Sliding Mode Control and Applications

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Lecture-32

So, welcome back. In the previous class, I was talking about the existence and uniqueness of the solution of a differential equation with a discontinuous right-hand side. And that is also the so-called differential inclusion. And commonly during sliding mode control, that kind of differential equation comes into the picture. And we have learned about two different ways to tackle that problem. One is Filippov regularization method and another is equivalent control method.

In this lecture, I am going to extend the concept of first-order sliding mode control to higher-order sliding mode control. So, I am going to give you the definition, and after that, I will actually prepare sufficient background so that you can understand the higher-order sliding mode control, and we are also trying to understand the motivation for the higher-order sliding mode control. So, the purpose of the discussion. I have already told you that higher order sliding mode control and we all are already well aware that one of difficulty of classical sliding mode control that is also called first order sliding mode control that is called chattering.

So, during implementation we have somehow finite amplitude and frequency vibration and that is actually dangerous for the actuator and in that way this kind of somehow oscillation or vibration can affect the overall performance of the system. And first-order sliding mode control is limited to the relative degree-one system. What is the meaning of the relative degree? We have already seen during the input-output linearization that if I have a system that is somehow affine in the control system we are discussing, and suppose I have an output equal to $y = h(x)$. So, if I take this as a sliding variable, you can see that control explicitly appears on $\dot{y}$ and then basically we are applying the discontinuous control to maintain $y = 0$, and after that, the equivalent value of y is also equal to 0 if we apply some equivalent control method, but it is possible to show that several times control does not explicitly appear in the first-order derivative.

Now, in that particular case, how to maintain all subsequent output and their higher order derivatives equal to 0 is one of the challenging problems. One of the ways we have seen that is by some kind of control that is called terminal sliding mode control. And terminal sliding mode control has some limitations that we have actually learned. Due to this reason, we are now going to extend the notion of sliding mode control itself for the higher relative degrees. The system means that if I have a sliding variable equal to h and if control explicitly appears in the r-th derivative, then we are going to define sliding mode control, which is called higher-order sliding mode control.

So, we are going to learn in this particular lecture, and most of the time it is possible to show that particularly some class of algorithms is such that even if you have partial knowledge of the derivative, you can still be able to design the control. Due to that reason, this approach has become very, very popular. This approach was somehow coined by Professor Levan and Emiliano, and subsequently by Professor Leonid Friedman and several researchers who actually worked in this particular direction. So, in this lecture, I am first going to extend the notion of sliding mode control for arbitrary relative degree. This approach also gives us the beauty that we can somehow mitigate or remove the chattering.

And obviously, I have already told you that some class of algorithms just requires the approximate knowledge of the derivative. So, what is the main idea of sliding mode control? What we have seen is that we are defining some kind of manifold in higher dimensional space; the manifold might be a function of time and state. Most of the time we are taking σ as a function of x only. And after that, what we are basically doing is assuming that this is continuous and differentiable, and after that, by taking the derivative, control actually comes into the picture. We are assuming that the system is actually affected by some kind of matched or unmatched uncertainty; then we are defining some kind of discontinuous control action.

So, in this way, we are basically talking about sliding mode control. Now, in this class, I am going to extend that notion where discontinuous control will appear at the r-th derivative degree. What does it mean? If σ is the output of some system, then if I take the r-th derivative, control is going to explicitly appear. So, now what is our aim? So, suppose that I have some kind of autonomous system and this is the output; this is now a sliding variable, and now, by using input-output linearization, I am going to calculate the subsequent derivative. So, since I am going to calculate the subsequent derivatives, $\sigma, \dot{\sigma}$ up to $\sigma^{(r-1)}$, all are continuous.

Otherwise, I cannot proceed towards the derivative. And now, at the $\sigma^{(r)}$ power, control is basically going to explicitly appear. So, somehow I will gain some kind of control. Now, what is our goal? To design control such that I can keep $\sigma, \dot{\sigma}$, and $\sigma^{(r-1)}$ equal to 0 in finite time. So, obviously again this control is discontinuous control.

I am going to define that using continuous control you are not able to solve this problem. So, those kinds of things I am going to actually quantify in this particular lecture. So, this set is also called the Filippov set. What is the meaning of the Filippov set? So, whenever discontinuous control comes into the picture, where the solution is not defined or the right-hand side of the differential equation is not defined, then we are going to apply the convexification, and after that, we will complete the definition of the vector field at that particular point. So, now, what is the main motivation of the sliding order? We will take the output, and after that, we will try to understand when the derivative or at what derivative control is going to explicitly appear, and that is going to give us the order of the sliding mode control.

So, during this process, we have already learned input-output linearization and we have understood that whenever I am going to take the derivative of σ, so σ is nothing but

$$\frac{d\sigma}{dx} \cdot \frac{dx}{dt} = \frac{d\sigma}{dx} \cdot \dot{x}.$$

So, I have $d\sigma/dx$, and after that, some term $b(x)$ comes into the picture. So, if that is equal to 0, then I will proceed towards the second derivative of σ, and second derivative control will explicitly appear. But if this condition is again satisfied, then the third derivative control is going to explicitly appear.

In this way, if I proceed up to this point, then it is possible to show that the $\sigma^{(r)}$-th derivative control is going to explicitly appear. And in this way, we are going to define the sliding order. So somehow, relative degree of the system and sliding order is related. Now, the same kind of things I have actually written here. So, the meaning of the Lie derivative of σ with respect to f is nothing but the gradient of σ multiplied by f.

So, the same language I am going to use here. So, I am going to take the first derivative, and if this term is 0, then I will proceed to the next derivative. So, r is the smallest number when the term which is multiplied with control is non-zero, and once this is non-zero, then only control will explicitly appear. So, with help of Lie derivative, I can able to define the relative degree of the system.

Now, suppose that I have a time-varying system that is also called a time-dependent system. So, you can see here that again I am able to proceed just like the previous case, but here just what we have to do is assume one kind of state that is a fictitious state, and that state is nothing but

$$\frac{dt}{dt} = 1,$$

and that state I am going to observe here, and then actually now I have a state like

$x_1, x_2, \ldots, x_n$ and one more state I have t. In this coordinate system, if I represent this system, then again the system looks like an autonomous system, and then again I can proceed towards the relative degree of the system. Due to that reason, I am now going to talk about the regularization problem. What is the meaning of the regularization problem? I have a dynamical system that looks like this.

Here I am assuming that this $a(t, x)$ and $b(t, x)$ are not completely known to us. So, whenever you are doing mathematical modeling, a lot of approximation comes into the picture. Now, what is our job? Our job is to design a control such that I can be able to force σ to equal 0. If the relative degree is r, then I have to set $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ equal to 0 using some kind of feedback, and here it is possible to show that this feedback is nothing but some kind of discontinuous feedback. So, somehow this is a black box problem.

It is possible to show that if I have just information of σ, then I need some kind of differentiator or some kind of observer to estimate the higher order derivative and then in presence of noise and then we can able to design some kind of discontinuous control to achieve this particular job. And what is the difficulty? These terms, A and B, are unknown. We have only partial knowledge of that, and for that reason, this problem is called the black box problem. Now, what is the meaning of relative degree or sliding order? So, I have already told you that you can start from the sliding variable and take the r-th derivative; then, only control will explicitly appear, where h is defined by the Lie derivative and similarly g is defined by the Lie derivative. Here you can see, since this term $A^T x$ and $B^T x$ is not known and due to that reason, I can able to assume that $H^T x$ is also not known and due to that reason, I am going to assume something like this.

Why am I assuming that $K_M > 0$? The lower bound is also 0, because I want the control not to change the sign. If they change the sign, then I need some extra care. Now, h can contain any value between $-C$ and $+C$. For that reason, I have represented it with absolute value. Now, I have to design control using the information of this such that $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)} = 0$ in finite time.

That is our regularization problem. Now you can see that I am able to design some kind of dynamic output feedback. Why is this called dynamic output feedback? Because I am assuming that σ is an output and I know its higher-order derivative, and if that is not known, I have to calculate it. What do we have to do? You can see here that $G^T x$ is going to lie between $-K_M$ and K_M, but here $H^T x$ can lie between $-C$ and $+C$. So, I have two extreme kinds of dynamics.

One dynamic is given like this; another dynamic can be given like this. Okay, you can also be able to take the upper bound here. Now, what do we have to do? I have to design u such that I can stabilize this system as well as this system because now C is uncertain. So, C can take any value within some time interval and due to that reason control is capable enough to stabilize both system together. So, now using continuous control, I am not able to achieve this.

This kind of thing is already proven in the literature. And due to that reason, we now have to look into the discontinuous control. And obviously, now our objective is to make all variable $\sigma, \dot{\sigma}, \ldots, \sigma^{(r)} = 0$ for a finite time. So, now the overall problem is expressed like this, and this is the symbol of the differential inclusion set-valued map. So, I have infinitely many differential equations; now you can see here between $-C$ to C and similarly $-K_M$ to K_M.

In this inclusion, what am I going to do? I am going to design u, and obviously, since this problem is converted into an inclusion problem, I now need some kind of method to deal with the solution, existence, and uniqueness of the solution. In the previous class, we learned about how to deal with the solution of this class of systems. One more important point you can see now is that we have started from this particular mathematical model. Now, in this particular mathematical model, we do not have any perfect knowledge of the system dynamics. Still, I can able to stabilize all the physical variable because σ is some kind of physical variable and they are higher order derivatives.

I can able to stabilize in the finite time. And for that reason, this problem is also called the black box problem. Now, whenever we have discontinuity in the right-hand side of differential equation and discontinuity lie at the 0 measure, we have already seen one can able to apply the Filippov procedure and then one can able to talk about existence and uniqueness of the solution. So, the same kind of things I am going to apply for this class of the system. Now, whenever we are going to design the control, obviously in this lecture I am not going to talk about how to design this control.

For that, I need sufficient background and one of the easiest ways is if you follow some kind of scaling property. What is the beauty of the scaling property? If you design some kind of control such that you are able to stabilize your system in some region, then it is possible to show by scaling that the same kind of property will satisfy everywhere in the space. So, if I have a linear system, then everything is easy because the linear system is going to satisfy the properties of additivity as well as homogeneity. But that is not true for the nonlinear system. Now, we are going to extend some notions that are related to linear systems to the non-linear system, and then we are going to design the control; that notion is called weighted homogeneity or simply homogeneity. So, we are going to learn weighted homogeneity in the next lecture, and at that time we will try to understand the physical interpretation and how to design r-sliding homogeneous control such that I can stabilize this particular system in finite time. So, you can see that several times I am giving emphasis to the finite time. So homogeneity basically gives us a systematic way to proceed toward that class of problems.

Now, obviously, I have a discontinuity. So, in the previous class, we explored that whenever I have some kind of manifold, a discontinuous manifold, and if I have just a single input system, then I am going to divide the whole state space into two regions. One region is $S > 0$; another is $S < 0$. And then what I am going to do is apply the regularization method so that the entire trajectory is basically maintained along this particular sliding surface. So, after that, we have also calculated the projection of the control, and after that, we are defining convexification such that I will talk about a unique solution in all forward time. Here, the problem is just like I will start on σ^r. So, before σ^r, somehow I have a structure; you can write if you define σ as σ_1.

So,

$$\dot{\sigma}_1 = \sigma_2, \quad \dot{\sigma}_2 = \sigma_3, \quad \ldots, \quad \dot{\sigma}_r = \text{some discontinuous term.}$$

So, again, now here you can see that the Filippov kind of solution comes into the picture. So, and again I can able to apply the convexification method. This is nothing but a projection. We already learned during the previous lecture. Why am I repeating again? Because we are trying to understand whether I have some kind of situation like this in which we have to design discontinuous control. And then how to talk about the existence and uniqueness of the solution, because that is very, very important. This will give you the room to design several different kinds of higher order sliding mode control for this uncertain system.

That is our main goal. Now, in the previous class, we have already seen that during the convexification process, two terms, α and $1 - \alpha$, come into the picture, and in order to calculate α, what do we have to do? I have to actually set the inner product or dot product of $\frac{\partial S}{\partial x}$ and f_0 equal to 0. So, this represents the inner product. So, if that equals 0, then we will get α equal to this, and in this way, I can put it inside the sliding dynamics, and this is the equation during the sliding. So, this equation is also called the convexification equation or convex equation because this equation we have actually obtained by the convexification of the vector fields f_n^+ and f_n^-.

And how basically sliding mode control comes into the picture, if we are able to maintain f_n^+, which is here, and f_n^-, which is here, of opposite sign like this, then I can maintain trajectory along this particular sliding surface. So, this is the condition of sliding. I am going to extend the same kind of things for the higher sliding mode control as well. Now, let us try to see how differential inclusion comes into the picture. There are several ways in which you can replace the classical system with differential inclusion.

So, let us try to take the example of a perturbed system. So, here $d(t)$ is perturbation and $d(t)$ is going to lie between $-a(x)$ to $+a(x)$. Then you can see now for all $-a(x)$ to $+a(x)$, I have to talk about the stability of this system. So, how do we talk about the stability of the whole system? Again, I am able to represent the whole system in the form of the differential inclusion. And obviously, I am able to apply the Filippov construction method whenever a discontinuity point comes into the picture.

So, in this way, this will give us the room to actually model some kind of uncertainty which is entered inside the system. Almost all system is actually affected by the perturbation. At that time, the automatic differential inclusion representation comes into the picture. So, differential inclusion is very common, because each and every system is actually affected by disturbance.

So, at that time you can actually be able to replace your ordinary differential equation with disturbance by differential inclusion, and then you can proceed towards the Filippov construction for the existence and uniqueness of the solution in the presence of the disturbance. It is possible to show that whenever discontinuous control or some kind of switching kind of control comes into picture for the forced system, then still actually you have to realize solution using some kind of Filippov regularization way. So, here this control is switched based on some kind of state-dependent logic, then it is possible to show that again I can able to handle this system at least two different way in this particular course. So, one way is the Filippov construction. So, what basically Filippov construction is suggesting that this is suggesting that how to maintain unique solution in all forward time.

So, they are defining the vector field along this line through convexification. We have just seen what kind of vector field is going to act whenever we slide by this particular expression. Before that we have classical problem. So, existence and uniqueness are satisfied due to classical Lipschitz kind of continuity. Now, whenever we are at the discontinuous plane, then at that time by convexification, I can able to talk about the existence and uniqueness of the solution.

So, using some examples, we are trying to understand this. And what is the benefit of the equivalent control method? This ensures that if some kind of matched uncertainty comes into picture, then how to actually maintain system trajectory along this sliding surface in case of the matched uncertainty. And how do you do that? So, for that, we have to take the average value of $\dot{S} = 0$. How do you calculate the average value? You have to pass through a low-pass filter. So, basically we are talking about this kind of dynamics during the sliding whenever we are going to utilize the method of equivalent control. So, equivalent control method suggests us that $\dot{S} = 0$.

So, how do you calculate $\dot{S}$? So, $\frac{\partial S}{\partial x}$ multiplied by $\frac{dx}{dt}$. So, $\frac{dx}{dt}$ is $f(x)$, and we are going to convert it into an equivalent, and due to that reason, this relation comes into the picture. So, what is an important note, Filippov and the equivalent control method may lead to different solutions, unless F satisfies some kind of specific regularity assumption. So, we are going to discuss the regularity assumption; then you will be able to show that both solutions, the Filippov construction, which is based on the convexification or regularization method, and equivalent control are equal.

And we are going to highlight that class of system. So, let us first take this class of systems. Obviously, this is not affine in control systems, because of the presence of u^3.

Suppose that the sliding variable I have defined is a linear combination of x_1 and x_2, and this is the control. Inside control, you can see that this is sliding surface $x_1 + x_2$ coming into the picture, but I have defined logic such that it is multiplied by x_1. So, what is my suggestion? You can take this system and you can simulate this system in MATLAB.

It is possible to show that $s_s < 0$. So, whenever $s_s < 0$, it is possible to show from Filippov that all trajectories are basically attracted towards the sliding surface. Now, I have two different directions. So, attraction is not only enough because the trajectory may be different from the equilibrium point. So, I have to show that once the trajectory converges here, they will maintain along this line.

So that is also required. So, you can see in this particular example, this is a very good example, which will also tell you that even if all trajectories are attracted towards the equilibrium point, we will experience some other phenomena. So, $\dot{s}$, I have calculated how to calculate $\dot{s}$ simply as $\dot{x}_1, \dot{x}_2$. I am going to substitute the dynamics, and then you just contain the information of the sign. So, I can approximate that like this, and after that, if I multiply by s, then I will get this kind of expression.

So, you can see that the sliding condition is satisfied. So, now that the sliding condition is satisfied, I am able to calculate the equivalent control method, and using the equivalent control method, you can see here that the equivalent control is $\frac{1}{2}$. So, I can able to substitute equivalent control in the original system. So, if you substitute in the original system, then you have a reduced order system like this: $S_1 + X_2 = 0$. So, you can see here that what kind of things happens, all trajectories are attracted towards the sliding surface, but reduced order dynamics is unstable, because now if you plot x_1 dynamics, then what happens, this is positive number. So, if you start, if you take any non-zero number, you are going to diverge.

Now, let us try to see the Filippov construction for the same problem. So, how are we basically proceeding towards the construction? We are calculating α with the help of the sliding variable, and after that, we are going to replace it here and here. In this way, it is possible to show that after replacing, I have resulting dynamics like this. And what is your conclusion now? Your conclusion here, you can see that negative sign comes into picture. It means that our system is asymptotically stable because x_1 is tending towards 0.

So, $x_1 + x_2$ tends towards 0. So, x_2 is also tending towards 0 as $t \to \infty$. Utkin's regularization method or Utkin's equivalent control method basically tells us that overall system is unstable, but what Filippov construction was telling is that overall system is stable. So, actually, the stability of your system fully depends on the way you are going to construct the solution. Now it means that system differs based on the method you are going to use. Origin is locally unstable, and in the case of Filippov, the origin is asymptotically stable.

And this actually gives us the idea that Filippov method provides a more robust analysis for the system with sliding mode, and you can able to simulate this system, but if you change the example, then your conclusion will change. You can see here using the equivalent control method, now what happens is that we have sliding with respect to x_1 because control is only applied with respect to x_1. So, reduced order dynamics comes like this.

So, this becomes reduced order system after the sliding and now this is stable, but if you apply Filippov regularization method then this becomes unstable and if you actually simulate this system then this system is stable. It means that Utkin's equivalent control method gives us the proper way to talk about the solution of this system, and due to that reason you cannot rely on one way of the solution. So, now I have to find some kind of common class where both solutions are equal, and it is possible to show that if you take an affine control system, and if regularity condition, what is the meaning of regularity condition: $\frac{\partial S}{\partial x}g(x)$, if that is non-singular enclosed to the sliding manifold, then it is possible to show that both solutions will give us the exact same result. So, this is the regularity condition.

Now it is time to conclude this lecture. So, what have we seen? We have seen how to define higher-order sliding mode control, how to construct the Filippov solution, as well as the equivalent control method. So, which method is better, we can comment on notably. So, depending on the problem, we have to actually select the method, that is the conclusion, but at least for affine control systems, basically both methods are the same, and due to that reason, you might have seen that during the previous modules, whenever I have a system, I am converting it first into some kind of affine control form, and then we are basically designing the first-order sliding mode control. So, with this remark, I am going to end this lecture. Thank you very much.