu \geq 1$, then if you start from any closed ball, it is possible to show that for all initial conditions, the time of convergence is uniform or independent with respect to the initial conditions; again, you can achieve the same result.

Several mechanical systems are going to satisfy this kind of condition; you can see that this is the representation of a mechanical rigid body with friction, and if you solve it within time, I am telling you this is fixed time because this is independent of the initial conditions. And here if you analyze this system, so nearby of the $v = 0$ that is homogeneous with 0 and far from the 0 that is homogeneous by 2. So, this is called the bi-homogeneity concept. And with classical bi-homogeneity, it is possible to show that $v = 0$ at some fixed time, that this time is greater than or equal to $t_{\max}$, and that it is also independent of the initial condition. So in this way, the classical homogeneity property is related to stability.

Now, in 1958, a new kind of homogeneity property comes into the picture that is called weighted dilation. And what is, how is that actually different from the classical one? In classical one, we uniformly dilute each and every component. Here we are giving different weight to different components. So, here you can see that I have n number of state and I am going to give different weight and that weight I am going to represent by d_k. So, homogeneity dilation scales each coordinate by an unequal amount, and m_i is nothing but the degree of homogeneity for that particular coordinate.

And now, this function that is a map from $\mathbb{R}^n$ to $\mathbb{R}$ is called a weighted homogeneous function because I am giving some kind of weight if that kind of relation will satisfy. So, here in case of classical one, I am scaling with same weight, here I am scaling with different weight and that is given like this. So, it is possible to show that this function is not classically homogeneous, but that it is weighted homogeneous of degree 6. You can easily able to check by substituting the degree of x_1 by 2, it means that λ^2 you can put or k^2 you can put and after that x_2 in place of x_2, you can actually scaled by. This kind of scaling you can do for the x_2 and x_1, then you will be able to check this kind of property.

Now, it is possible to show that weighted homogeneity will satisfy several arithmetic properties. So, a and b are homogeneous if the degree of a is equal to the degree of b, and if you multiply them, then they will satisfy a simple addition property. If you divide them, then they will satisfy a corresponding subtraction property. If you scale a homogeneous function by some positive number, then the degree is not going to change. If you have a partial equation, then this kind of degree also comes into the picture.

So, now if you have an expression like $\mathrm{d}x/\mathrm{d}t$, then you can easily calculate the degree of x by using the degree of t in this way. A similar kind of thing one can easily see is that, at that time, if I am calculating the partial derivative of some function a with respect to x, and x is scaled in a weighted way, then finally I will get the corresponding expression for the degree. So, you can easily check this.

Now, I am actually trying to extend this kind of concept to the vector field. So, I am considering some kind of vector field that is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. After that, I am going to give some kind of dilation. So, with weighted dilation, it is possible to show that the degree of that dilation is defined as the degree of x plus q.

So, again, if I have this kind of function, then it is possible to show that the degree is 3, the degree of the function is 3, but if I club this with some kind of differential equation, then I have to talk about the degree in this particular way.

So, you can see here that if I have a differential equation and a weighted degree of time, suppose that is $-q$, then the degree of $\dot{x}_i$ is given by the degree of x_i minus the degree of t, which is equal to the degree of x_i. That is exactly the same as this. So, the degree of f_i is the degree of x_i, and the degree of t is q. So, plus q comes into the picture.

In this way, if I have some kind of differential equation, and if I map the solution in this particular way, k^{-q}, where q is nothing but the degree of time, and $D_k x$ I am going to map in a weighted way, then you will be able to see that the solution again becomes a differential equation that is invariant. That means I will get the same differential equation.

So, any differential equation $\dot{x} = f(x)$ is called a homogeneous differential equation provided that it satisfies this kind of property. This is basically derived from this particular definition.

Now, you can see that if the degree of x is equal to 1 and if I have a function whose degree is 2, but if that function is a vector field, then what is the meaning of vector field? If I have a differential equation like $\dot{x} = x^2$, then x^2 has degree 2 and x has degree 1. So, basically, $2 - 1$ comes into the picture, and that gives minus 1.

In another way, in order to match the degree of homogeneity of both sides, you can see that one side is $\mathrm{d}x/\mathrm{d}t$ and the other side is $\lambda^2 x^2$ or $k^2 x^2$. So, how do you match them? It means that the degree should be minus 1. So, the degree of homogeneity is minus 1. In this way, we can basically define the degree of homogeneity.

Now, suppose I have this equation. Here, you can see that x_1 is scaled by 3. So, you can apply λ^3 into x_1, and a square into x_1, or k^2 into x_1, or k_2^2 into x_2.

So, this is x^2 and this is 3. If you replace it here, then what happens is that on this side, I have λ^3, and on the other side, I have λ^2. So, how do you match this? In order to match, you have to take a negative degree now. Then only $3 - 3 - 1 = 2$. In this way, the degree of homogeneity becomes minus 1.

Here, you can see that $\lambda^{1/3}$ comes into the picture, and here also λ^1 comes into the picture, but on this side, I have λ^2. So, how do you match this? It means that I have to subtract 1, and for that reason, the degree of homogeneity is minus 1.

So, in this way, we are able to define the degree of homogeneity. It is possible to show, and in the next class I am going to establish, that if a system is asymptotically stable and the degree of homogeneity is negative, then the whole system is also finite-time stable. It means that using asymptotic stability, I am able to talk about finite-time stability.

Now, it is possible to show that all degrees of homogeneity can be normalized to plus or minus 1. For that, you have to apply this kind of transformation. This transformation is also called the Euler transformation, and then you can scale all degrees by plus or minus 1.

So, what we have understood so far is that we have understood homogeneous differential equations, weighted homogeneity, which is basically invariant under scaling transformation, and weighted homogeneity degree, which basically depends on what kind of weight you are giving to x and t. We have also seen several examples.

To calculate the weighted homogeneity degree of a differential equation, it is possible to normalize the weighted homogeneity degree. This is a very, very powerful tool because I have already told you one of the results without proving it: that asymptotic stability in a stable system with negative degree of homogeneity leads to a finite-time stable differential equation.

Now, I am going to extend this result to differential inclusion, because in this course, I am going to deal with differential inclusion. In exactly the same way, one can talk about the degree of homogeneity for differential inclusion. Differential inclusion will also satisfy the exact same kind of invariant property under transformation.

You can easily check that this differential inequality is basically less than or equal to zero. This is not a differential equation. Again, you can easily check the degree of homogeneity, where the degree of homogeneity is plus 1. Please do it by yourself.

Another important aspect is stability. We know how to talk about the stability of a differential equation. Using epsilon and delta, we are able to talk about the stability of any continuous differential equation.

So, autonomous differential equation and what is our main intention whenever I have autonomous differential equation. So, this differential equation I am giving an initial condition, and after that, we are generating the trajectory. After that, we assume that 0 is the equilibrium point. So, if I start nearby of the 0, so meaning of stability suggests us that you are going to remain in the nearby of the 0.

So, how are we basically doing that? We are constructing two balls. We are constructing an epsilon ball. So, I can select an epsilon ball as large as possible. So, corresponding to the epsilon ball, there exists a delta ball. And after that, what are we basically doing?

We are making an initial condition x that is going to lie inside the delta ball. So, that is less than delta, that will imply that $x(t)$ is going to lie in the epsilon ball. In this way, we are basically defining for all $t \geq 0$. In this way, we are basically defining the Lyapunov stability for the autonomous system.

Now, I basically have differential inclusion. What is the meaning of differential inclusion? I have infinitely many differential equations at a certain point. So, again stability can be visualized using the Lyapunov way exactly the same as this, and I am able to talk about the convergence of the solution as t tends towards infinity, with all $x(t)$ tending towards 0. In this way, we are able to talk about global asymptotic stability. It means that I will start anywhere inside the space and I can converge asymptotically to 0.

Another notion is globally uniformly asymptotically stable. Uniform means for all initial conditions. It is possible to show that whenever I am talking about uniform, the word uniform, then we are guaranteeing that for any initial condition in this region, $x(t)$ is going to enter inside this region in finite time. I am not saying that $x(t)$ is tending towards 0 in finite time. I am just giving the guarantee that inside this ball, the trajectory is going to actually enter in some finite time t, and t is depending on R and ϵ.

So, actually I am going to construct some examples. In this example, you can see that first I have proved the Lyapunov stability. How can one prove Lyapunov stability? First, you have to solve it. You have to choose an epsilon ball, and after that, corresponding to the epsilon ball, you have to select a delta ball. Then you will be able to show that for every epsilon, there exists a delta such that if you start from the delta ball, you will remain inside the epsilon ball.

Convergence as t tends towards 0 and t tends towards infinity shows that this is also established, and due to that reason, this is globally asymptotically stable. This is true for all initial conditions. But if you change the example, now here in this example, you can see that the solution can be written like that.

So, whenever you are talking about stability, Lyapunov stability, first you have to solve the equation. But here a new phenomenon comes into the picture. After this time, you can see that I am able to guarantee that all solutions are going to enter inside this particular ball. As epsilon is going to decrease, then this time is going to increase, and due to that reason, after infinite time only I can converge to the equilibrium point.

So, somehow in this way, global uniform asymptotic stability is a stronger notion than asymptotic stability, and this is not uniform. You can take this example again, you can solve it, and it is possible to show that the size of the ball depends on the initial condition, and for that reason, this is not uniformly asymptotically stable.

I have constructed another example of a second-order system. This is a fully decoupled second-order system. Again, you will be able to calculate the time, and then you will be able to give the same kind of guarantee that if you start from a ball of radius r, all initial conditions inside this ball will enter the epsilon ball in finite time. Due to that reason, a globally uniform stability condition comes into the picture.

So now it is time to conclude this lecture. So, what have we seen in this lecture? We have seen the concept of homogeneity, after that weighted homogeneity, and after that homogeneous differential equation, homogeneous differential inclusion, and how to extend this kind of concept to talk about the stability. That is the topic of the next class.

So far, I have just introduced two different notions of stability: asymptotic stability and global uniform asymptotic stability. So, with this remark, I am going to end this lecture. Thank you very much.