Observer for Linear Systems with Binary Output

Welcome back. In the previous module, I was talking about the observer design for the linear system, the linear system with disturbance, and the non-linear system. We have also discussed the Luenberger observer, high-gain observer, and sliding mode observer. Everywhere, I assume that we know the output completely. But in a realistic situation, this assumption is not satisfied. So how do I design an observer if I don't know the complete information of the output, which is the topic of this particular lecture? The purpose of the discussion is to observe the linear system with binary output.

What is the meaning of binary output? You might have seen that in several places I have observed and suppose that in this environment, if I have to detect the level of oxygen and if I place the sensor, we always have some kind of critical limit. So, beyond that limit if oxygen level or any methane gas or any other gas will go, then our sensor will give us the positive result. So, it means that either our sensor is going to give us two pieces of information: something is there or something is not there, something like that. So, now, in that particular situation, how to actually get the information of all states based on that particular boundary operation is the topic of this particular lecture.

So, what is the challenge? Whenever we talk about conventional observers, even if I talk about the Luenberger observer or any other observer, what is our assumption? That we know that we have the full output information. So, complete output information is given, and from the input and output information, this is the input $u(t)$, and this is the output $y(t)$. So, from this information, I have to estimate the $x(t)$, which is the prime objective. But in the case of boundary sensors, one of the examples I have already discussed is the oxygen sensor, which only provides sign information. Now it means that a standard observer design method failed with such limited output.

So, now how to design an observer in that particular case, and this class of sensors has several practical applications, and due to that reason, I am going to include this particular topic in the sliding mode because it is possible to show that some sort of sliding mode transformation comes into the picture such that you are able to design an observer for the linear system. So, basically, the solution is a two-stage observer deployment. So, how do you deploy a two-stage solution? So, I have to develop two different kinds of observers. One observer is based on the discrete time varying principle and is going to estimate the state at each sensor at the switching instant. So, that is actually going to be the job of the discrete sensor or discrete observer.

Now, I am also going to design the second part that is based on the philosophy of the continuous time system and that is going to estimate the state trajectory between switching instants. So, suppose that I have installed some kind of sensor in some kind of open environment, and suppose that it is a moisture sensor. So, initially I gave some kind of water, then the moisture sensor turned on. After that, once the moisture quantity is low, it becomes off. Again, I have given some water, then that became on.

So, in this way, this on-off interval is most of the time not uniform. Now, based on that particular information, I am going to estimate the status. So, for that, we need the design principle that is based on the discrete as well as continuous time philosophy. So, this is the practical problem statement. I am going to start with the linear system.

You can see here that I am assuming a linear system, which is an LTI (linear time-invariant) system without any disturbance. And I am assuming that the output just contains the sign information. Here, b is nothing but some kind of scalar threshold. So, every sensor has some kind of threshold. People are setting that threshold based on some kind of experimental data, and after that, they are deploying it in the field.

So, that limit you have to keep here. Now, what I am assuming is that $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and for simplicity, I am assuming that $y \in \mathbb{R}$. But you will be able to extend this kind of result if you have more than one sensor. Now, here I have to actually deploy an observer based on the discrete as well as continuous-time philosophy. And due to that reason, I am going to first define the switching instant.

So, what is the meaning of switching instant? Suppose that at the $t_k$instant, I have some kind of state that I have defined as z_k, and after that, if our sensor is going to maintain its state between $t_k$and t_{k-1}, it will then become either on or off. So, this is nothing but the time interval between two switchings. So, the same kind of things I have written is that $\tau_k$is the switching interval that is between the $(k{-}1)$th and kth switch. So, switching instant I am going to represent, switching instant state I am going to represent by $z_k$and time interval I am going to represent by τ_k. Now, since this system is a linear time-invariant system.

So, one can easily see that I can write the exact solution provided I know the z_k. So, I hope that most of you are already aware. If you are not aware, then for that reason, I am going to write here. Suppose I have this system, $u(t)$. So, this is a linear time-invariant system.

It is possible to show, particularly for linear time-invariant systems, that I always have a closed-loop solution. What is the meaning of a closed-loop solution? I can always express a solution in some kind of form. So, I can write a solution like $x(t)$, and after that, this term is $e^{-t_0}x(t_0)$, and after that, I have an integral from $t_0$to t, $e^{-\tau}Bu(\tau)\,d\tau$. In this way, I can always be able to write the closed form solution.

And what is the physical interpretation? If you know the initial condition $x(t_0)$, and you know the control, then you can be able to know the futuristic evolution of a state; that is the physical interpretation. Now, you can see here, I am assuming that the switching instant is z_k; it means that our initial state is z_k, and for that reason, in place of x, I have substituted z. At that time, the initial time is t_k; based on this particular definition and for that reason, I have substituted $t_k$here. And I have to integrate from t_0 to t, but I know that our sensor is going to switch at t_k, and for that reason, I am going to integrate from t_k to t. So, this is the exact solution between any two switching instants.

How does the output basically come into the picture? Output measurement I can do using this kind of expression because I know here that the output is the sign function. Here I am assuming that you can define sign $(\cdot)$based on your specification. So, suppose that sometime you want to switch between 1and 0; then you have to show that if this is > 0, then that is 1. Less than 0, then that is 0or $+1, -1$. So, based on that, you can actually define your sign function.

Now, during this, you can see that since I do not know z_k, I just know the on and off. So, what is z_k? That is nothing but $x(t_k)$. Here I am assuming that all information about $z_k$is available to know what $y_k$is, but I do not have information about z_k. So, for that reason, I have to actually estimate that information from the initial condition. So, for that, I have to construct an observer.

So, how do you construct an observer? What am I going to do? I am going to take this information. So, if you see the switching instant between $t_k$and t_{k+1}, then actually here I have this become $z_k$and now this becomes z_{k-1}. I hope that you are able to understand this. So, this is nothing but somehow you can assume that this is z_{k+1}. What I am assuming is that z_k, so in order to know z_k, I need z_{k-1}, and I have to integrate from $t_{k-1}$to t_k.

Now here, basically $t_k - t_{k-1}$comes into the picture, and that I have already defined as τ_k. So, at the end, due to that reason, based on τ_{k-1}, this becomes τ_{k-1}, and t_{k-1}becomes τ_{k-1}. And now, how do I design an observer? You can just copy this dynamic. So, we have copied this dynamic. Now, since this time interval, I have already told you that we should suppose our sensor is switched here, and after that, either you can represent this as t_{k+1}or t_{k-1}, $k - 1$.

So, basically, this interval and this interval are most of the time not the same in all practical applications, and due to that reason, this is time-varying. If you calculate the difference between this discrete system and this discrete system, you can see that $z_k - \bar{z}_k$is defined as $\tilde{z}_k$, which is the error variable, okay. Now, I have another term, since $e^{A\tau_{k-1}}$is the same on $z_k$and $\hat{z}_k$, and due to that reason, now this becomes $z_k - \hat{z}_k$, and $\tilde{z}_k$comes into the picture, and due to that reason, this has become $\tilde{z}_k$. Now, in this part, you can see that exactly this and this part match, and due to that reason, this cancels out. Now, here you have to become very careful since this b.

And I am going to switch. So, before that, that is nothing but $Cz_k = b$, and that I am going to substitute here. Okay. And if you substitute this, then you are going to get this dynamic. Okay. Now, similar to the Luenberger philosophy or sliding mode philosophy, what do we have to do? We have to show that the error dynamics are stable.

Now you can see that this is not a simple pole placement problem. Why? Because this torque interval is time-varying. Okay. Fully dependent on the practical situation. Due to that reason, the error convergence problem is not a simple pole placement problem.

So, due to that reason, you have to treat in some different way. So, the same kind of things I have written: the observer gain selection, because the main methodology is to select the gain of the observer such that the estimated dynamics and the original dynamics both match; that is our main goal. But what happens due to this τ_k, which is actually changing time-varying? I have to design $k$in such a way that this closed-loop error dynamics should be stable, at least asymptotically converging. So, I have to design in such a way that this will tend towards 0as $t$tends towards some k, which is a very large number or infinity. At least those kinds of things I have to guarantee.

Obviously, I have written that conventional eigenvalue placement will not work. Why? Because this is a time-varying system. So now I have to make sure that for all t_k, and $t_k$is coming from the practical situation, I can able to show that this system is stable. So how do you do that? So obviously, I have to limit my system. And due to that reason, I am not going to give a generalized result.

So, whenever you have a second-order system or a third-order system, you can actually develop the criteria for the gain criteria based on that. So, mainly I have to develop the gain criteria because everything else is very easy. So, what am I going to do for that? I am going to consider a second-order system. So, you can consider $\dot{x} = ax + bu$, and here $x \in \mathbb{R}^2$. It means that $x$contains two pieces of information: a state $x_1$and x_2.

So, correspondingly error become $z_{1k}$and z_{2k}. Here I am assuming that I have just one piece of information, which is that the information of $x_1$is present. So, somehow our sensor is going to give us $x_1$minus some kind of information like b. So, this kind of output I will get in a realistic situation. Based on that, I basically have to design the observer now.

So, what will I do? Lastly, I will achieve this kind of dynamics only if I express everything in the coordinates of the error. So, I am assuming that this term is $a_{ij,k-1}$that is associated with this term. So, after that, I have another term L_{k-1}multiplied by C; $C$is here, and I have taken $[1 \ \ 0]$. So, basically, L_{k-1}comes into the picture.

So, based on that, I have selected this. So, similarly, I have defined it everywhere. Now, I have to prove that this will and this will both tend towards 0 as t tends towards or k tends towards very large values. So, I have selected like this. Now, you can see here that the sliding mode philosophy comes into the picture.

So, basically, this is a discrete time system. So somehow, you have to apply the philosophy of discrete-time sliding mode control. So far, we haven't understood what the meaning of discrete-time sliding mode control is, but you can think of it like this. In place of $x_1 = x_2$and $x_2 = u$, now I have some kind of dynamics I can think of where $x_{1,k+1} = x_{2,k}$. And after that,

$x_{2,k+1} = u_k$, something like that. Now, what am I going to do? Exactly the same as here, I am designing $x = cx_1 + x_2$; here, $c$is some kind of gain.

So, here I am going to define k, which is nothing but containing the information of x_{2k}plus some kind of gain. So, αor some value you can use, and x_2, x_{1k}. So, this is x_{1k}. So, in this way, I am able to design the sliding surface, and then I have to force the sliding surface in the next instant to equal 0, or at least in the subsequent instant, I have to show that it will converge to 0. So, I have two different philosophies: either I can exactly converge to the sliding surface or remain in the vicinity of the sliding manifold.

So, that kind of sliding mode I am going to teach you is called the quasi sliding mode control. So, now what am I going to do? Since I have a discrete time system, it is exactly the same as this. So, I have a system in the coordinate frame of $\tilde{z}_{1k}$and $\tilde{z}_{2k}$. So, I have designed a sliding surface.

α, I have to decide that based on this variable. I will tell you. I will tell you why I have selected it this way, so that you can easily understand when I am going to substitute the whole term inside the system dynamics. Okay, now what I am going to do is transform my system into the coordinate frame of $x_k$and α_{1k}, and after doing that kind of transformation, you can see here that the whole dynamics is transformed into the coordinate frame of $x_{k-1}$and x_k. Now what I have to do is actually design this gain.

I can easily select these two gains. That is always in my hands. And due to that reason, here you can see that I have selected this gain in such a way, such that somehow I can able to, and similarly here I have such that I can able to transform system in this way. And why is this way important? You can see that in discrete sliding mode, I am present. So, if I converge to 0, which means $x_k = 0$, then this term is actually slowly disappearing; this is equal to 0.

So, now I just have this particular term. And if I apply the theory of discrete time systems, let us try to understand this point, which is αx_k. So, now if you put $k = 0$, $x_1$is αx_0. What is x_2? That is αx_1, and after that, if you substitute this here, then this becomes $\alpha^2 x_0$; and similarly, if you proceed up to $k$instant, then $\alpha^k x_0$comes into the picture, and here you have x_k. So, how can we make sure that $x_k = 0$? For that, I need $\alpha$to lie between -1 and 1. And due to that reason, you can see here, whatever r and q, I am going to strictly telling that that should lie less than 1.

In this way, I can ensure the convergence. So, by designing this kind of gain, I am able to convert a system like this, and once the conversion happens, our job is done because $x_k$is tending towards 0by the selection of gain, and once x_k is tending towards 0, then ultimately $\tilde{z}_{1k}$ is tending towards 0. If $\tilde{z}_{1k}$ tends to 0and $x_k$tends to 0, then there is no choice; $\tilde{z}_{2k}$ also tends to 0. So, in this way, both error dynamics, that is, $\tilde{z}_{1k}$ and $\tilde{z}_{2k}$, are tending to 0. So, once I am able to estimate $\tilde{z}_k$, what I am going to do is utilize this information.

To design the continuous part of the observer. So I know the solution. So now what I am going to do is substitute $\hat{z}_k$ inside the solution. So in that way I can be able to get $\hat{x}_k$. And once you get the $\hat{x}_k$, you can put that $\hat{x}_k$ to generate another signal.

Once the signal is generated, you can proceed in the same way. So basically, whenever our sensor is based on the boundary principle, several sensors—several moisture sensors, several gas sensors—are based on this philosophy. And you can see here that one is able to integrate the philosophy of sliding mode control to design the complete observer, and here our assumption is that the system is linear. Obviously, you can extend this result to the non-linear system as well. But at that time, we have to guarantee that there exists some kind of solution, and the solution you can write in some kind of closed form.

Otherwise, you again have to consider the robustness effect. Now several times in practical situations, since nowadays all sensors are being placed in the field and after that data collection is done through the cloud. And due to that reason, some kind of delay is always associated. So, in the presence of a delay, how can I modify this binary observer so that I can again estimate the state easily and extend the previous development? So, what I am assuming here is that our state is delayed. So, once a state is delayed, you can just substitute here $t - \theta$. What do we have to do? I have to estimate $x(t - \theta)$ because I know that whatever information I am getting is delayed information.

Again, you can utilize the same principle to design the $\hat{z}_k$ that switches between instant and continuous parts using convolution, allowing you to decide easily. So, in this way I can extend this result for the time-delayed measurement also, and for that reason this particular lecture is very useful. From the practical utility point of view, whenever you are trying to deploy some kind of binary observer in your working field. So, I have considered a very simple example of a DC motor, but you can change the example based on your project or whatever you are working on.

So, I am assuming a matrix is given like this: B and C. Several times I have discussed the DC motor, and due to that, reason mathematical modeling is not considered here. And what is our specification that the boundary sensor will detect if the position is greater than 0.5? So, that kind of means that I have just one piece of information, and this is the value of b. Suppose that at the switching instant like this, we are trying to understand how one can design an observer, and we have already seen that two parameters are very important: r and q. I have selected the initial condition, and this is actually the control input.

So, I can easily calculate $e^{A\tau}$ by considering $\tau = 0.3$, then $a_{11}$and a_{21} I can construct, and using that I can design $\alpha_1$because finally, what is our objective? I have to design the gain of the discrete observer. So, that will provide us when you are at switching instant, then how to observe that particular state. So, I have to design L_{11} and L_{21}. So, for that reason, these kinds of terms are required, and due to that, you can calculate each and every term and substitute it.

So, what is my suggestion? Please check all calculation. Several time calculation mistakes occur, and for that reason, whenever you are playing with the numerical example, you have to check it very carefully. And in this way, I am able to show how convergence is actually going to progress. And this is the actual state between the switching instant. And it is possible to show that the performance of the final position error is just this; the velocity error is like this.

In 1.5 seconds, we will be able to get the actual predicted value. In this way, this observer will work very fine for this system. So, now let us come to the conclusion. So, what is our conclusion? In several practical applications, observer nature is binary in nature. And due to that reason, I have to design an observer for that class of system.

Okay. So, somehow I have used the philosophy of discrete sliding mode control to design the first part. Okay. The second part is coming from the solution. Okay.

And in this way, I can measure the full state based on the sign measurement. And after that, we can be able to deploy this kind of observer for several practical applications. So, in this way, I will conclude this lecture. Thank you very much.