Sliding Mode Control and Applications

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Week-07

Lecture-31

Welcome back. In the previous modules, I was discussing the sliding mode control and its application. So, particularly, I was talking about the first-order sliding mode control, also called classical sliding mode control. So, what were our observations in order to talk about the application of sliding mode control, for example, for a stabilization problem, tracking problem, observation problem, and after that, implementation issues everywhere? What have we done? We have taken the help of some kind of differential equation, and its right-hand side is actually discontinuous. So, in order to achieve the occurrence of sliding mode control, we need some kind of differential equation with a discontinuous right-hand side. So, in this lecture, I am going to talk about some analytical aspects of the differential equation with a discontinuous right-hand side.

So, for the purpose of the discussion, we are trying to understand the existence as well as the uniqueness of the solution with continuous and discontinuous right-hand sides. What does it mean? That suppose I have some kind of equation like $f(t) \cdot x(t)$; in this course, most of the time I am going to talk about this system $\dot{x} = f(x)$. So, this system is called an autonomous system, while this is called a non-autonomous system. So, if you are going to use some kind of gain which is also dependent on time-varying gain,

So, at that time, this class of system comes into the picture; otherwise, I have this class of system after the substitution of control, and our original control system looks something like $f(x, u)$; obviously, we have a disturbance also. So, but in order to talk about the existence and uniqueness I am going to restrict myself without disturbance and after that obviously I will incorporate disturbance. And what is our methodology? We are designing some kind of control $u = \alpha(x)$; here, α may be continuous or discontinuous depending on the situation. And then we are talking about the closed-loop system, the stability of the closed-loop system, like this or this. So, one of the very important questions some of you may learn during the non-linear control course is, is it possible that all differential equations will represent physical systems? The answer is negative.

Not all differential equations are going to represent the physical system. So, in order to represent some kind of physical system, two requirements are there. First, I have to give guarantee that a solution exists, and after that we have to also give guarantee that a unique solution exists. And both aspects now we are going to look into if I have a differential equation with a discontinuous right-hand side. Because, in order to maintain sliding, we are basically using an equation that is somehow for the first-order case, which is given by this.

Okay and you can see that here the right-hand side is discontinuous so how to talk about existence and uniqueness that is very very important and using this approach that is called so-called Filippov approach I am going to handle the differential equation with discontinuous right-hand side although before starting sliding mode control I have to taught this, but actually these things are so much theoretically involved. So, the newcomer is not feeling motivated whenever you talk about this, and for that reason, I have postponed the discussion in the middle of the course. So, what is our main motivation? I am going to look into the sliding mode again, as well as equivalent control. What is the meaning of equivalent control? We have seen that once a trajectory is on the sliding surface, the rate of change is also equal to 0. So, what is the outcome? Obviously, we are trying to understand or revise the existence and uniqueness of solutions of differential equations where the right-hand side is continuous.

Then I am going to involve the differential equation where the right-hand side is discontinuous, and at that time we are going to utilize the notion of Filippov convexification. And, then we are going to understand the sliding mode behavior such that suppose that if you come up with some new algorithm, whatever I taught in this course that is not same like your algorithm, then is that algorithm correct or not. So, how do you check? So, for that reason, this lecture is very, very important. We are also trying to understand the notion of equivalent control, the mathematical notion of equivalent control, and their invariance principle. So let us start the discussion.

I have already told you that in this course, we are going to start with some kind of system where, basically, I have X_t, U_t, and obviously D_t, and this disturbance we are talking about is the matched uncertainty. We have also talked about the unmatched uncertainty, those kinds of uncertainty that enter through the control channel. So, for simplicity of presentation I am assuming that $d(t) = 0$ and $u(t)$ is a function of $x(t)$, then obviously I will get this system. So, now this is a differential equation. So, one equivalent representation of a differential equation is the integral equation, and it is possible to show that these two are not always equivalent.

Some kind of condition on the right-hand side of the differential equation is required such that I can give a guarantee that the differential representation is equal to the integral representation, but only if f is continuous. So, a continuous function is always integrable. So, in this way I can able to give guarantee that if this system is situated here, I will give the initial condition that is $x(t_0) = x_0$ in this particular problem and after that I will generate $x(t)$. So, the only necessary condition that will give us the existence of solution is the right-hand side is continuous. But what is our observation in this course? Right-hand side is discontinuous.

So, let us try to understand the issue. Here, what I am going to do is take a very, very simple system. So, here I am going to assume that $x \in \mathbb{R}$. You can think of it this way: suppose that you have a system like this and you want to design some kind of on-off control. On-off control is very, very popular in practical scenarios.

Suppose that your engineer visited your site and they decided this kind of logic. What is logic? If $x \geq 0$, at that time they set controller such that that will behave like -1 and $x < 0$, then that will behave like $+1$. So, now I am going to check whether this logic is correct or not. So, how can we check the correctness if a solution exists? So, one more important thing you can see is that the left-hand side of the differential equation is differentiable. It means that $x(t)$ should be continuous because without continuity I cannot able to talk about the differentiability.

So, whatever solution that comes into picture that should be continuous. So, let us analyze this system. You can see that I have selected the initial condition equal to 0. So, once you will keep 0, then at that time what happens? That $\dot{x}(t) = 1$. So, if $\dot{x}(t) = 1$, then for all sufficiently positive time, you can see here that $\dot{x}$ becomes -1.

Why? Because $x \geq 0$ at that time, that becomes -1. So, now this is going to contradict about the continuity of $x(t)$. Why? For continuity, the left-hand limit, right-hand limit, and limit at this point should be equal, but what happens is that when I start with $x = 0$, suddenly $\dot{x} = -1$ comes into the picture. So, somehow no classical solution exists.

So, if you apply this kind of logic, your system will obviously not work. Now, what am I going to do? I am going to change the logic. So, now, I am going to make a simple modification. What am I going to do? At $x = 0$, I am going to put 0. Now, in this situation, you can see that as $x \geq 0$, the solution is given like this.

You can simply check the solution of the differential equation, which is given like this. Now, $x < 0$, at that time you can able to see that this is the solution. So, $t = 0$, if you substitute, then automatically you can generate $x(t) = 0$. What is the conclusion? If we modified the right-hand side of the differential equation like this, then I have all classical solutions for every initial condition. It means that even if the right-hand side of the differential equation is discontinuous, we can talk about the solution.

So, I know the solution, classical solution I know on this side, classical solution I know on this side. So, how can we confirm the classical solution? You can just remove the discontinuity point, and after that, you can apply the Lipschitz condition. Now, what am I going to do? I am going to take the convex hull. I will define the meaning of convex hull using some figures in the next slide.

So, this will represent the convexification of these two vector fields. So, whatever is on the right-hand side of the differential equation, I will also define as a vector field. So, in this way, whenever discontinuity comes into the picture at the 0 measure, what is the meaning of 0 measure? You can see that this side and this side are going to separate by 0 distance. At that time, I will utilize this convexification process. And after that, I will be able to define the solution at each and every point. So, one of the particular case, one can able to define convexification like this.

In literature, there are several other ways, at least 8 to 9 different ways to replace discontinuous points with continuous ones. In this course, I am going to utilize two ways: one is the Filippov way, another is equivalent control way. So, now convexity, what is the meaning of convex? It means that α should lie between 0 and 1. Now, you can, if you are here, you can see that you have two vector fields and after that, now I have to maintain f_f such that I can able to move along this line. So, this process is called Filippov convex method, because we have used convex combination and you can see that now even if disturbance comes into picture.

So, these two vectors are going to adjust themselves such that I can always be able to slide along this line. So, this is the definition of the Filippov differential equation; I have already told you that requirement. So, at least I have some solution because it is a non-empty closed convex locally bounded set with upper semi-continuity. It means that, somehow, I will talk about the forward solution. So, if you read through the literature, one of the very good books that is differential equations with discontinuous right-hand sides is actually the PhD thesis of Professor Filippov.

So, it is possible to show that I am able to represent whatever I have told you so far in this way. So, this is just a mathematical representation, and what this mathematical representation is confirming is that $\mu_n = 0$; it means that the set with distance 0 has measure 0; then I am going to exclude that. And after that, suppose that this point is discontinuous. So, through that, I am going to create a ball. So, I have the solution here and here, and after that, I am going to take the convex hull.

So, this is the interpretation of this entire language. Now, I have to give guarantee that under which situation I have sliding mode control. So what are our common observations? If I have one control, then this σ plane or summation plane is going to separate the whole space into two parts. So I will calculate the projections of F_1 and F_2. And after that, I will check this condition. If this condition comes into the picture, then, if you start anywhere, all trajectories are attracted towards this surface provided the projection of F_1 is greater than 0 and the projection of F_2 is less than 0.

The whole manifold becomes attractive. It means that if you start anywhere, you are going to be attracted to the manifold, and after that, you will be able to maintain it. And if this condition is satisfied, then a repulsive kind of thing comes into the picture. You can see that four different situations come into the picture. So, obviously in this situation there is no sliding; in this situation there is no sliding, and all trajectories will move towards x_2; here all trajectories will move towards x_1. Here you can see that all trajectories are going to diverge, even if some trajectories start on the sliding surface, which is also going to diverge.

So, in this case, I only have sliding mode control; please do not confuse the meaning here: one trajectory will go in this direction and this direction, and then their projection in this direction. So, in this way, this is the condition for the sliding mode, and that is given like this. The same kind of things, basically, I have written here: the projection of normal along F_1 is given by > 0 and F_2 is < 0; then I have sliding mode control and how to define the Filippov solution by taking the convex hull. Since I am going to maintain, I am going to guarantee that the entire trajectory is actually aligned along this surface. So, how do we give that guarantee? So, you have to make sure that normal and F_F are both perpendicular.

It means that their dot product is equal to 0. In this way, you can able to calculate the $\alpha(x)$. So, and if you apply $\alpha(x)$ here, it is possible to see. So, all trajectories start anywhere in the state space. Now, they are going to confine along this line.

And obviously, if this condition is satisfied, then we have repulsive sliding mode. So, we have to become very, very careful. Obviously, this kind of condition is actually based on the definition of which side I am taking: F_1 and F_2. So, please be careful whenever you are defining F_1 and F_2. So, now whenever we are talking about sliding mode control, you might have seen that I am defining it manyfold.

I have told you that various ways I am defining, various language I am using. So, sliding mode due to that reason this language we are preferring. $x > 0$. So, I have to design this on/off control. So, after that, we have to do on and off with a very high switching frequency so that I can maintain along this line.

It might be possible that sometimes these two vectors are not equal; that is quite possible. And by taking the convex combination, I can maintain along this particular line. So, what is the physical interpretation? Basically, what happens is that in every system there exists some kind of time delay, there are several impurities, and due to that reason, most of the time we are not actually aligned along this line. So, what happens is that the trajectory will come here, and they try to deviate like this, there, and after that, they try to deviate like this. So, for that reason, you are able to see this kind of motion.

So, now, I am in this principle; I can also calculate the value of α. So, how do you calculate? So, suppose some trajectory comes here, and I am going to actually analyze the time, which is $\Delta t_1 + \Delta t_2$. What does it mean? That $t = t$ somewhere that will hit this trajectory; at that time, I am going to isolate Δt_1. When at that time, I am going to apply $u + \Delta t_2$; then I am going to apply $u-$. $u-$ is on this side and $u+$, I am assuming, is on this side.

You can also assume in a reverse way. So, total distance since F_1 is somehow represents the vector field $f(t) = F_1$ in right-hand side. And due to that reason, you can see that $\Delta x(t)$, this is $\Delta x/\Delta t$. So, since this is finite, if $\Delta t \to 0$, then this is equal to this.

And due to that reason, I have written $F_1 \Delta t$. Similarly, $F_2 \Delta t$. So, total distance is Δx and total time is $\Delta t_1 + \Delta t_2$. And due to that reason, the average is $\Delta t/\Delta t_1$. So, in this way, you can also see that the same convexification comes into the picture. So, this is called the Filippov solution by the regularization method. Now, how can we achieve sliding mode control? What do you have to do? You have to calculate the derivative of this, and you have to force the rate of change along the sliding surface to equal 0; then I can maintain it along the sliding surface.

In this way, I can express α in terms of the sliding variable. So, during sliding, this vector actually comes into the picture. And, the condition of sliding is that $S^+ < 0$ and $S^- > 0$. So, in this way, in several places, I have used the idea that $\dot{S} \leq 0$ to enforce the sliding mode control. Another physical interpretation you can able to give based on the observation that is called equivalent control. So, this notion of equivalent control is proposed by the professor Utkin, who is one of the founders of sliding mode control.

So, what is their idea? He was saying that once you are at the surface, instantaneous velocity is basically not 0, but the average is equal to 0. So, I am going to utilize the same kind of idea to maintain along the sliding surface. And for that, I have adopted this system. In the next class, I will tell you why I am taking this class on the system called affine in control systems.

And what is our goal? Our goal is to maintain $s = 0$. And after that, what I have done is calculate the derivative of s. So, s is a function of x. So, if you calculate the derivative. So, here dx/dt comes into the picture and x is also a function of time.

So, due to that reason, dx/dt's. So, whenever we are writing this kind of notion, physical interpretation is x is also function of time, and due to that reason, first time I have to calculate the derivative with respect to $x(t)$, and then here just time is there, and due to that reason full derivative I am going to calculate, why partial here, because s is also function of x as well as t, t is implicit here. And in this way, I have defined this as $g(x)$, and that I am going to substitute here. So, whenever you are on the sliding manifold, at that time, equivalent control, some kind of continuous control, acts on the system such that the trajectory is going to maintain the sliding along the surface; that is the physical interpretation of the equivalent control. Several times we have utilized this kind of concept during the design process.

We have already seen several designs. So, what is the physical interpretation? Same kind of things I am telling that ideal sliding mode never occurs. Why? Because what happens? There are several imperfections. For that reason, I always have some kind of oscillation near the sliding manifold. So, it is possible to show that the overall system trajectory is governed by some kind of dynamics, which is low switching frequency dynamics.

So, how do you calculate that? You can just pass through the low-pass filter. If you pass through the low pass filter, whatever control comes into the picture is called equivalent control, and using equivalent control we have just the slow component, and using the slow component I am able to achieve the sliding mode. So, that is the physical interpretation, and you can also able to get that control equivalent control. So, how can you physically realize equivalent control? You can take any first-order filter or a higher-order first-order filter.

And then you can adjust the time constant; Δ is the width of the switching dynamics. So, it means this width. So, you have to make sure that Δ/τ, where τ is the time constant of the filter, is tending to 0. It means that you have to tune the low-pass filter perfectly in order to generate the equivalent control. So, we have already seen equivalent control generation and how to utilize this for control design, which we have also discussed during the chattering separation. Now, let us try to see one of the conditions why sliding mode control is insensitive with respect to matched uncertainty.

So, for that, I have taken this defined in the control system. Here, I have also defined this as matched uncertainty. What is the meaning of matched uncertainty? Those classes of uncertainty which enter through the control channel. Here, I am assuming that Δt is bounded. Now, I am going to take the derivative of sliding surface and after that what I am going to do, I am going to calculate the equivalent control.

How do you calculate equivalent control? It means that the rate of change of the sliding variable should equal 0. In that particular way, you can able to generate this control. Now, what can you do? You can substitute that control during sliding at this particular place. So, you can see that some beautiful phenomena come into picture. These two terms are going to cancel each other and basically, now I have whole dynamics that is free from the disturbance. It means that during sliding, by equivalent control method, it is possible to show that our system is insensitive, that is not going to feel the disturbance, insensitive with respect to any class of disturbance.

So, this is the mathematical proof of why sliding mode is going to handle the matched uncertainty. And this property is also called the invariance property. It means that if

I have matched uncertainty and once I am on the sliding manifold, and if our gain is sufficiently large, that is greater than the disturbance, it is possible to show that you are not going to lose this manifold. It means that if you start from this manifold, you will remain on that manifold for all forward time.

So, this property is called the invariance property. So now it is time to conclude this lecture. So, what have we seen? How to guarantee that a classical solution will exist? So, for that, we have seen the role of the Lipschitz condition, and somehow, if the right-hand side is Lipschitz, then I am able to utilize the idea of the completion principle, and then we can give a guarantee that there exists a unique solution. In case of the discontinuous right-hand side, we have seen two approaches that is Filippov approach of convexification as well as Utkin's method that is called equivalent control. So, using both philosophies, it is possible to show that if I have a differential equation with a discontinuous right-hand side, then I can tackle that part, and during sliding, I have some kind of continuous control that is going to act on the system, and we have also seen the invariance principle. So, with this background, I am going to end this lecture. Thank you very much.