Chattering Suppression via Boundary Layers & Observer Design

So, welcome back. In the previous class, I was talking about chattering. And what is the meaning of "chattering"? Chattering is a phenomenon that occurs whenever we apply some form of on-off control with a very high switching frequency, and for that reason, finite amplitude and frequency variations appear in the output or input; such variations can be dangerous, particularly from the actuators' point of view. Due to that reason, people start exploring solutions to mitigate or reduce the chattering. So, in this class, I am going to discuss some of the strategies that are commonly used in the literature to suppress the chattering. So, for the purpose of the discussion, we have to somehow minimize the harmful effects that basically come due to the implementation of the sliding mode control.

And this is also one of the limitations. So, whenever we talk with any industry person and suggest they apply sliding mode control, their first answer or question is usually, "No, no, no, this is not suitable because sliding mode control is very harmful; it will create chattering." So, now you have to convince them that chattering is just due to some kind of implementation restriction. So, now what is the immediate solution? Try to replace that discontinuous control with the continuous one.

So, how do you do that? So, if you see the signum function, the graph of the sign (x)function looks like this, where this is $x$and this is sign (x). For $x > 0$, this is $+1$; for $x < 0$, this is -1. For $x = 0$, it will lie between $-1$and 1. In some books, people are using this kind of notation; in some books, people are assuming that $x = 0$is not defined. So, in reality, that kind of situation will never occur, and for that reason, we are just switching with either $+1$or -1.

So, now what kind of suggestion actually came into the picture? Because due to this high switching frequency near $x = 0$, some kind of finite amplitude and frequency vibration comes into the picture, both on the output side as well as the input side. So, people have suggested what you can do: since due to the discontinuity at $x = 0$, chattering kinds of phenomena come into the picture, you can replace it like this. So, in this region, I now have basically some kind of linear function, and beyond that, I have a saturation function. But it is possible to show that during this replacement, I can lose the sliding mode beauty during this linear region.

But in several practical problems, it is possible to show that that kind of compromise is okay. And in that particular situation, what you can do in order to reduce chattering is replace the sign $(\cdot)$ function with the saturation function. So, this approach is known as a boundary layer approach. Another approach is actually based on the observer; I have partially discussed it whenever I was talking about the DC motor case study and the sliding mode observer design for the DC motor. So, in order to prevent the high-frequency switching, which causes the wearing and tearing inside, particularly the mechanical actuator, we are now going to play some kind of game between robustness and smooth operation.

So, obviously, I can lose sensitivity with respect to perturbation whenever I am in this particular region. With respect to match uncertainty. But, it is possible to show that one can deduce by replacing the signum function with the saturation function. So, in order to design a controller, which is based on the saturation function, what am I going to do? Again, I am going to start with a very simple plant. Obviously, you can able to extend this result for any order of the system.

Here, I am assuming that the actuator through which I am going to apply control is represented by a second order. Most of the actuators actually fall in this particular category. So, for that reason, we have made this assumption. We are also assuming that whatever parameter a is an unknown parameter, but its lower bound and upper bound are known to us. A similar kind of thing is associated with b, but you can see that b, I am assuming, is not going to change the sign; for that reason, the lower bound of b is also greater than 0.

And what is $\omega(t)$? $\omega(t)$is nothing but a control variable, because now what I am going to do is plant, and now I have an actuator. So, whatever output of the actuator that becomes the input to the plant is nothing but w. So, I am going to implement the original control to the actuator. Whatever disturbance I am considering is uniformly bounded. What does it mean? So, it means that this bound exists for all $x$and for all $t \geq t_0$.

t_0 is the starting time of the experiment. These are the representations of the unmodeled actuator. So, that is actually represented like this. And here, $p$is nothing but the Laplace variable. I have not used $s$because, in sliding mode literature, people frequently use $s$as a sliding variable.

So, in place of s, I have just used p. And $u(t)$ is the actual control input. So, the actual control input we are going to give to the actuator, and the actuator actually works like some kind of operator that is going to provide us with the output $\omega(t)$, which is physically applicable to the plant. Now, here I am assuming that $\omega$is the unknown actuator bandwidth and $\mu = 1/\omega$, which is a small time constant. So, that kind of assumption I have is.

So, this is the actuator model. So, this is the plant model; this is the actuator model. Now, I am going to define the control objective because I have to formulate the control problem. So, what is the control objective? So, I have to force $x(t) = x_d(t)$, and for that, I am assuming these two things: that whatever the desired trajectory is, it is also bounded, and that its rate of change is also bounded. So, if you want to simulate, then you can utilize these kinds of examples.

Here, you can see that all assumptions, whatever I have stated so far, are satisfied. Now, first assume that there are no actuator dynamics. So, if there are no actuator dynamics, then whatever ideal sliding mode control I am going to design is implementable. So, what am I going to do? I am going to define the sliding surface, and after that, I am going to design the control. Control is based on switch and gain, and then using the Lyapunov method, you can see this Lyapunov function.

That is positive definite for all $s \neq 0$; when $s = 0$, this is 0. Now, what am I going to do? I am going to take the derivative, and here are all kinds of non-linear parts that are coming from x, x_d, and everything that I am going to substitute here, and this is the gain part. So, what is our aim? So, whenever I have some kind of sliding surface, I have a finite time to reach here, infinite time, and after that, I have to maintain it. So, I have to show that infinite time is equal to 0. So, for that I have to design gain, I have selected gain like this, and it is possible to show that this expression comes into the picture.

So, just please do the substitution and then you can easily be able to get this expression by substituting $s$in terms of V. Now, if you solve, then we will get this solution, and what this confirms is that as $t \geq$this value, then basically $s = 0$. So, in this way, I can design an ideal sliding mode control, but in reality, what happens? I want to change this control, the sign $(\cdot)$control, by some kind of boundary layer approximation. So, I have to define the meaning of the boundary layer approximation. So, I am going to change the discontinuous control to some kind of saturation function.

And, I hope that you all are already well aware of the saturation function. So, the saturation function is exactly like the signum function when $s(t) \geq \varepsilon$. In this boundary, that is basically nothing but some kind of continuous or linear function. So, I have already stated this kind of thing. So, suppose that this is $\varepsilon$and this is $-\varepsilon$.

So, in this region, this acts like the signum function, and in this region, that acts like some kind of linear function. So, in exactly the same way I have represented here, what I have done is initially this is $+1$and -1; now I am going to shift this by magnitude $M$and $-M$. That is the meaning of this particular gain, and M/εI have actually kept here. In this way, you can minimize ε, but the gain is going to increase; you can see here. So, this is the explanation: the boundary layer I have obviously controlled is exactly like sliding mode control, but inside the boundary, I have some kind of linear control.

So, here you can apply the stability based on linear control theory. So, now I am going to write the exact same thing in the form of a block diagram. So, basically, I have a plant; this is the actuator. So, second-order actuator dynamics, $\ddot{\omega}$, this is the controller, and now I have to track $x = x_d$. So, I have designed the sliding surface that is based on $x_d - x$.

So, what is the key benefit of the boundary layer solution? That control becomes continuous now. And now a smooth control comes into the picture, and for that reason, the question of the chattering, which is due to the high switching frequency–based control near the sliding surface, does not come into the picture. And in this way, people are actually happy with sliding mode control. They do not want exact sliding, but they want real sliding. It means that based on this approximation, εwill be reduced.

Inside that bound, they will apply linear control, and beyond that, they will apply sliding mode control. And analysis is again easy if you have a system. So, obviously due to sliding mode control, we have just proved that $-\varepsilon V^{1/2}$comes into the picture, but inside the boundary, since $M/\varepsilon$and $x$is, as I am going to apply linear control, I am going to apply why linear I am telling because the sliding surface is the linear combination of the state, something like cx, I am going to apply. So, now you will get this kind of condition. You can see that now, obviously, I have to analyze the stability of the linear control in order to show that our entire control system is stable.

So, for that, that is also again required. So, now let us come here. I want to analyze the stability here because the sliding part stability is already established. So, what am I going to do? I am going to apply this control $\frac{M}{\varepsilon}\, s(t)$, and now I am going to express $s(t)$. So, the sliding surface you can see how I have basically defined the sliding surface as $x_d - x(t)$; in this way, I have selected it.

So now I am going to write in terms of the frequency domain. So, how do you write? So, this will convert to the frequency domain; this will convert to the frequency domain, and this will also convert to the frequency domain. And I will apply frequency domain analysis here, and then I am going to substitute. So, in this way, if you will do all kind of substitution, then finally, you will end up here.

So, I somehow skipped that part. So, what do you have to do? You can do the calculations. Here p is the Laplace variable, and it is possible to show that this kind of criteria will satisfy. What is the interpretation of this system? This whole term is nothing but a kind of disturbance, and this part acts like a linear system. So, now I have linear uncertain system.

So, this will also bounded. So, this is the input I have bounded. So, bounded kind of disturbance is acting on the linear system that is the interpretation. So, now what happens? Unmodeled dynamics must be stable and faster than the system dynamics. So, it means that actually overall response is governed by the slow dynamics. If you see, whatever we are locating is the steady state response, and for that reason, we have to make sure of this.

Maximum feedback gain for linear stability. So, I have to define how to select M/εsuch that I can preserve stability inside this boundary layer region. So, after that, higher gain leads to instability and chattering near $x(t) = 0$. You can easily see that what happens is that as $\varepsilon$is very small, $M/\varepsilon$is very high, but we are very close to the equilibrium point. And in this way, again, chattering kinds of things come into the picture.

And this is the critically damped gain that one can calculate based on the Hurwitz criteria. So, these gain calculations I am going to show you now. So, what kind of system do I have? I have somehow represented the system like this, and how to get this system. I have already told you that you can design the control because your control is $(M/\varepsilon)\, s(t)$. So, the $s(t)$part is nothing but $x(t) - x_d(t)$or $x_d(t) - x(t)$; both are fine, which means you can apply the definition, and after that, you can take the Laplace transform of $x(t)$.

I am using the same variable p; you can maintain it. So, in this way, you are getting a third-order differential equation, and $D(p)$can be treated as uncertainty. Now, what is our next step? So, I will first remove the disturbance, and then I will show the stability of the homogeneous characteristic equation using the Hurwitz criterion. So, how do I apply it? If you have a polynomial like this, it is possible to show that the coefficients of all the polynomial terms should be greater than 0and that the determinant is also positive.

So, applying this kind of criteria, I will get this kind of condition. Please check it, and after that, I also have to check the Hurwitz matrix formulated by this polynomial, which should be greater than 0. So, based on that, I will use this kind of criteria. Now, in order to achieve critically damped conditions, I am going to use this formula. And once you substitute everything, then you can get the critically damped condition.

So, what happens is that in this way I can choose the gain such that if I replace the signum function with the saturation function in the vicinity of the sliding variable, then in the presence of the unmodeled dynamics, our system is still stable. Let us come incorporate the effect of disturbance. So, actually disturbance is bounded. And now, I have a linear system, a third order linear system. So, it is possible to show that if without any disturbance or any input, if system is stable, then always I can able to confirm the bounded input, bounded output stability.

A similar kind of analysis, if you have a linear system, can be performed using the Lyapunov method. So, in the tutorial, I will discuss that. So, what I have done is implement sliding mode control; the parameters I have already written at the beginning. So, I have used the same parameter. So, in the MATLAB environment, I have simulated, and you can see that now our trajectory $x(t)$is not equal to $x_d(t)$because I have some kind of sinusoidal disturbance.

So, there is always some oscillation here, which is actually determined by the perturbation. So, now what is the benefit? Control is continuous and we have partially invariance condition now, because now I cannot able to give guarantee that system is insensitive with respect to perturbation everywhere or just you can assume that matched perturbation in first order system all kind of perturbation is matched only. Global uniform ultimate bondedness is the kind of condition one can establish in this particular case whenever you are going to actually replace the switching function with some kind of saturation function. So now, I am going to discuss the next approach. So, now, what people found is that if you are going to design a sliding mode or apply sliding mode directly to the system with an actuator, then what happens is that it will create difficulty.

Several times, suppose that actuator dynamics are very, very fast, but the sensor is not accurate; then chattering also comes into the picture. And this is the idea of Utkin, that is what you can do. Now, you can design any asymptotic observer, like the Leuvenberger observer, and then generate ideal sliding in the observer loop. And whatever the whatever the equivalent kind of things comes into picture you can apply to the system. So, equivalent control now apply to the system and equivalent control is basically not a switching based control and in this way you can able to minimize the sliding mode control.

So, we already familiar that whenever system is in sliding then equivalent control acts. So, basically, once this first loop is designed, then practically only equivalent control is applied to the plant, and in this way, you can minimize the direct effect of the switching control on the plant. So, here you can see that this is the plant; again, I have an actuator. So, I have created one auxiliary observer loop, and what I am going to do now is the main loop. So, whatever control omega that appears here, I am trying to force that control as an equivalent control.

I am going to apply an on–off control in this particular loop. So, basically, in a state of exact measurement, I am now going to utilize the estimated measurement. So, in this way, the observer is free from unmodeled dynamics, which are actuator or sensor dynamics. So, let us try to understand the philosophy. So, for simplicity, I am going to assume that I have a system like
$\dot{x}(t) = a\,x(t) + b\,u(t)$.

And here again, I am assuming that I have exact knowledge of a and b. Then, observer design is possible, and I have a certain kind of uncertainty in the plant as well. So, now by removing uncertainty, I am going to design the Luenberger observer first, and you are already familiar with how to design the Luenberger observer. What can you do? You can just copy the dynamics, and after that, you can add the correction term.

Now, you have to design the correction term L. Based on any pole placement technique, because this is a linear system. If you have a nonlinear system, then you already understand how to design L_1using the Lyapunov method. And once you see the error coordinate, the error is $x(t) - \hat{x}(t)$. So, in this error coordinate, what happens now? This kind of expression comes into the picture.

So, $d(x, t)$is coming from here. This is nothing but $d(x, t)$, and after that, $-L\,e$comes into the picture. Now, if $\dot{e}(t)$I am able to control, then what will happen? Somehow, $e(t)$remains bounded and I am able to see the bound of $e(t)$based on this condition. Now, what am I going to do? I am going to design an observer loop. And in that particular loop, I am going to implement the sliding mode.

Okay. And what is our main focus? Our main focus is to design sliding such that whatever $x(t)$is forced to $\hat{x}(t)$. So, here now since we are in the observer loop, I have $\hat{x}$here. So, now I am assuming that I do not, and there is no need to know $x(t)$. If I know $\hat{x}$, then I can design the sliding surface based on $\hat{x}$and $x(t)$.

So, for that reason, I have designed it like this. And after that, I am going to apply the philosophy of sliding mode control m sign (x). Again, I will take the Lyapunov function the same as the boundary layer. After that, I will calculate the derivative, and it is possible to show that $s(t) = 0$. One can show by this analysis, and then one can also show what kind of gain I can select, and then we can achieve this sliding. So, since I know that $e(t) \leq d + L_1$by the design of the observer, because the Luenberger observer in the presence of uncertainty obviously will not always give very good results if I have an error, and if I have a one-dimensional system, then this is d^+plus, this is d^-, so the error somehow converges in this band.

So, I am going to utilize that information from the Luenberger observer, and after that, I am going to place it here. Now, you can see that after simplification, if I select g^+, which contains all the other information, then I will get this. In this way, I can analyze. Now, once we are sliding, what happens? Basically, our $x$is nothing but $x_d - \hat{x}$.

Now, during sliding, the equivalent value of $\dot{x}$ is also 0. So, I have to substitute $\hat{x}$; $\hat{x}$can be substituted from the Luenberger observer. And then you have to put $\dot{x}_{\text{eq}} = 0$, then you will get this expression. Now, once you get this kind of expression, we already know that in this particular system, I have unmodeled dynamics. So, in unmodeled dynamics, I have actuator dynamics, and we know the relation between actuator dynamics $\omega(t)$and $u(t)$, as I learned from the previous discussion.

So, here you can just for clarity; I will come to that slide. So, basically, we are very far. So, I have to wait. So, this is the model. Now, what am I going to do? Let us go back to the original slide again.

So, basically, I am going to come back here. That I am going to substitute. So, you can see that dynamics is exactly like the previous one; again, I have third-order dynamics. So, how to design the gain, and this is treated as a disturbance. Again, I am able to design the gain like this, and what happens is that now stability is gone. So, in this way, I can reduce the chattering without directly applying discontinuous control to the main system.

So, only the equivalent control is applied to the system. Now, you can also improve the performance by designing this kind of extended disturbance observer, and what is the philosophy? Suppose that if the disturbance and the rate of change of the disturbance are very slow, then you can write $\dot{d} = 0$ and design the Luenberger observer. So, please do this simulation by yourself. So, this is called the extended Luenberger observer. So, what is our conclusion? So, we have actually seen two approaches, the boundary layer approach and the observer-based approach, to mitigate the chattering. Practically, whenever we have a mechanical system and want to design classical sliding mode control, most of the time we are using either the first approach or the second approach.

For an electrical circuit, it is possible to show that the second approach is much better. And due to that, in the DC motor, we have utilized the second approach. So, practical benefits include reducing mechanical wear and energy loss, and maintaining system stability and performance. Practically, we are able to implement the sliding mode control based on these two methods. So, with this remark, I will end this class. Thank you very much.