

## Chattering Analysis

Welcome back. In the previous classes, I talked about sliding mode control as well as sliding mode observers. In this class, I am going to discuss some implementation issues of the sliding mode controller as well as the observer. And that issue is basically called chattering. So, in this lecture, we are trying to understand what the actual definition of chattering is and how chattering can be minimized. So, those kinds of things I am going to explore in this particular lecture.

So, the main purpose of the discussion is to first understand the advantages of sliding mode control, and obviously, then I will come to the limitations. Basically, I am going to explore the practical limitations. So, what we have seen so far is that sliding mode control is obviously one of the robust control techniques and is applicable for both classes of systems: linear as well as non-linear. Obviously, in the presence of the plant parameter variation and disturbances, we are able to talk about insensitivity with respect to the mesh perturbation.

And due to that reason, one can tell that sliding mode control is very less sensitive with respect to plant parameters and variations. We have also seen that whenever we are actually in the sliding phase, because in sliding mode control we have two phases, and this is actually the philosophy of classical sliding mode control. So, we are basically using discontinuous control. We first land on the sliding surface and after that we are going to maintain. So, during this phase, our system order is going to reduce, and now you can able to design the feedback based on any existing control methodology, and somehow these two phases are decoupled, and that kind of advantage basically we are getting from the sliding mode control.

After that obviously, In several practical application, operation is on-off. Particularly, if you take the example of a power converter, a power converter now has several uses in electric vehicles, drones, or various places. So, if you see the control action, that inherently on-off in nature. So, sliding mode is basically quite suitable for that particular class of system. In this particular course, we have already seen several applications that are related to robotics, electrical drives, process control, as well as motion control.

So, what is the conclusion so far? That sliding mode control is one of the powerful robust control techniques and basically relies on the discontinuous control action. Now, let us come to one of the obstacles of the sliding mode control implementation that is so-called chattering. So, first I am going to define chattering. So, somehow chattering is not a desirable phenomenon, and it is characterized as the oscillation with finite frequency as well as finite amplitude, and why this comes into the picture is that whenever we are constructing a sliding surface or sliding mode control, the first step we take is modeling the plant. And during this modeling, we are somehow neglecting some part that has a time constant that is very, very fast.

After that, during the implementation, two subsystems also come into the picture. On this side, we have an actuator; on this side, we have a sensor. So, these two subsystems has also some kind of dynamics. And basically, we are never caring about those dynamics during the design of the sliding mode control, and due to that reason, these two dynamics are going to create obstacles during the implementation that I am going to explore in this lecture. The second obstacle is the discrete time implementation.

If you see the current hardware scenario, most of the hardware, either macro processors or computers, that actually accept digital signals also generate digital signals. And somehow, since our design is based on continuous time philosophy, whenever we discretize, we lose something. So how is this basically going to impact our system? So, our accuracy control is low, and if we have a mechanical actuator, it is possible to show some kind of wear and tear inside the mechanical actuator. And now we have already seen that robotics is one of the current fields, and that particular robotic system falls into the electromechanical class of systems. So, somehow these kinds of strategies create some issues like chattering.

If you are going to implement classical sliding mode control, first-order sliding mode control, that is actually developed by the Utkins and their subgroup. Now, what is the key point? Chattering is a harmful phenomenon caused by the unmodeled dynamics and discrete time implementation. So, if you look closely, sliding mode control does not have any problems. What are we basically doing? Since we are neglecting in order to design sliding mode control, that negation is basically going to create a problem. So, now another question is how to incorporate that so one can minimize.

So, we are going to look into the next chapter. Now, let us try to see the challenges of chatter. So, I have already told you that some systems inherently need on-off kind of control. So, now, on-off control is required. So, I cannot say that I can replace on-off control everywhere with some kind of continuous control.

because operation is based on the on-off control. So, the only thing I can do is control the switching frequency in such a way that I can minimize the chattering. So, in literature, several methodologies have been proposed by Lee and Utkin, and after that, in the next module, we are also going to see higher-order sliding mode control. So, somehow that methodology comes into picture due to somehow minimize this effect that is called chattering. So, what is the key point? Chattering poses significant challenges, especially in power electronics, where the switching frequency must be carefully controlled.

So, chattering with unmodeled dynamics, because I have already told you that one source of chattering is unmodeled dynamics. And what is the source of chattering? Actuator. And during the complete control system, we need sensors, a data processor, and several other things. And every subsystem has some kind of small time constant, and during a steady state, we are somehow ignoring this. But due to the nature of the on-off control with high frequency, these dynamics may be perturbed, and that will cause sufficient degradation of

performance.

So, in order to understand the effect of unmodeled dynamics on chattering, what am I going to do? I am going to take a linear system, and in this system, what is  $\omega$ ? Since I need to implement some kind of control, this is the plant equation; after that, I need an actuator. So,  $\omega$  is nothing but the input of the plant, some output of the actuator, and the control I am going to apply here. So, we are going to implement the control logic we have designed through hardware called an actuator. So, somehow, whatever output of hardware that can signal, this plant is going to receive. So, I have represented it like this.

Here,  $\sigma$  is called the sliding surface, which is the linear combination of all states. Now, what is the key point here? Unmodeled dynamics from the actuator can lead to chattering in sliding mode control. We are going to verify that we can. So, this is the block diagram that I have planned. This unmodeled dynamics is nothing but the actuator, and here I have control logic.

So, based on logic, we have designed some kind of control action. Now, you can see here that this  $\omega$  and  $u$  are not exactly the same. So, whatever control we have designed, the same control is not implementable here. Why? Because this actuator has some kind of dynamics. So, now we are going to see its effect after designing the sliding mode control.

It is possible to show that if the system is linear, then this block is almost equivalent to this block. What can you do? Now, you can put the unmodeled dynamics on the right-hand side of the plant. Again, this unmodeled dynamics shows that whatever control I have designed is exactly applicable to the plant, but now the difficulty is that the output of the plant is somehow modified by this unmodeled dynamics, and our sliding philosophy is based on  $\sigma$ . So, somehow these two linear systems are equivalent; you can also think of it like this: I have learned. So, this dynamic corresponds to the sensor unmodeled dynamics; this corresponds to the actuator unmodeled dynamics.

So, somehow both are the unmodeled dynamics only, but from the theoretical analysis point of view, one can see the effect of the sliding mode control here very easily and obviously after I perform the same kind of analysis I am going to conduct here. So, here you can see that in the equivalent system, this system I am telling that that is equivalent to this. So, in this equivalent system, now  $u$  sliding mode control I can able to design and due to that design in place of  $\omega$  now I have replaced the  $u$ , because in this upper block diagram, a block diagram you can see that input to plant is  $\omega$ . Due to that reason, we have written the equation like this, but in the equivalent plant, the input is  $u$ , and due to that reason, we have written the equation like this. However, how is this equation different from this equation? You can see that in place of  $\sigma$ , I now have to apply  $\sigma^*$ .

Why? Because the output is  $\sigma^*$ . Now, what I am going to do, you can see here that  $\sigma$  and  $\sigma^*$ , and here I am just going to utilize the information of sign ( $\cdot$ ). So, if  $\sigma$  and  $\sigma^*$  are very close to

each other, it means that if they do not change sign, then I can be able to get exact sliding. So, somehow equivalent representation tells us that how to get exact sliding in case of the unmodeled dynamics. Now, I have to understand the impact.

So, unmodeled dynamics can cause high-frequency oscillation. So, now I have to characterize this high-frequency oscillation. So, I am going to characterize this. I have already told you that this chattering is very dangerous whenever our actuator is mechanical, because if our actuator switches with very high speed, then that will create wear and tear, and somehow heat is generated. That will cause some kind of difficulty, and for that reason, proper selection of gain is required; if you neglect module dynamics, you may also be able to see the instability.

So, this is the key point: unmodeled dynamics introduce chattering, which must be addressed through proper gain selection. It is possible to show that if you design  $M$ ,  $M$  is the gain of the sliding mode controller, and if you also account for the unmodeled dynamics, you can reduce the chattering. Now, up to now, we have discussed the effect of an actuator and sensor with a small time constant, and our observation that if we have just a sensor, and somehow if the output  $\sigma$  is approximately equal to  $\sigma^*$ , then I can achieve exact sliding in the presence of unmodeled dynamics as well. And now it is possible to show that the gain of the sliding mode controller is very crucial to reduce the chattering and that somehow theoretically we have to establish how gain is related to the chattering magnitude; obviously, design is required because if I am able to establish that gain is responsible for the chattering magnitude, then I can reduce the gain. Reducing gain is not always possible.

Why? Because the magnitude of that gain should be greater than all kinds of uncertainty. So, there is only one approach. What can you do? Try to rely on the feedforward control. So, designing feed forward control, it is possible to show that some kind of known part we are able to cancel. And in this way, I can minimize the level of gain.

Now, let us first try to see the equivalent structures. In equivalent structure, what have we done? Again, please return to the representation. So, here I am going to take this system, and here  $\sigma$  is the input to the unmodeled dynamics, and I am going to assume that unmodeled dynamics is represented by second-order dynamics, and where the time constant is very, very small. So,  $\sigma$  is the input to the unmodeled dynamics, and  $\sigma^*$  is actually the input to the controller.

So, I have written the same kind of things in the mathematical language here. So, I am considering the unmodeled dynamics that we have neglected. Why are we neglecting? You can see from here that if you are going to put  $\mu = 0$ , and  $\mu$  is very, very small, then the effect of these first two terms,  $\dot{z}$  and  $\ddot{z}$ , will disappear, and  $z = R\sigma$ . Now, here  $\sigma$  and  $\sigma^*$  are not the same.

So,  $\sigma^*$  is some kind of value. So,  $z$  is the output of the unmodeled dynamics. So, here I have

unmodeled dynamics. So,  $\sigma$  is the input, and after that, what is the output? The output is  $z$ , and after that, whatever things I pass through this, this is somehow some kind of gain that is  $Q$ , and after that, from this, I will get  $\sigma^*$ , and based on this  $\sigma^*$ , I am going to design this sliding mode control. Now, what am I going to do? I am going to create a state space representation of this system. So, how do you create a state space representation? So, you can define  $z_1 = z$ , and after that,  $\mu \dot{z}_1$  I am going to define as  $z_2$ .

So, if you take  $\mu \dot{z}_1$ , that is nothing but  $\mu^2 \ddot{z}$ , then I will get this dynamics. So, basically, the unmodeled dynamics are represented as a second-order system with a small parameter  $\mu$ ; this is the small parameter. So, for this system, I can apply the theory called singular perturbation theory, which suggests that if the derivative  $\dot{\sigma}$  is also bounded, then I can be able to substitute  $\mu$ , which is very, very small. So,  $\mu \approx 0$  since  $\mu$  is small, and due to that reason,  $z$  is approximated by 0.

Similarly, the right-hand side of the last differential equation is also approximated as 0. So, basically, this is already 0,  $\dot{z}_2$ . So, I am the only one who has the choice that  $R\sigma = z_1$ . Approximated by, I have to write, and some more terms will appear here because that is just an approximation. So, the same kind of things I have written here: after applying the singular perturbation, what happens? After a finite time, once the transient is over, it is possible to show that  $z_1$  is going to converge here, where  $\sigma_1$  shows the accuracy, and if  $\mu$  is very, very small, tending towards 0, then  $z_1$  is exactly equal to  $R\sigma$ .

And what is  $\dot{\sigma}$ ? That is  $Qz$ , and  $z = z_1$ . So, if I substitute now, you can see here that  $QR = \sigma$ , and some small term comes into the picture. So, as  $\mu$  tends towards 0, now what is our observation?  $\sigma^*$  is exactly equal to  $\sigma$ . Okay, so exact sliding I can see in the vicinity of this particular bound.

Why? Because always some kind of perturbation is present. So, these two actually, I am using the sign ( $\sigma^*$ ) = sign ( $\sigma$ ). So, everything actually depends on the perturbation parameter, which is a function of  $\mu$  and  $t$ . Due to that, since  $t$  is done, I am saying that I will get a sliding mode kind of behavior in this vicinity; that is the physical interpretation. And if  $\mu \rightarrow 0$ , there is no unmodeled dynamics. So, what happens? I will actually see the sliding mode with respect to  $\sigma = 0$ .

Now, I have already analyzed the equivalent system. What is an equivalent system? Let us come back again. This is the equivalent system. Now, I am going to analyze this original system. So, what is our observation of the equivalent system?  $\mu$  is not very small, so I cannot achieve the exact sliding.

In the vicinity of some kind of  $\varepsilon_\mu$ , I will get this, I will get the sliding, and this vicinity is responsible for the chattering. Now, I will take the original one, and we are going to analyze stability in this vicinity and that confirms the chattering. So, by analyzing the dynamics below, I can give a result for the  $A_1$  dynamics. This is  $A$  and this is  $B$ . So, let us now see the

mathematical model of that system.

So, here in this system, in this representation, the actuator will get the input from the sliding mode control, and after that, whatever input is here, the plant is going to get input  $\omega$ . So, the same kind of things I have written:  $u$  is the input to the unmodeled dynamics, and  $\omega$  is the input to the plant. Again, I will do the state-space representation. You can see here that the state-space representation is a little bit different because I do not want to apply the singular perturbation theory.

Now, I am going to study Lyapunov stability theory. So, how do you achieve stability? I have three state variables. So, I have first defined  $\sigma_1 = \sigma$ , which is the sliding variable. After that, what am I going to do? I am going to select  $CB$ , where the eigenvalue of  $CB$  is equal to the identity. I have selected sliding mode in such a way that this will be preserved. It is possible to always show that these kinds of assumptions are satisfied.

After that, what have I done? You can see here that I have calculated the derivative; the first derivative,  $\dot{\sigma}$ , and  $\sigma$  is nothing but  $cx$ . So, what is  $\dot{\sigma}$ ?  $c\dot{x}$ , and what is  $\dot{x}$ ? That is nothing but  $Ax + CBw$ , and  $CB = 1$  by design; due to that reason,  $w$  will appear here.  $w$  is nothing but the input to the plant, and we have seen here that  $w = Qz$ . Now, I am going to take one more derivative, and if you take one more derivative, then what happens? Now, after taking one more derivative, here  $CA\dot{x}$  comes into the picture, and  $\dot{w}$  comes into the picture. What is  $\dot{w}$ ?  $\dot{w} = Q\dot{z}$ , because  $Q$  is some kind of constant matrix.

Now, I know that here  $z_1 = z_2$  from this particular expression. So, for that reason, this term comes into the picture. Similarly, if you calculate  $\dot{\sigma}_3$ , then control explicitly appears, whatever non-linear term I am going to keep here. Now, it means that our unmodeled dynamics is represented by this third-order equation:

$$\dot{\sigma}_1 = \sigma_2,$$

$$\dot{\sigma}_2 = \sigma_3,$$

and  $\dot{\sigma}_3 =$  this.

So, now, I am going to construct a Lyapunov-like function. This is not exactly a Lyapunov function. Why? The Lyapunov function should be positive definite. So, for all  $\sigma > 0$ , when  $\sigma \neq 0$  and is only equal to 0 at  $\sigma = 0$ , this condition is not satisfied. So, for that reason, I am saying this is a Lyapunov-like function. Now,  $\dot{V}$ , if I calculate and if I design  $M$ , and  $M$  is larger than this whole value, then it is possible to show that  $\dot{V} < 0$ ;  $\dot{V} < 0$ .

And here we are also assuming one more thing: that if  $M$  is very, very high, then I can be able to get sliding. What kind of sliding will I get? In this way, all possible variables  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$  are equal to 0, and that is nothing but all higher-order derivatives being equal to 0. So, basically, equivalent value, because whenever I am in the sliding phase, I have already told you that one equivalent control  $u_{eq}$  comes into the picture, which is actually confirmed by Utkin's philosophy. So, basically,  $z$  is nothing but  $R u_{eq}$  during the sliding.

So, I have basically written the same kind of things here. Since I have some kind of uncertainty, I can see this expression in the vicinity. So, I have to select a gain that is larger than  $Qz_1$ . So, in this way, you can see that what I have observed is that  $V$  is a sign of indefiniteness, but  $\dot{V}(x) < 0$ , and where this kind of thing is conforming in some vicinity, and due to that reason, this system is attractive, but not stable. What does it mean? That all trajectories will converge to some vicinity, and after that they can actually diverge within this. So, all trajectories are attracted towards this, but we lose stability and due to that reason, chattering is confirmed.

In this way, you can show theoretically that chattering is there. Now, what is our next goal? Our prime goal is to calculate the amplitude and frequency of the chattering because that is finite. So, for that, I am going to apply on-off control again. You are free to select any system. So, I have selected here; you can see that it is an  $n$ th-order system with only one control input for simplicity, and this is the sliding surface.

It is possible to show. That is how to realize sliding, so for that, you have to take the derivative. Now, once you are going to apply the on-off control, it is possible to show that in a steady state, this and this are approximately constant, and due to that reason, the whole plant can be approximated by this plant,  $\dot{x} = a + bu$ , where  $a$  and  $b$  are the constants, and due to that reason, in order to analyze the chattering, even if these two are uncertain, I can approximate that by a constant one. So, this is our plant; these are the sensor dynamics. Now, I am assuming that since I have an equivalent kind of circuit,  $H\dot{z} = 0x^*$ . So, whenever we are actually tuning the sensor, so we are adding one extra block.

Why? Because somehow we want that in a steady state  $z$  will be approximated by  $x^*$ . So, by designing or tuning the sensor for a specific application that is very much essential. Now, we are going to design sliding mode control based on the information of  $x^*$ , and for that reason, you can see that the sliding surface is designed based on  $\sigma^* = cx^*$ . This is the actual dynamics of the sensor, so I am assuming that here the sensor is represented by just first-order dynamics.

For simplicity, we are assuming that whatever sensor is stable. So, how do you make a matrix stable? You can make it stable. Okay, now I have already assumed that whenever high-frequency chattering occurs, at that time system parameters become constant, or approximated by a constant, and  $u$  is actually the sliding mode control at that time, which is  $\sigma^*$ . Whatever output comes from the sensor that passes through this gain, the user has to design this gain, and in a steady state, the user has to confirm this. Then I can actually show that  $x^*$ , or whatever sliding, real sliding or actual sliding, are both the same.

That is the philosophy. So, now you can see that in a steady state, the rate of change is going to cease. So, at that time, I basically have  $z = Bx$ . At that particular moment, I can show that since what I have to do is show  $x^* =$ , I know that  $x^* = Hz$ , and for that reason, I will calculate  $z$  from here. So,  $z = A^{-1}Bx$ . Now, I am going to substitute  $z$  here, and now you can

see that I will get this expression.

So, this expression tells us how you can tune your sensor such that  $x$  is approximated by  $x^*$ . So, if I keep the identity, then this problem is solved. So, the static mode controller's perception of the state should be aligned with the state to avoid distortion. So, this kind of tuning you have to do whenever you are doing the practical implementation. Now, whenever I am applying the chattering phenomenon and high switching phenomenon, we have seen that the system is approximated by a linear system.

With some kind of uncertainty. So, I can apply the Laplace transform, and I want to characterize the harmonic function, what kind of actually finite frequency or finite amplitude variation comes into the picture, and due to that reason, I have to express the solution in terms of harmonic functions. So, for that, I am going to utilize the combination of Fourier series and Laplace series in the next part of this slide. So, what I am going to do now is have a sliding surface  $\sigma$ ,  $\sigma^* = c x^*$ . So, what am I going to do? I am going to apply the Laplace transform to both sides.

So, this is the Laplace transform of  $\sigma^*(t)$ . So,  $\sigma^*(t)$  is our sliding surface. So, I have either you can represent, because this is just a function of  $x$ ; for that reason, I have written it, but obviously,  $x^*$  is also a function of  $t$ , and for that reason, people are writing it like this. So, please do not be confused. Now, you can see here that  $X^*$  is nothing but  $HZ$ . And what is  $Z$ ? If you look carefully here,  $Z$  is nothing but  $X^* = HZ$ , and I can calculate  $Z$  from the sensor dynamics.

So, you can take the Laplace transform here, and after that, you can substitute here. And after that substitution, you will be able to get  $U$ ;  $U$  is basically what we are going to apply to the plant. And what is the equation of the plant?  $\dot{x} = a + b u$ . Again, you can take the Laplace transform, and then you will get this kind of expression.

So, please do it by yourself. Now, what am I going to do? I am going to show you the solution to this. The  $\sigma$  is because I am taking the Laplace transform; finally, I have to show what the expression of  $\sigma(t)$  is. So, I am assuming this is the solution. Now I have to calculate  $\alpha$  and  $\beta$ ; then I will be able to understand the harmonic solution exactly.

So, after that control, I am going to design what kind of control I am going to apply. I am assuming that there is no distortion because I have designed  $H$  such that there is no distortion. Then I can design  $u^* = M \text{sign}(\sigma)$ ; again, I am going to calculate their components. How do you calculate this component? So, for that one can be able to apply the Fourier series analysis. So, Fourier, this is the formula for the Fourier series. If you substitute this whole quantity here and take the integral, you will be able to get this expression.

So, this is the work you have to do. What I am going to do is that I know the expression for  $u^*$ , I know the expression for  $\sigma^*$ , so I will take the Laplace transform of  $\sigma^*$  and  $u^*$ . The

Laplace transform of  $\sin(\omega t)$  comes into the picture here again;  $\sin(\omega t)$  comes into the picture, and after that, I am going to substitute it here. So, those kinds of things we have basically done. We are also assuming that  $\mu = 0$ . The time constant is very small because we are chattering whenever it comes into the picture, especially during the sliding.

So, already system is in some kind of a steady state. So, during that I am assuming that  $\mu = 0$ . Now, after taking the Laplace transform of this, I have this, and the right-hand side is this. You can see here that I have already tuned the gain of the sensor in such a way that all three parameters are equal to identity. So, what is our conclusion now? I have to match the left side to the right side.

How do you match? It means that this term should not be present. So, if this term is not present, because if you do partial fractions, then I can generate this kind of term, but I have to remove it. And you can see that physically I will get an equivalent control. So, in sliding mode control, since  $\sigma = cx$  and  $\dot{\sigma} = ca + cbu$ . So, how do you calculate equivalent control? You can set  $\dot{\sigma} = 0$ .

So,  $u_{eq}$  is nothing but  $\frac{ca}{cb}$ . So, the same kind of things actually came here. This is done by designing the sensor gain. I am able to tune it. Now, you can see here that since I know  $u_0$ . So, I am going to substitute  $u_0$  inside this expression, and then I will be able to write the expression for  $u_1^*$  because they are high, meaning finite-frequency oscillation comes into the picture, and for that reason, I have calculated  $u_1$ , and after that, I am going to substitute the whole control inside the system.

So, if you see the system dynamics, you can now see that one frequency  $\omega$  comes into the picture and  $M$  is here. So, due to that reason, the chattering amplitude depends on  $M$ , and in order to mitigate the chattering and reduce it, I have to somehow select this  $M$  properly. Now, I also have to see the frequency. So, how do you see the frequency? Again, I will come to this expression, and in this particular expression, what am I going to do? I am going to neglect this, I am going to neglect all initial conditions, because I have to calculate the transfer function, and after that I am going to look into the argument. So, you can see here that one term, here the term should be  $jsI$ , but instead of that, what happens? I have jumped into  $\omega$ , and  $\omega$  comes into the picture.

So, this is responsible for some kind of  $P$  or  $S$ , as you can tell;  $P$  and  $S$  I am not using, because in the sliding variable I am representing it like this. It means that now  $\omega^*$  comes into the picture, and  $\mu$  into  $\omega$ , and what is  $\omega$ ? So,  $\omega$  is nothing but here in place of  $\omega^*$ ; you can also write here basically nothing but  $\omega^*$ . So,  $\omega = \mu \omega^*$ . So, if  $\mu$  is small, then you can see that the frequency is very, very high. If the time constant of the sensor is very small, then the chattering amplitude is higher; that is the conclusion.

So, in this way, I can characterize the chattering and how to reduce it; obviously, I have to control the gain, as that is the only way to actually reduce the chattering. That is the

conclusion. So, now it is time to conclude this lecture.

So, whenever we are designing sliding mode control. So, at that time we have to tune sensors. We have just considered sensor dynamics here because we have already looked at actuated dynamics, and after that, I am going to understand the effect of the sensor. So, I have to actually decrease it suitably and figure out how to do so. So, we are going to see in the next part of this course how to properly reduce the chattering. So, there are several methodologies for that. So, in the next two classes, I am going to discuss that, and after that, in the next part of this course, I am going to talk about higher-order sliding mode to mitigate this kind of phenomenon. So, with this remark, I am going to end this class. Thank you very much.