High Gain Observer for Nonlinear Systems

Welcome back. In the previous class, I was talking about the extension of the Luenberger observer for non-linear systems. And we have assumed that the non-linear dynamics are completely known to us. And then we proceed to the observer design. In reality, that assumption is never satisfied, and for that reason, in this class, I am going to look into another kind of observer that is called a high gain observer. So, the purpose of the discussion is to design a high-gain observer for the non-linear system.

So, like in the previous class, I am assuming that I do not have a linear system, but here our assumption is that we have several model uncertainties and disturbances. It means that our practical system is running in some kind of environment where we have disturbances as well as perturbations. And our job is exactly the same as in the previous class. Now, I again have information about some part of the output; you can represent it as $y = h(x)$,
or, for simplicity, I am going to assume that it is represented in this form.

Now, using this particular information, I have to estimate the entire state. Obviously, the state estimation task should finish as quickly as possible. Due to that reason, convergence is one of the very important criteria. So, in order to design an observer for a non-linear system, again some theory is required. So, what kind of theory am I going to apply? So, I am going to apply some kind of theory that is inspired by the time scale separation.

That is also called a singular perturbation. So, during the first module, you have already learned about singular perturbation and what our observation was. Suppose that if you have a system like $\dot{x} = u + d(t)$,
and if you design

$$u = \frac{1}{\varepsilon} x$$

or $u = \frac{k}{\varepsilon} x$,
you can substitute it into the dynamics of x. And if you choose $\varepsilon$to be very, very small, at that time you can see that this dynamics, once $\varepsilon \to 0$, is going to collapse, and $\varepsilon k x \to 0$
in spite of any kind of disturbance. So, this property is called the time separation property, if you are going to look for the same kind of property for higher-order systems.

So, that kind of things I am going to apply in this particular lecture and obviously, system is non-linear. So, I am going to apply the Lyapunov framework. This observer, similar to the high-gain feedback, suffered from some kind of practical limitation that is called peaking phenomena. And due to that reason, we now have to explore a realistic observer called the sliding mode observer. We are also going to look into the sensitivity to the measurement noise because noise is always present; whatever sensors I deploy, I cannot assume that all information is perfect.

And if we are going to apply high gain, noise may or may not be amplified. So, we have to actually take care in the selection of gain. I am also going to perform a robustness analysis. So, let us start this lecture. Here, I am going to actually start with a two-dimensional system.

So, what is the physical interpretation? I am assuming that I have some mechanical system or electrical system, where the system is represented in coordinates such as position- or velocity-like quantities. Here, I am assuming that the system is represented in a state-space form and that the term ϕ is not completely known. However, I have to guarantee that, since this is a nonlinear system, there exists a unique solution. So, for that, this assumption is very, very crucial.

So, again, just like in the previous class, I have to show that this particular term ϕ is a locally Lipschitz function with respect to $x$and u for all $t \geq 0$.
We are also assuming that this term remains bounded in order to prevent blow-up as time tends to infinity.

Now, the observer design is exactly like in the previous class. What am I going to do? Since I have second-order dynamics, I am going to apply one correction term here and another correction term here. So, this is exactly like the Luenberger observer.

Suppose that I have a system like

$$\dot{x}_1 = x_2$$

and $\dot{x}_2 = u$, and there is no uncertainty or disturbance. If I know
$y = x_1$, then this information is available to me.

So now, how do we design an observer? What can you do? You can copy this dynamics. But copying exactly is not possible, and due to that reason, $\hat{x}_1$ and $\hat{x}_2$ come into the picture. So, the observer dynamics will involve $\hat{x}_1$, $\hat{x}_2$, and the known input u.

Now, what is our main goal? This is our original system, and this system is going to run on the computer. Now, our main goal is to force this system to behave like the actual plant. How do you force it? I can only design some kind of correction term based on the information from whatever measurement I have. And for that reason, I am going to use gains like

$$h_1(y - \hat{x}_1)$$

and $h_2(y - \hat{x}_1)$. After that, I will calculate the error, and based on the error, I can determine $h_1$and h_2. So, this is the philosophy of the Luenberger observer.

Here, what I am going to do is design $h_1$and $h_2$based on the principle of high gain. Here, one more term comes into the picture because the dynamics of $x_2$are not completely known. And due to that, I am assuming that $\phi_0(\hat{x}, u)$ is going to contain the nominal model of $\phi(x, u)$.

I have this assumption because, in the end, I also have to provide a guarantee. So, I cannot assume that any arbitrary class of nonlinear terms is acceptable in order to design a high-gain observer.

So, I am only able to design a high-gain observer provided this nominal model, or this original model, satisfies this kind of condition, where $L$and $M$are some kind of nonnegative bounds. Now, what is our main target? To force this kind of system to run inside the computer by designing $h_1$and $h_2$to connect it to the original system. So, these two gains are the design parameters. So, how can we guarantee that the system running inside the computer behaves the same as the original system? For that, I have defined the estimation error. Now, I have to make sure that $\tilde{x}(t) \to 0$ as $t \to \infty$.

So, again, you can apply the idea of Newton. What can you do? Since I have to control this error, I am going to control its rate of change. For that reason, I have calculated the rate of change. I have substituted the value of $x$from the original dynamics and $\hat{x}$from the observer dynamics that is running inside the computer. And then I obtain a model of the form
$\dot{\tilde{x}} = A\tilde{x} + B(\cdot)$,
where $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the matrix A is actually the design matrix. The gains $H_1$and $H_2$are the parameters that I have to design.

And δ is nothing but the mismatch between the original nonlinear term and the nominal term. So, what is our main intention now? I have to force at least this system to be asymptotically stable. Why? Because we know that if there is no disturbance and no mismatch, then this is a linear system. And what is the property of a linear system? If a linear system is stable, then for any bounded input that you apply, you will obtain a bounded output.

This is the property of a stable linear system. So, in the error coordinates, whatever system I have should be stable; only then can I apply this kind of philosophy. And due to that reason, I am going to make sure that A_0
is Hurwitz in nature. Now, what is our second goal? Since a bounded input gives a bounded output, if the input range is higher, then the output range is also higher. It means that if this input, which is the unknown input δ, has a certain magnitude, then the resulting error response will also scale accordingly.

So, if δ is worse, then our precision is also worse. So, by design, I have to minimize its effect. So, how do you do that? For that, I now have to understand the dynamics more carefully. Since this is a linear system, where δworks like some kind of bounded perturbation, I can analyze it using standard linear system tools. Now, I am going to apply the Laplace transform, because for a linear system, you can analyze the behavior either in the time domain or in the frequency domain.

It is always easier to look into the frequency domain. I am assuming that the initial condition is equal to 0, because this will only contribute to the transient response. What is our objective? Our objective is to wait for some time such that $\tilde{x}(t) \to 0$ as $t \to \infty$.
For that reason, the initial condition does not matter, and during the analysis, I am going to

apply this assumption. Now, I am going to create some kind of transfer function that treats δas the input. So, this is the Laplace transform of the small signal $\delta(t)$, which I denote by $\Delta(s)$. And now, whatever transfer function comes into the picture will map this input $\Delta(s)$to the corresponding output in the error coordinates.

Now, our main problem is to work on the transfer function and minimize it so that I can suppress the uncertainty. That is our main goal. So, what am I going to do for that? I am going to calculate the transfer function. So, how do you calculate the transfer function? Again, the steps are very simple because I have a second-order system. So, you have a 2×2 matrix, and if you have a 2×2 matrix, during your first-year mathematics course, you have already learned how to deal with it.

And after that, I will be able to calculate the inverse. You can easily calculate the inverse when you have a 2×2matrix. So, I have applied the same formula here. Now, after that, I obtain the transfer function, because in order to calculate the transfer function, I have to multiply by B and $B^\top$. If you see, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
So, after carrying out this multiplication, this gives the final transfer function.

Okay, so once I get the final transfer function, I have to analyze it further. What is our job? The transfer function is nothing but the Laplace transform of the output divided by the Laplace transform of the input. And what is the input here? In this system, I am assuming that, in the Laplace coordinate frame, $\Delta(s)$
is the unknown input, which is the mismatch between the nominal model and the original model. And here, $\tilde{X}(s)$ represents the Laplace transform of the error dynamics that I have to force to 0. So, by looking at this expression, you can easily see that if I am able to dominate a particular part of the transfer function, then I can minimize the Laplace transform of the error corresponding to the unknown input. Now, terminology like "high gain" comes into the picture.

So, what is the motivation for high gain? We are trying to minimize this transfer function. During that minimization, I am going to select two parameters, $h_1$and h_2. I am going to take the help of a very, very small parameter $\varepsilon \leq 1$; that is the physical interpretation of a small quantity. You can see that when $\varepsilon < 1$, the gain $h_1$becomes very, very large. Even if ε^2appears, it becomes further larger. Similarly, if you have a third-order system, then I will select terms involving ε^3, and for an n-th order system, I will select terms involving ε^n.

So, somehow you can see that I am placing the gains at different scales. If you do that, particularly for a second-order system, you can see that after substituting

$$s' = \varepsilon s$$

into s, I will obtain an expression of this form. Now, if I keep the gains very, very high, how do I keep the gains high? You can fix α_1 and α_2.
Then, you can simply tune this particular parameter ε.

If εtends towards 0, you can see that this forces everything towards 0. So, you can easily verify that the transfer function tends towards 0. It means that if the transfer function is equal to 0, then your system becomes insensitive with respect to uncertainty. And this property is exactly the property of high gain. Why does this philosophy come into the picture? Because I am designing the gains in such a way that $\varepsilon$is very, very small. Due to that reason, these classes of observers are called high-gain observers.

Now, since I have already told you that if you have a linear system, then you are very lucky: you can carry out the analysis either in the time domain or in the frequency domain, and both domains will give you consistent results. The interpretations may look different, but the conclusions are the same. So, now I am going to interpret the error in the time domain in order to understand the time-separation property of the error. So, what have I done? Since I have a second-order system, I have defined
$\tilde{x}_1 = x_1 - \hat{x}_1$,
$\tilde{x}_2 = x_2 - \hat{x}_2$,
and now I have defined new variables η_1 and η_2.

You can see here that η_2 is exactly the estimation error of x_2, but here η_1 is a scaled version of the estimation error, scaled by the design parameter ε. So, in particular,

$$\eta_1 = \frac{\tilde{x}_1}{\varepsilon}$$

and $\eta_2 = \tilde{x}_2$.

And in these new coordinates, the dynamics of the system can be represented in this form. It is possible to show that A_0 is the system matrix corresponding to the original error dynamics. Now, after applying this transformation, I obtain a new matrix F_0. So, the eigenvalues of A_0 are equal to $\frac{1}{\varepsilon}$ times the eigenvalues of F_0.

In this way, I can now design ε to be very small, and then I can increase the speed of convergence. So, I am now going to explain the step-by-step process of how to proceed. So, from the frequency-domain analysis, we have already seen that the gains can be designed as
$h_1 = \frac{\alpha_1}{\varepsilon}$,
$h_2 = \frac{\alpha_2}{\varepsilon^2}$. At that time, if you substitute these gains, then the matrix A_0 is represented in this form, because here $h_1$and $h_2$are replaced by these expressions. Now, if you look at $h_1$and h_2, that is exactly why I have designed them in this way: because I want very fast convergence, and I also want some degree of insensitivity with respect to disturbances.

So, if ε is truly small, then I can actually talk about insensitivity with respect to perturbations. Otherwise, I can only talk about boundedness or uniform boundedness. And for that reason, I have introduced this kind of coordinate transformation.

So, what is the role of this coordinate transformation? We are going to understand that in the subsequent discussion. What I have done is define $\tilde{x}_1 = \varepsilon\eta_1$. So, I have taken the derivative of this expression, and again I have taken the derivative of the corresponding second state. Then, I am going to express everything in terms of the η_1 dynamics and the η_2 dynamics.

Now, by using further analysis, I obtain this state-space representation. So, I have two states, and after that, I have represented them in the matrix form. Now, I have to prove that

$$\eta = [\eta_1 \ \ \eta_2]^\top$$

tends towards 0 as $t \to \infty$. That is our aim. I also have to show that the effect of the perturbation term is minimal.

Since $\varepsilon$is very, very small, one can naturally expect that the theory of singular perturbations can be applied. This theory shows that as $\varepsilon \to 0$,
the fast dynamics dominate, and the dynamics of the form

$$F\eta$$

drive ηtowards 0. So, if you are able to make the eigenvalues of this nonsingular matrix stable, then you obtain a unique solution, which is $\eta = 0$.

So, in this way, you can apply singular perturbation theory and directly arrive at this conclusion. However, sometimes, in order to clearly understand the role of the design parameters, Lyapunov analysis is more helpful. And due to that reason, I am going to proceed with Lyapunov analysis to explain how to select F, because now Fcontains the parameters $\alpha_1$and α_2. So, you can also determine them by eigenvalue calculation, but you have to ensure that the resulting system is stable.

And obviously, we have already discussed this, so now we are just summarizing. So, what is our key insight? If $\varepsilon = 0$, then from singular perturbation theory, I can conclude that $\eta \to 0$. And if $\eta \to 0$,
then the state estimation satisfies $x = \hat{x}$,
since

$$\tilde{x} = x - \hat{x} \to 0$$

due to the high gain.

So, this kind of conclusion can also be obtained using the Lyapunov method. So, you can define an energy-like function. Here, $P$is nothing but a 2×2 matrix, and I am assuming that this matrix is symmetric.

After that, this matrix is positive definite. What is the meaning of positive definite? It means that for the matrix P, all eigenvalues are strictly positive. Now, this is the Lyapunov equation. Since Q has already been fixed as $-I$, and F has already been tuned, due to that reason the matrix P is unique.

In this way, you can design the Lyapunov function, and then you can take the derivative of the Lyapunov function. After rearranging the terms and multiplying by ε, you obtain a result of this form. I multiplied by εjust to simplify the analysis, because the sign property of $\dot{V}$and $\varepsilon\dot{V}$is the same when $\varepsilon > 0$. So, even if you multiply by ε, the Lyapunov stability property does not change, and for that reason, in Lyapunov analysis, you are allowed to incorporate it.

Now, what I am assuming is that the mismatch between the nominal model and the original model is expressed in this form. Now, what am I going to do? I am going to apply the Cauchy–Schwarz inequality. If you have a function of this kind, then you can take its norm. So, I have taken the norm, and after that, I am going to substitute the value of δ from the above expression. Then I will obtain an expression of this form.

If you tune your parameter ε^2 appropriately, then you can factor out a term involving $\parallel \eta \parallel^2$. By taking it out from the expression, you will finally obtain a term of this form, which allows you to directly analyze the sign of the Lyapunov derivative and conclude about the convergence of η.

Now, if I maintain this quantity to be greater than or equal to $\frac{1}{2}$, then at the same time this term becomes equal to $\frac{1}{4}$, and in this way, I obtain a condition of this form. So, what is our objective? Our main objective is to ensure that
$\varepsilon\dot{V} \leq 0$.

Now, if I select ηsuch that $\parallel \eta \parallel \geq 2\varepsilon cM$, where c comes from the Lyapunov function and Bcomes from the system dynamics, then it is possible to show that
$\dot{V} \leq 0$, or equivalently, $\varepsilon\dot{V} \leq 0$.

Under this condition, the Lyapunov function is nonincreasing, which guarantees the boundedness of the estimation error and convergence of the observer error dynamics.

So, this bound on $\eta$is nothing but the ultimate bound. What is the meaning of ultimate bound? It means that for all $t \geq T$,
for some finite time T, it is possible to show that the system trajectory remains within this bound. Here, what is meant by the system trajectory? I am referring to the error dynamics, which are always going to lie within this particular band. And by making $\varepsilon \to 0$,
you can see that $\eta$is forced towards 0. Now, the design part is over. So, what have we observed? We have observed that, in the presence of mismatch, a high-gain observer still works effectively.

But whenever you apply a high-gain observer, some practical difficulties come into the picture. One such practical difficulty is termed the peaking phenomenon. So, what is the meaning of the peaking phenomenon? I am going to set ε to be very, very small. The peaking phenomenon refers to the fact that, during the transient phase, the observer states can exhibit very large peaks. And this transient response can amplify several unmodeled dynamics present inside the system.

So, those kinds of things I am going to look into. So, what I have done again is look in the time domain framework, and after that, I calculated the error. So, now system can be expressed like this. Now, here you can see this system and what happens? Now, I am able to solve this equation. The solution is exactly like the solution of a linear time-invariant system (LTI).

This is nothing but δ. I am assuming that I have an unknown control input. So, I can easily solve the system, and after solving it, you can see that as $\varepsilon \to 0$,
the variable η_1 becomes very, very large.

So, if you take $\varepsilon = 0.01$, then you can see that
$\eta_1(0) = \frac{1}{\varepsilon} = 100$.

What is the physical interpretation of this? You can see that the solution has the form
$\eta_1(t) = \frac{1}{\varepsilon} e^{-at/\varepsilon}$.

Now, the first part of the dynamics contains the term
$\frac{1}{\varepsilon} e^{-at/\varepsilon}$.
So, at time $t = 0$,
this term becomes $\eta_1(0) = \frac{1}{\varepsilon}$.

So, if ε is very, very small, then what happens? Even if you start very close to the equilibrium point, your initial transient becomes very large. However, this large transient decays very fast due to the exponential term.

Obviously, there is an exponential term, so it is going to decay. But during this initial period, the system dynamics may behave in an undesirable way, especially because the system is nonlinear. As a result, finite-time blow-up can occur. That means the trajectories may move towards infinity, and our physical system can get damaged due to this kind of observer behavior. And due to that reason, we have to select ε in a reasonably careful way. Otherwise, what happens? If the system is sensitive to uncertainties, then by aggressively designing a high-gain observer, one can actually destabilize the overall system instead of stabilizing it.

So, in order to address this kind of issue, several new observer theories have appeared in the literature. One of these theories is based on sliding mode control. And what is the beauty of this approach? It is possible to show, in the case of sliding mode control—which will be the topic of future lectures—that the error dynamics $\tilde{x}$ tend towards 0 in finite time, that is, as $t \to T < \infty$, by designing some kind of finite gain. This is one of the key contributions of the sliding mode observer. Another important result is that the gain requirement is not infinite. So, by using a reasonably high gain, rather than an extremely large gain, one can successfully solve the sliding mode observation problem.

The theory of high-gain observers is quite attractive. They are able to suppress any kind of uncertainty or mismatch between non-linear models, but they suffer from the picking phenomenon. Obviously, there are several ways one can tackle the picking phenomena. This kind of observer is actually contributed by Professor Khalil, and you can see this kind of generalization for higher-order systems for high-gain observers in the literature on non-linear control systems written by Professor Khalil. So, with this remark, I am going to end this lecture. Thank you very much.